Universität
Bremen

Advances in Algorithmic Meta-Theorems with Applications to Solution Discovery

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Parameterized Complexity of Computational Reasoning (PCCR 2026)

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Solution Discovery

Motivation

In many real-world settings, the current state of a system is already almost usable, but not yet perfect.

Goal: Find a new feasible solution by modifying the current one as cheaply as possible.

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Solution Discovery [FGMMRRSS 2023]

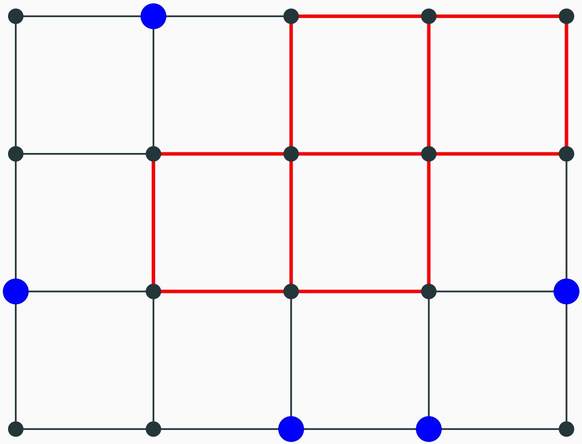
We model this as a graph problem inspired by combinatorial reconfiguration:

- we start with a given configuration (imagine tokens on vertices),
- we allow only small local changes (slides along edges),
- and we ask whether a valid solution can be reached within a given budget.

Feedback Vertex Set Discovery

Solution Discovery

- Input: A graph G, an initial set $S \subseteq V(G)$ of token positions, and a budget $b \in \mathbb{N}$.
- Question: Can we reach, by at most b token slides along edges, a set S' that forms a valid solution?



A discovery sequence for FEEDBACK VERTEX SET

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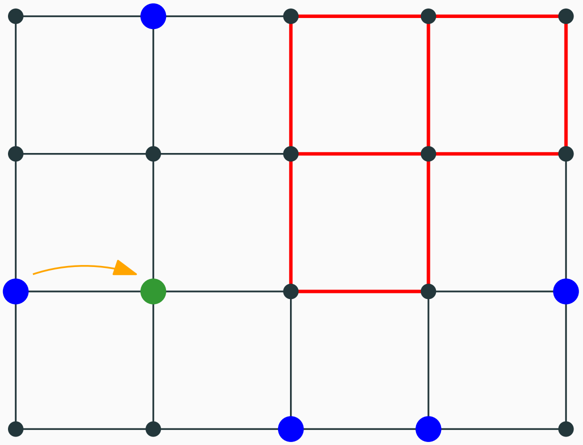

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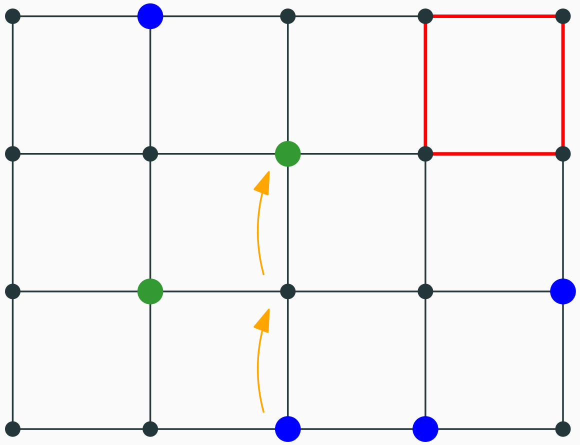

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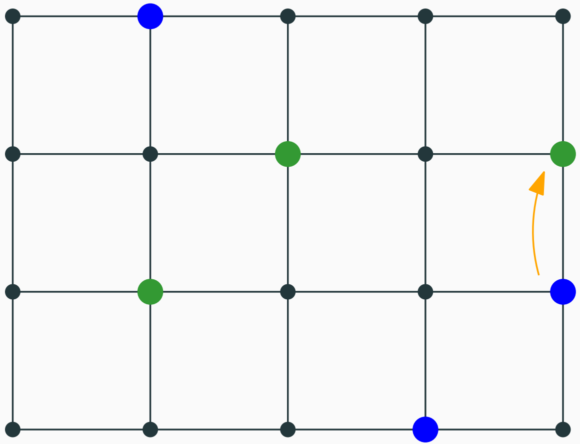

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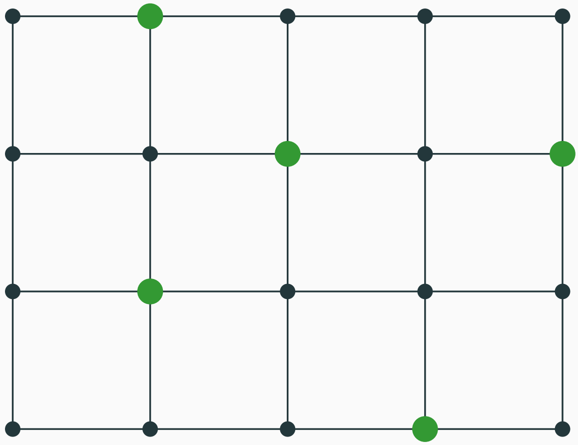

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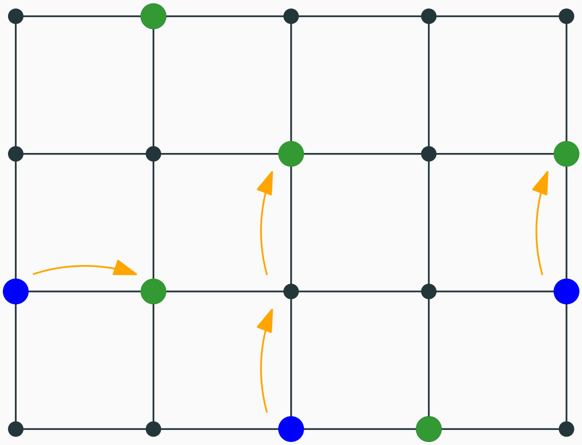

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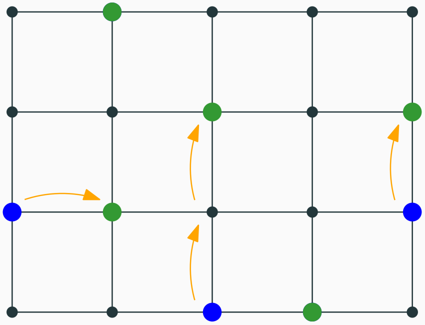

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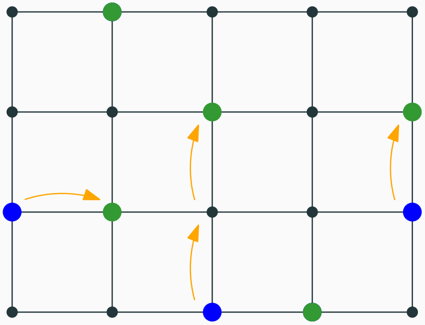

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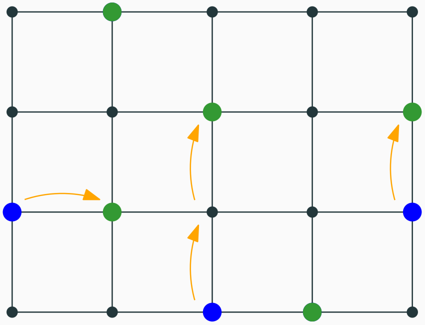

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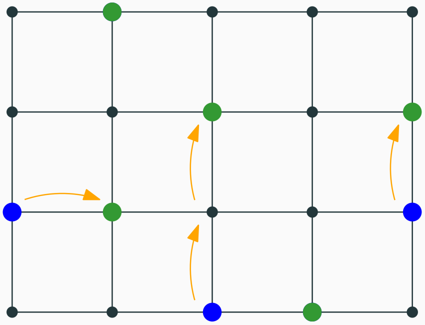
- What are the most general graph classes on which the problem is tractable?



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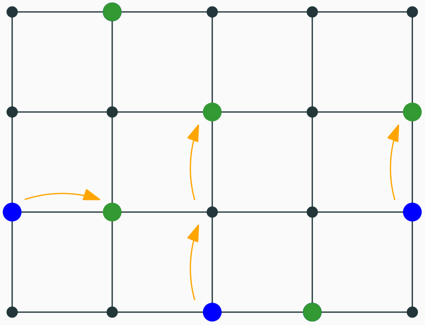

FVS-Discovery

Meta algorithmic approach

Can logic provide the right abstraction to reveal such tractability frontiers?

Logic as a tool to solve algorithmic problems

Model checking

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A formula for $\textsc{FVS-Discovery}$ (2 tokens move 2 steps each, a unary predicate T marks the initial token positions):

$$\varphi_{2,2} = \underbrace{\exists s_1, s_2, a_1, a_2, t_1, t_2 \left(T(s_1) \wedge T(s_2) \wedge \ E(s_1, a_1) \wedge E(a_1, t_1) \wedge \ E(s_2, a_2) \wedge E(a_2, t_2)\right.}_{\text{First-order logic (FO)}}$$

$$\left. \wedge \ \underbrace{\mathrm{Acyclic}(G - (T - \{s_1, s_2\} + \{t_1, t_2\}))}_{\text{Monadic second-order logic (MSO)}} \right).$$

Algorithmic meta-theorems

Courcelle's Theorem

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- $\mathscr{C}$ has bounded treewidth $\Leftrightarrow$ MSO cannot transduce all graphs from $\mathscr{C}$:
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- $\mathscr{C}$ has bounded treewidth $\Leftrightarrow$ MSO cannot transduce all graphs from $\mathscr{C}$:
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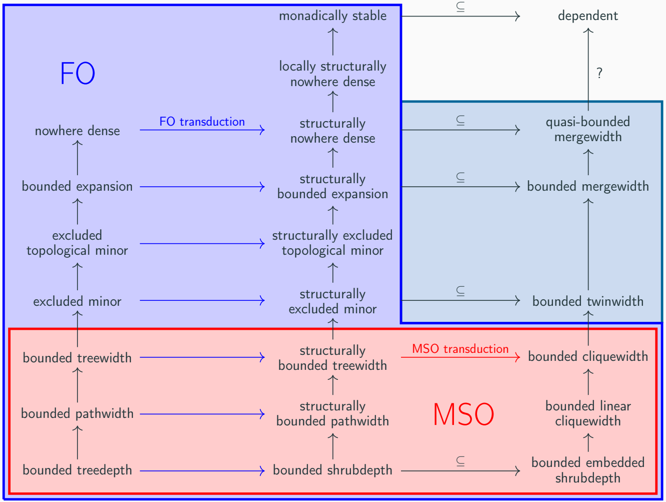
- Courcelle's Theorem captures the limit of tractability for MSO.
  (under a mild condition on efficiency of encoding)

Tractable graph classes



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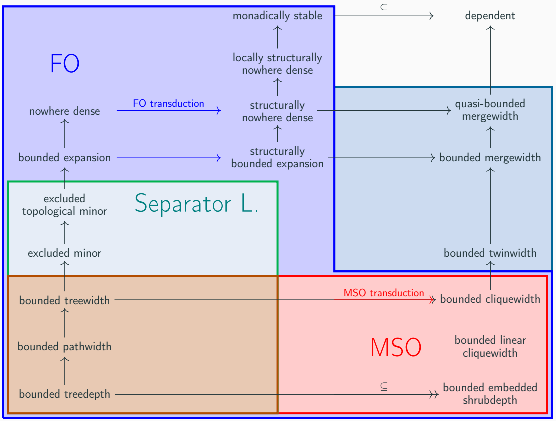
To formulate FVS-discovery, we need to talk about the disappearance of cycles.

- MSO can formulate the problem, but then we have tractability only on classes with bounded treewidth/cliquewidth.

Separator logic

- FO is too weak for connectivity and acyclicity.



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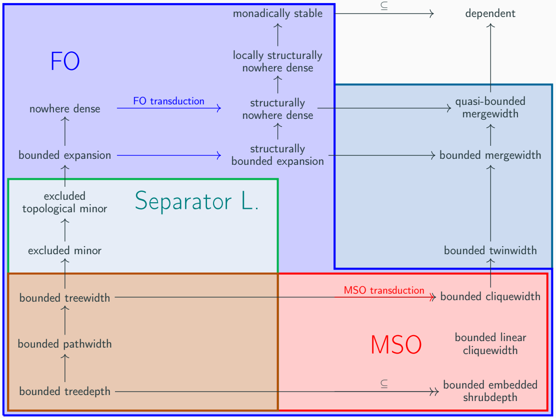
$$\mathrm{conn}(x, y; \bar{z}),$$

that is, connectivity after deleting finitely many vertices [B 2022, SSV 2022].



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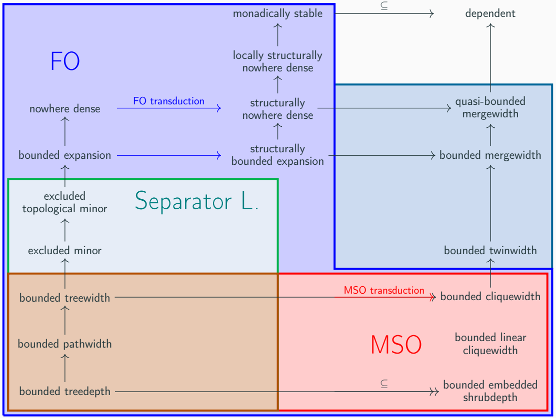
- On weakly sparse classes:

  $\mathscr{C}$ dependent for separator logic $\Leftrightarrow$

  $\mathscr{C}$ excludes a topological minor and model checking is efficient on $\mathscr{C}$ [PSSTV 2022].



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Subconnectivity logic [MOdMSSSTV 2026+]

FO+subconn extends FO by atoms

$$\mathrm{subconn}(x, y; \bar{z}),$$

with the meaning: x and y are connected in G_Z after deleting $\bar{z}$.

Subconnectivity logic

Theorem [MOdMSSSTV 2026+]

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Corollary

$\mathrm{FVS\text{-}D{\scriptsize ISCOVERY}}$ on classes with excluded minors is FPT with respect to parameter b.

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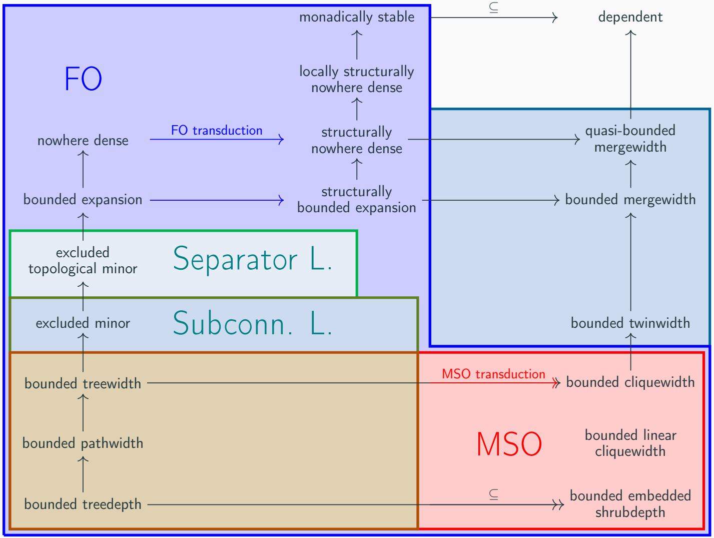

Proof sketch

Key observation

Let Y be a depth-first search spanning tree of a (connected) graph G and let $\preceq_Y$ be the corresponding tree-order.

Then there is a formula $\psi \in \mathrm{FO}[E, \preceq]$ such that

$$G \models \mathrm{conn}(x, y, \bar{z}) \Leftrightarrow (G, \preceq_Y) \models \psi(x, y, \bar{z}).$$

Proof of the observation

Tree-ordered graph classes

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- A linearly ordered hereditary class of graphs $\mathscr{C}$ is dependent for FO $\Leftrightarrow$ $\mathscr{C}$ has bounded twinwidth.

  Furthermore, FO model checking is FPT on such classes.
  [Bonnet et al. 2024]

Tree-orders vs linear orders

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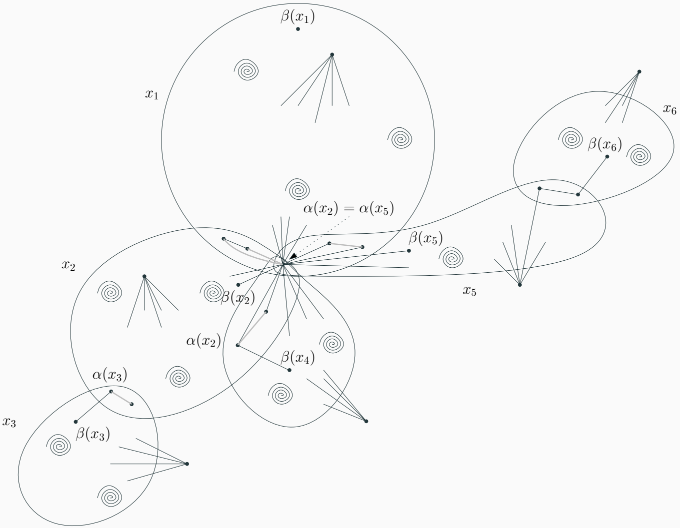
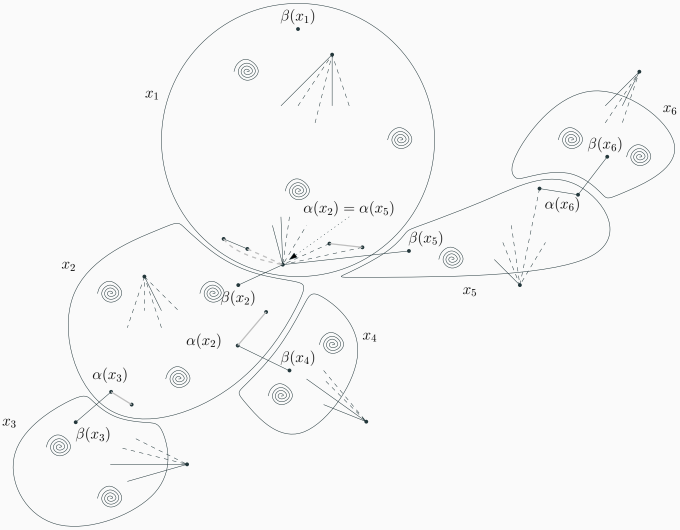
- If $\mathscr{C}$ is apex minor free, then there exists an apex minor free class $\mathscr{D}$ of subcubic graphs such that for every $G \in \mathscr{C}$ there exists $H \in \mathscr{D}$ with $G \leq_m H$. [Milolidakis, 2025]

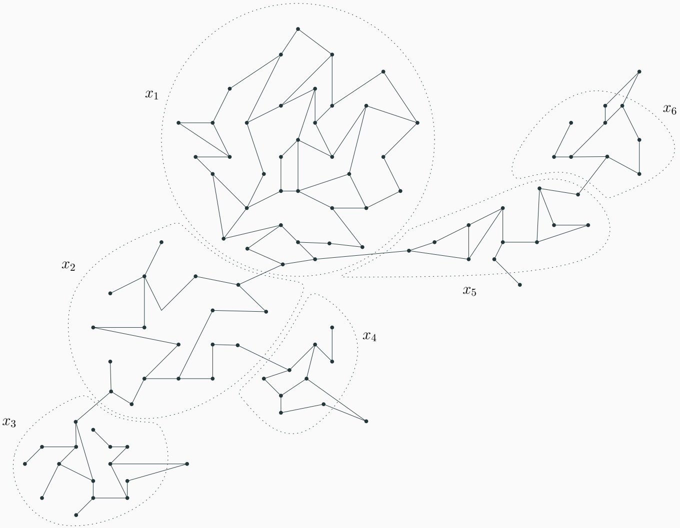
The structure theorem of Robertson and Seymour



Killing the apices



Apex minor free graphs [Milolidakis, 2025]



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- Given a graph $G \in \mathscr{C}$, compute a subcubic graph H excluding a fixed apex minor and a spanning tree Y_H of H such that G and a dfs tree Y_G of G is definable in H.
- The class of all (H, Y_H) can be linearly ordered in a definable way and hence has bounded twinwidth and model checking is tractable.

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Corollary [MOdMSSTV 2026+]

Let $\mathscr{C}$ be a weakly sparse hereditary class of graphs. The following are equivalent.

1. $\mathscr{C}$ excludes a minor.
2. The expansion of $\mathscr{C}$ by all spanning forest orders is dependent.
3. The expansion of $\mathscr{C}$ by all spanning forest orders has bounded twinwidth.

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Theorem [MOdMSSTV 2026+]

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Meta-theorem for solution discovery

For every fixed FO+subconn-formula $\psi(T)$, ψ-Modulator Discovery on classes with an excluded minor is fixed-parameter tractable with respect to b.

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Meta-theorem for solution discovery

For every fixed FO+subconn-formula $\psi(T)$, ψ-$\textsc{Modulator Discovery}$ on classes with an excluded minor is fixed-parameter tractable with respect to b.

- In particular, $\textsc{FVS-Discovery}$ is FPT on classes with an excluded minor, parameterized by b.

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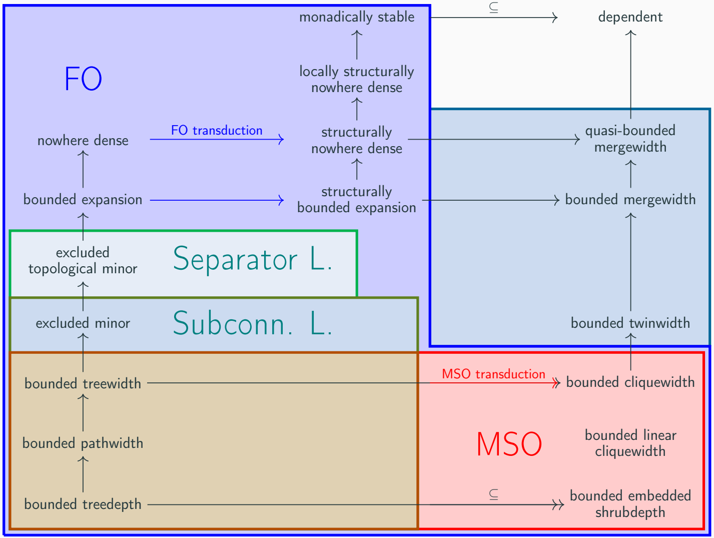

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