Gateways to Tractability for Satisfiability in Pearl's Causal Hierarchy

PCCR 2026, Lisbon

Robert Ganian, Marlene Gründel, Simon Wietheger

TECHNISCHE UNIVERSITÄT WIEN

Causality

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Causality

Man's mind cannot grasp the causes of events in their completeness, but the desire to find those causes is implanted in man's soul.

(Leo Tolstoy, War and Peace, 1869)

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but the desire to find those causes is implanted in man's soul.

Without considering the multiplicity and complexity of the conditions [...]
he snatches at the first approximation to a cause that seems to him intelligible
and says: 'This is the cause!'

(Leo Tolstoy, War and Peace, 1869)

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

Recent Publications:

- [Dörfler, van der Zander, Bläser, Liskiewicz, ICALP '25]
- [Dörfler, van der Zander, Bläser, Liskiewicz, ICLR '25]
- [Mossé, Ibeling, Icard, Review of Symbol Logic '24]
- [Gupta, Bläser, AAAI '24]
- [van der Zander, Bläser, Liskiewicz, IJCAI '23]

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

probabilistic
(*seeing*)

Given the fact, that I am feverish ($F = \text{yes}$), what is the probability that I have a headache ($H = \text{yes}$)?

$$\Pr(H = \text{yes} \mid F = \text{yes})$$

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

causal
(*doing*)

probabilistic
(*seeing*)

What is the probability that my headache will be cured ($H = \text{no}$) if I take aspirin ($A = \text{yes}$)?

$$\Pr([A = \text{yes}]\ H = \text{no})$$

$\uparrow$

intervention

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causation: $\Pr([A = \text{yes}]\ H = \text{no}) > \Pr(H = no)$

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What is the probability that my headache will be cured ($H = \text{no}$) if I take aspirin ($A = \text{yes}$)?

causation: $\Pr([A = \text{yes}]\ H = \text{no}) > \Pr(H = no)$

no causation: $\Pr([A = \text{yes}]\ H = \text{no}) = \Pr(H = no)$

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

counterfactual
(*imagining*)

causal
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probabilistic
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What is the probability that it was the aspirin ($A =$ yes) that cured my headache ($H =$ no)?

$$\Pr(([A = \text{no}]\ H = \text{no}) \mid (A = \text{yes} \land H = \text{no}))$$

partial and chained interventions

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

counterfactual
(*imagining*)

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base

comparisons

$$\Pr(\circ) \geq \Pr(\odot),$$
$$\Pr(\oplus) \geq 1$$

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

counterfactual
(*imagining*)

causal
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probabilistic
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base

linear

sums over probabilities

$$\Pr(\circ) - \Pr(\oplus) \geq \Pr(\odot),$$
$$2\Pr(\odot) + \Pr(\otimes) \geq 3\Pr(\circ)$$

The Languages of Pearl's Causal Hierarchy [Judea Pearl '00, '09]

counterfactual (*imagining*)

causal (*doing*)

probabilistic (*seeing*)

base

linear

polynomial

polynomials over probabilities

$$\Pr(\circ)^2 \geq \Pr(\odot) + 0.5,$$
$$2\Pr(\oplus) - 0.1 \geq 3\Pr(\odot)^3$$

Satisfiability in Pearl's Causal Hierarchy

SATISFIABILITY IN PCH

Input: A PCH-expression ϕ over variables $\mathbf{V}$.
Task: Decide if there exists a model $\mathcal{M}$ that satisfies ϕ.

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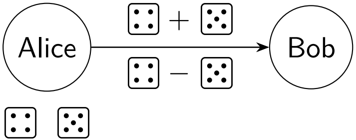
- a probability distribution $\mathbb{P}$ over $\mathbf{U}$.

Example—Satisfiability in Pearl's Causal Hierarchy



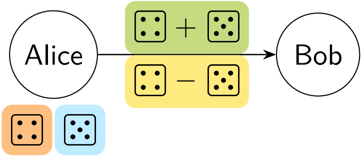
Example—Satisfiability in Pearl's Causal Hierarchy

$$\mathcal{M} = (\mathbf{V} = \{V_1, V_2\}, \mathbf{U} = \{U_1, U_2\}, \mathcal{F} = \{f_{V_1}, f_{V_2}\}, \mathbb{P})$$

$$f_{V_1} = U_1 + U_2,$$

$$f_{V_2} = U_1 - U_2,$$

$$\mathbb{P}(U_k = i) = 1/6 \text{ for } k \in [2] \text{ and } i \in [6].$$



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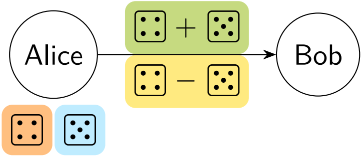

A set of equalities satisfied by $\mathcal{M}$:

$$\Pr(V_1 = 2 \wedge V_2 = 0) = 1/36,$$
$$\Pr(V_2 = 0 \mid V_1 = 2) = 1,$$
$$\Pr([V_1 = 2]\ V_2 = 0) = \Pr(V_2 = 0).$$

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$f_{V_1} = 2$,

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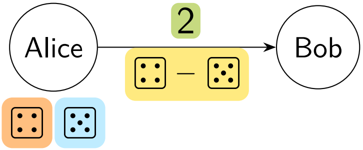

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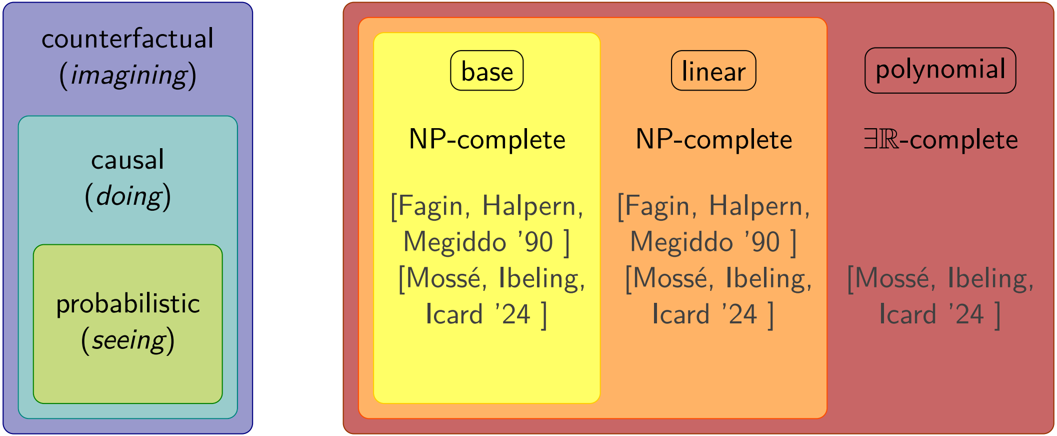
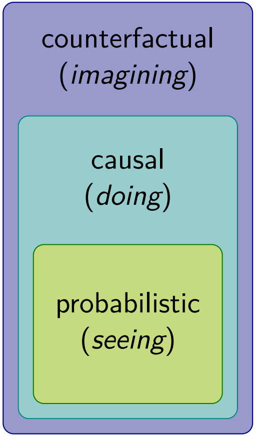
Classical Complexity of Satisfiability in Pearl's Causal Hierarchy



Our Results—Tractability in Pearl's Causal Hierarchy



$\boxed{\text{polynomial}}$

$\exists\mathbb{R}$-hard
even if

$|\mathbf{V}| = 1$ or

$|D| = 2$ and
all variables
occur isolated

Our Results—Tractability in Pearl's Causal Hierarchy

counterfactual
(*imagining*)

causal
(*doing*)

probabilistic
(*seeing*)

causal & counterfactual

FPT by $|D| + |\mathbf{V}|$ (XP by $|\mathbf{V}|$)

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Can we improve? — NO

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Proof Sketch: Tractability for probabilistic & linear constraints

Input: set of constraints ϕ

$$\Pr(V_1 = 1 \land V_3 = 1) \geq \frac{1}{2}$$
$$\Pr(V_2 = 1 \lor V_3 = 1) \geq 2\Pr(V_3 = 1 \lor V_4 = 1)$$
$$\Pr(V_4 = 1) \geq \frac{1}{3}$$

Output: Model $\mathcal{M} = (\mathbf{V}, \mathbf{U}, \mathcal{F}, \mathbb{P})$ with $\mathcal{M} \models \phi$, if it exists.

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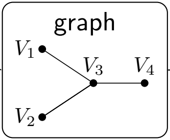

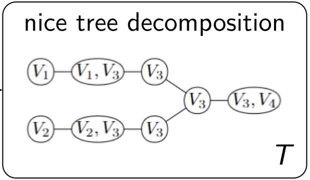

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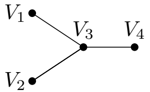
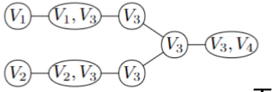
graph



nice tree decomposition



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$\mathbf{U}$ encodes edges in T,
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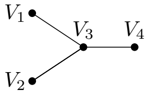

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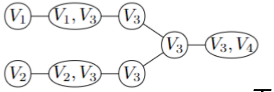

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Linear Program

LP-variables capture bagwise assignments

LP-constraints (1) rephrase ϕ & (2) ensure consistency

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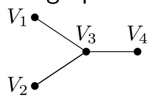

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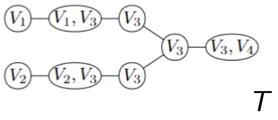

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We covered

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- further applications of treewidth-guided Linear Programming

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