Claw-free β-perfect graphs

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For a graph G, let $\beta(G)$ be the maximum of $\delta(G')+1$ taken over all induced subgraphs G' of G. A graph G is *β-perfect* if $\chi(G') = \beta(G')$ for every induced subgraph G' of G. β-perfect graphs are even-hole-free, and the greedy coloring algorithm applied to a linear-time computable vertex ordering produces optimal colorings for β-perfect graphs in polynomial time. The class of β-perfect graphs was first studied by Markossian, Gasparian and Reed in 1996, and since then several subclasses defined by forbidden induced subgraphs have been identified, but a forbidden induced subgraph characterization remains unknown. It also remains open whether β-perfect graphs can be recognised in polynomial time. The *claw* is the complete bipartite graph $K_{1,3}$. In this paper we give a forbidden induced subgraph characterization for the class of claw-free β-perfect graphs. Furthermore, we give an algorithm that decides in polynomial time whether a given claw-free graph is β-perfect. An intermediate result, which may be of independent interest, is a forbidden induced subgraph characterization of β-perfect rings and an algorithm that decides in polynomial time whether a ring is β-perfect.

1 Introduction

All graphs in this paper are finite and simple. For a graph G, let

$$\beta(G) = \max\{\delta(G') + 1 : G' \text{ is an induced subgraph of } G\},$$

where $\delta(G)$ denotes the minimum degree of a vertex in a graph G.

A *k-coloring* of a graph G is any function $c\colon V(G) \to \{1,\dots,k\}$ that has $c(u) \neq c(v)$ for all adjacent vertices $u, v \in V(G)$. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest integer k for which there exists a k-coloring of G. The parameter $\beta(G)$ is an upper bound on the chromatic number of a graph, i.e., $\chi(G) \leq \beta(G)$ for

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every graph G. A graph G is *β-perfect* if $\chi(G') = \beta(G')$ for every induced subgraph G' of G [10], and is *β-imperfect* if it is not β-perfect. A number of classes of β-perfect graphs have been identified in literature (see, e.g., [4, 6, 7, 8, 9, 10]), but for the class of all β-perfect graphs, the complexity of their recognition and characterization in terms of forbidden induced subgraphs remains open.

A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H, and G is *H-free* if G does not contain H. We extend this terminology to a family of graphs $\mathcal{H}$ by saying that G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. A *hole* is a chordless cycle of length at least 4, and is *even* or *odd* depending on the parity of its length. The class of β-perfect graphs forms a subclass of even-hole-free graphs (a class that has been the object of much research; see [14] for a survey), for if H is an even hole, then $2 = \chi(H) < \beta(H) = 3$.

A graph is *minimally β-imperfect* if it is β-imperfect but all its proper subgraphs are β-perfect. Even holes, for example, are minimally β-imperfect.

A class of graphs is *hereditary* if it is closed under taking induced subgraphs. A *clique* of a graph G is a (possibly empty) set of pairwise adjacent vertices of G. A vertex is *simplicial* if its neighborhood is a clique, and is a *simplicial extreme* if it is simplicial or has degree 2. It is known that no minimal β-imperfect graph besides even holes contains a simplicial extreme, so one may prove β-perfection of graphs in a hereditary class of graphs $\mathcal{C}$ by showing that every graph in $\mathcal{C}$ contains a simplicial extreme (see, e.g., [4, 8, 9, 10]). Though not all β-perfect graphs contain simplicial extremes.

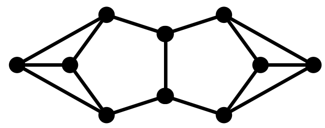

Figure 1: A minimal β-imperfect graph which is the clique-sum of two β-perfect graphs.

A *clique cutset* of a graph G is a clique $C \subseteq V(G)$ such that $V(G) \setminus C$ admits a partition (V_1, V_2) where V_1 is anticomplete to V_2. In this case we say that G is the *clique-sum* of $G[V_1 \cup C]$ and $G[V_2 \cup C]$, and that G *admits a clique cutset*. A *double clique cutset* of a graph G is a clique cutset C such that C admits a partition (C_1, C_2) and $V(G) \setminus C$ admits a partition (V_1, V_2) such that the only edges between $V_1 \cup C_1$ and $V_2 \cup C_2$ are those between C_1 and C_2; in particular, the sets C_1, C_2, V_1 and V_2 are all nonempty. In this case we say that G is the *double clique-sum* of $G[V_1 \cup C]$ and $G[V_2 \cup C]$, and that G *admits a double clique cutset*.

The *claw* is the complete bipartite graph $K_{1,3}$. In [2], Chudnovsky and Seymour observe that clique-sums do not necessarily preserve the property of being claw-free, but double clique-sums do. The same phenomenon occurs with respect to the property of β-perfection; that is, β-perfection is not necessarily preserved under the operation of clique-sums (see Figure 1), but it is under the operation of double clique-sums (the proof of which we give in Section 2).

In [8], the following subclass of claw-free β-perfect graphs was identified.

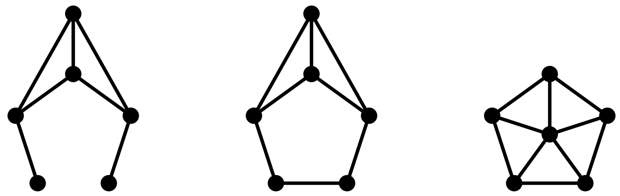

Figure 2: Forbidden induced subgraphs D_1 (left), D_2 (middle) and D_3 (right) that appear in a result of Keijsper and Tewes.

Theorem 1.1 (Keijsper and Tewes [8]). *Let G be a claw-free graph without even holes that contains no D_1, D_2 or D_3 (see Figure 2). Then G is β-perfect.*

A forbidden induced subgraph characterization for the class of β-perfect hyperholes (the class of hyperholes is defined in Section 3) is given in [7]. From that result, one can obtain an infinite family of claw-free β-perfect graphs that contain D_1 or D_2, yielding an infinite family of counterexamples to the converse of Theorem 1.1.

In this paper we generalize Theorem 1.1 by giving a forbidden induced subgraph characterization for the class of all claw-free β-perfect graphs.

- First, in Section 2, we show that no minimal β-imperfect claw-free graph admits a clique cutset, which allows us to restrict our attention to (claw, even hole)-free graphs that have no clique cutset.
- Then, in Section 3, we show that it follows from a known decomposition theorem that (claw, even hole)-free graphs with no clique cutset have quite a simple structure: they are obtained from "rings" (defined later, but briefly, they are graphs consisting of at least 4 cliques, arranged in a circular fashion, with no edges between nonconsecutive cliques, and with a rule governing the edges between consecutive cliques) by the addition of a (possibly empty) set of vertices that are adjacent to all other vertices. In fact, we may assume that this set of vertices adjacent to all other vertices is empty, for no minimal β-imperfect graph contains a vertex adjacent to all other vertices.
- In Section 4, we provide a forbidden induced subgraph characterization for the class of β-perfect rings. We use this characterization in Section 6 to recognize in polynomial time whether a ring is β-perfect (Theorem 6.2).
- In Section 5, we put together the results from Sections 2, 3 and 4 to obtain a forbidden induced subgraph characterization for the class of claw-free β-perfect graphs.
- Finally, in Section 6 we use our characterization to give an algorithm that decides in polynomial time whether a claw-free graph is β-perfect.

1.1 Terminology and notation

Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set of G by $E(G)$. For a vertex $x \in V(G)$, we denote by $N_G(x)$ the set of all neighbors of x in G, and by $N_G[x]$ the closed neighborhood of x, i.e., the set $N_G(x) \cup \{x\}$; and for $S \subseteq V(G)$, we denote by $N_S(x)$ the set $N_G(x) \cap S$. When the graph G is clear from context, we may write $N(x)$ instead of $N_G(x)$ and $N[x]$ instead of $N_G[x]$.

If A and B are disjoint subsets of $V(G)$ and every vertex of A is adjacent (resp. nonadjacent) to every vertex of B, then we say that A is *complete* (resp. *anticomplete*) to B.

The size of a maximum clique in a graph G is denoted by $\omega(G)$. A *stable set* is a set of pairwise nonadjacent vertices, and the size of a maximum stable set in a graph G is denoted by $\alpha(G)$. A *complete graph* is a graph whose vertex set is a clique.

For a set $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S, and by $G \setminus S$ the subgraph of G induced by $V(G) \setminus S$. In this paper, by a *path* we mean an induced (i.e., chordless) path. If P is a path, say with ends x and y, then P is called an *xy-path.* We denote by P^* the set $V(P) \setminus \{x, y\}$, and call this set the *interior* of P. The *length* of a path is the number of its edges. For an integer $k \geq 1$ we denote by P_k the path on k vertices, and for an integer $k \geq 4$, we denote by C_k the hole of length k.

2 Minimal β-imperfect graphs

The main goal of this section is to show that no minimal β-imperfect claw-free graph admits a clique cutset. We make use of the following results.

Lemma 2.1. *Let G be a graph, C a clique cutset of G, and (V_1, V_2) a partition of $V(G) \setminus C$ such that V_1 is anticomplete to V_2. Then $\chi(G) = \max\{\chi(G[V_1 \cup C]), \chi(G[V_2 \cup C])\}$.*

Lemma 2.2 (Horsfield and Vušković [7]). *If G is a minimal β-imperfect graph, then $\beta(G) = \delta(G) + 1$.*

Theorem 2.3 (Dirac [5]). *If G is a chordal graph that is not complete, then G contains at least two nonadjacent simplicial vertices.*

Lemma 2.4 (Markossian, Gasparian and Reed [10]). *If G is a minimal β-imperfect graph that is not an even hole, then G contains no simplicial extreme.*

The following is a consequence of Theorem 2.3 and Lemma 2.4, which we use later.

Lemma 2.5. *Chordal graphs are β-perfect.*

Lemma 2.6. *No minimal β-imperfect graph admits a double clique cutset.*

Proof. On the contrary, suppose G is a minimal β-imperfect graph that admits a double clique cutset C. Let sets C_1, C_2, V_1 and V_2 be as in the definition of a double clique cutset, and set $G_1 = G[V_1 \cup C]$ and $G_2 = G[V_2 \cup C]$. By minimality, G_1 and G_2 are β-perfect, and hence $\chi(G_1) = \beta(G_1)$ and $\chi(G_2) = \beta(G_2)$. By Lemma 2.2, $\delta(G) + 1 =$

$\beta(G) > \chi(G) \geq \chi(G_1) = \beta(G_1) \geq \delta(G_1) + 1$, and therefore $\delta(G) > \delta(G_1)$. Similarly, $\delta(G) > \delta(G_2)$. Let $v \in V(G)$ be such that $d_G(v) = \delta(G)$. Since $d_G(u) = d_{G_1}(u)$ for all $u \in V_1 \cup C_1$ and since $\delta(G) > \delta(G_1)$, it follows that $v \in C_2$. By symmetry, it follows that $v \in C_1$, a contradiction. □

Lemma 2.7. *Let G be a connected (claw, C_4)-free graph that contains no simplicial vertex. If G admits a clique cutset, then G admits a double clique cutset.*

Proof. Suppose G has a clique cutset. Among all clique cutsets of G, let C be one that minimizes $|C|$. Let (V_1, V_2) be a partition of $V(G) \setminus C$ such that V_1 is anticomplete to V_2. By the minimality of C, every vertex in C has a neighbor in both V_1 and V_2, and C is nonempty since G is connected.

(1) For every vertex $c \in C$, both $N(c) \cap V_1$ and $N(c) \cap V_2$ are cliques.

Proof of (1): If a vertex $c \in C$ has two nonadjacent neighbors u and v in V_1, then for any neighbor w of c in V_2, the set $\{c, u, v, w\}$ induces a claw, a contradiction. Therefore $N(c) \cap V_1$ is a clique, and by symmetry so is $N(c) \cap V_2$. This proves (1).

Set $N_1 = N(C) \cap V_1$ and $N_2 = N(C) \cap V_2$.

(2) At least one of N_1 and N_2 is a clique.

Proof of (2): Suppose that neither of N_1 and N_2 is a clique, and let x, y be two nonadjacent vertices of N_1. Fix $u \in N_C(x)$ and $v \in N_C(y)$; by (1), $u \neq v$, and u, y are nonadjacent, and v, x are nonadjacent. If $N_{V_2}(u) \neq N_{V_2}(v)$, then up to symmetry there exists a vertex $w \in N_{V_2}(u) \setminus N(v)$, yielding a claw $G[\{x, u, v, w\}]$, a contradiction. So $N_{V_2}(u) = N_{V_2}(v)$; set $N_2' = N_{V_2}(u) = N_{V_2}(v)$. By (1), N_2' is a clique, so there exists a vertex $z \in N_2 \setminus N_2'$. Fix $w \in N_C(z)$; clearly $w \notin \{u, v\}$. If w, x are adjacent, then $\{w, x, v, z\}$ induces a claw, a contradiction. So w, x are nonadjacent, and by symmetry so are w and y. It follows that there exists some $s \in N_{V_1}(w) \setminus \{x, y\}$, and s is adjacent to u, for otherwise $\{w, s, u, z\}$ induces a claw; and similarly, s is adjacent to v. By (1), s is adjacent to both x and y, but now $\{s, x, y, w\}$ induces a claw, a contradiction. So at least one of N_1 and N_2 is a clique, and this proves (2).

By (2), we may assume N_1 is a clique. Since C is also a clique, and since G is C_4-free (so, in particular, $G[N_1 \cup C]$ is C_4-free), it follows that $G[N_1 \cup C]$ is chordal. If $G[V_1 \cup C]$ is chordal, then by Theorem 2.3, some vertex belonging to V_1 is simplicial in $G[V_1 \cup C]$ and hence in G, a contradiction. Therefore $G[V_1 \cup C]$ is not chordal, so in particular $G[V_1 \cup C] \neq G[N_1 \cup C]$; that is, $V_1 \setminus N_1 \neq \emptyset$.

Let ℓ denote the maximum distance in $G[V_1 \cup C]$ from a vertex of V_1 to C. Since $V_1 \setminus N_1$ is nonempty, $\ell \geq 2$. Set $L_0 = C$ and $L_1 = N_1$, and for each $i \in \{2, \dots, \ell\}$ let $L_i = N(L_{i-1}) \setminus L_{i-2}$. Observe that $(L_0, \dots, L_\ell)$ is a partition of $V_1 \cup C$, and that if there is an edge between L_i and L_j, then $|i - j| \leq 1$. Let $k \in \{0, \dots, \ell\}$ be the smallest integer such that some hole H of $G[V_1 \cup C]$ intersects L_k. (Since no vertex of G is simplicial, $G[V_1 \cup C]$ is not chordal by Theorem 2.3, and hence k is well-defined.) So every hole of

$G[V_1 \cup C]$ is a hole of $G[L_k \cup \cdots \cup L_\ell]$, and since $G[N_1 \cup C]$ (or, equivalently, $G[L_0 \cup L_1]$) is chordal, $k \geq 1$.

(3) For every $i \in \{1, \ldots, \ell - 1\}$ and every $v \in L_i$, $N(v) \cap L_{i+1}$ is a clique.

Proof of (3): For otherwise the vertex v together with any one of its neighbors in L_{i-1} and any two of its nonadjacent neighbors in L_{i+1} forms a claw, a contradiction. This proves (3).

(4) For all $i \in \{2, \ldots, \ell\}$, if x, y are two nonadjacent vertices of L_i, then there exists an xy-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-1}$.

Proof of (4): The statement holds for $i = 2$ by (3) together with the fact that every vertex of L_2 has a neighbor in L_1. Let $i > 2$, let x, y be two nonadjacent vertices of L_i, and fix $x' \in N(x) \cap L_{i-1}$ and $y' \in N(y) \cap L_{i-1}$. By (3), x, y' are nonadjacent and y, x' are nonadjacent, so in particular $x' \neq y'$. If x' and y' are adjacent, then $xx'y'y$ is the desired path; and if x' and y' are nonadjacent, then by induction there is an $x'y'$-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-2}$, which together with x and y forms the desired path. This proves (4).

For brevity we introduce the following terminology: if $i \in \{2, \ldots, \ell\}$ and $x, y \in L_i$ are nonadjacent, then an *xy-link* is any xy-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-1}$; at least one such path exists by (4).

(5) For every $i \in \{1, \ldots, k\}$ and every $v \in L_i$, $N(v) \cap L_{i-1}$ is a clique.

Proof of (5): Fix $i \in \{1, \ldots, k\}$ and $v \in L_i$, and suppose $N(v) \cap L_{i-1}$ contains two nonadjacent vertices x and y. Since C and N_1 are cliques, $i \geq 3$. But now any xy-link together with the vertex v forms a hole that intersects L_{i-1}, contrary to the minimality of k. This proves (5).

Recall that H is a hole of G that intersects L_k.

(6) $|V(H) \cap L_k| = 2$, and the two vertices of $V(H) \cap L_k$ are adjacent.

Proof of (6): Suppose $G[V(H) \cap L_k]$ contains a 3-vertex path xyz, and let P be an xz-link. But now the graph induced by $V(P) \cup (V(H) \setminus \{y\})$ contains a hole that intersects L_{k-1}, contrary to the definition of k. So $G[V(H) \cap L_k]$ is P_3-free.

To prove the second statement, it suffices to show that $G[V(H) \cap L_k]$ has only one component, and that this component is of size two. By (3), every component of $G[V(H) \cap L_k]$ contains at least two vertices, and thus it follows from P_3-freeness that every component is of size two. Suppose $G[V(H) \cap L_k]$ has at least two components, let S be any one of its components, and fix $s \in V(S)$. Then there exists a path P from s to some $t \in (V(H) \cap L_k) \setminus V(S)$ with interior in $L_{k+1} \cup \cdots \cup L_\ell$, which together with an st-link forms a hole that contradicts the minimality of k. This proves (6).

In view of (6), let us say $V(H) \cap L_k = \{v, w\}$. Let u be the neighbor of v in H different from w, and u' the neighbor of w in H different from v. Note that $u, u' \in L_{k+1}$, and since G is C_4-free, u is not adjacent to u'.

(7) $N(v) \cap L_{k-1} = N(w) \cap L_{k-1}$.

Proof of (7): For if not, then up to symmetry there exists some $z \in L_{k-1}$ adjacent to v and nonadjacent to w, yielding a claw $G[\{u, v, w, z\}]$. This proves (7).

In view of (7), let $X = N(v) \cap L_{k-1} = N(w) \cap L_{k-1}$ and let $Y = \{y \in L_k : N(y) \cap X \neq \emptyset\}$. So in particular $\{v, w\} \subseteq Y$.

(8) $X \cup Y$ is a clique.

Proof of (8): The set X is a clique by (5). Fix $y \in Y$ and suppose that y is non-adjacent to some $x \in X$. Clearly $y \notin \{v, w\}$. Since v and w are complete to X and $N(y) \cap X \neq \emptyset$, it follows from (3) that y is adjacent to both v and w. By (3), and since uu' is not an edge, we may assume without loss of generality that y is nonadjacent to u. But now $\{u, v, y, x\}$ induces a claw, a contradiction. So X is complete to Y. It now follows from (3) and since $X \neq \emptyset$ that Y is a clique. This proves (8).

Let $Y' = \{y \in Y : N(y) \cap L_{k-1} \neq X\}$. By (8), $X \subsetneq N(y) \cap L_{k-1}$ for every $y \in Y'$.

(9) Y' is anticomplete to L_{k+1}.

Proof of (9): Fix $y \in Y'$ and $z \in N(y) \cap (L_{k-1} \setminus X)$, and suppose that y has a neighbor a in L_{k+1}. By (8), y is complete to $\{v, w\}$. If y is adjacent to u, then $\{y, u, w, z\}$ induces a claw, a contradiction. So by symmetry y is anticomplete to $\{u, u'\}$ and hence $a \notin \{u, u'\}$. If a and v are nonadjacent, then $\{y, a, v, z\}$ induces a claw, a contradiction. So a is adjacent to v and by symmetry a is adjacent to w. By (3), a is complete to $\{u, u'\}$. But now, since uu' is not an edge, $\{a, u, u', y\}$ induces a claw, a contradiction. So Y' is anticomplete to L_{k+1}. This proves (9).

(10) There is no path P from $L_k \setminus Y$ to $Y \setminus Y'$ with $P^* \subseteq L_{k+1} \cup \cdots \cup L_\ell$. In particular, $L_k \setminus Y$ is anticomplete to $Y \setminus Y'$.

Proof of (10): Suppose there exists a path P with ends $a \in L_k \setminus Y$ and $y \in Y \setminus Y'$ and with (possibly empty) interior in $L_{k+1} \cup \cdots \cup L_\ell$. Fix $x \in N(y) \cap L_{k-1}$ and $z \in N(a) \cap L_{k-1}$. Since $a \notin Y$, a is nonadjacent to x and $z \notin X$, and therefore (since $y \in Y \setminus Y'$) y is nonadjacent to z. But now an xz-link (or the edge xz, if x, z are adjacent) together with P forms a hole that intersects L_{k-1}, a contradiction. This proves (10).

(11) $X \cup Y$ is a clique cutset of G.

Proof of (11): By (8), $X \cup Y$ is a clique. Suppose that $X \cup Y$ is not a clique cutset of G. Observe that, by the existence of $u, u' \in L_{k+1}$, $k < \ell$, and therefore $L_{k+1} \cup \cdots \cup L_\ell \neq \emptyset$.

If $L_k = Y$, then it follows from (8) that Y is a clique cutset of G, and therefore so is $X \cup Y$, a contradiction. So $L_k \neq Y$, and hence there exists a path P in $G \setminus (X \cup Y)$ from u to some vertex z in $L_k \setminus Y$ and with interior in $L_{k+1} \cup \cdots \cup L_\ell$. But now $P \cup \{v\}$ contains a path that violates (10). This proves (11).

(12) If F is a component of $G \setminus (X \cup Y)$, then either $N(V(F)) \subseteq Y \setminus Y'$ or $N(V(F)) \subseteq X \cup Y'$.

Proof of (12): Let F be a component of $G \setminus (X \cup Y)$, and suppose there exist vertices $s, t \in V(F)$ such that s has a neighbor $y \in Y \setminus Y'$ and t has a neighbor $y' \in X \cup Y'$. By (9), $t \notin L_{k+1}$, and therefore either $k \geq 2$ and $t \in L_{k-2} \cup L_{k-1} \cup L_k$, or $k = 1$ and $t \in V_2$. Since $s \notin X$ and $y \in Y \setminus Y'$, it follows that $s \notin L_{k-1}$. Furthermore, $s \notin L_k$ by (10), and therefore $s \in L_{k+1}$. Since F contains an st-path, there exists a path in $G \setminus (X \cup Y)$ from s to some vertex in $(L_k \cup L_{k-1} \cup L_{k-2}) \setminus (X \cup Y)$. By choosing such a path of minimum length, we obtain a path P from s to some vertex $a \in L_k \setminus Y$ whose interior lies in $L_{k+1} \cup \cdots \cup L_\ell$. But now $P \cup \{y\}$ contains a path that contradicts (10). This proves (12).

Clearly $G \setminus (X \cup Y)$ contains at least one component some vertex of which has a neighbor in $Y \setminus Y'$, and at least one component some vertex of which has a neighbor in $X \cup Y'$. It now follows from (11) and (12) that the set $X \cup Y$, partitioned $(Y \setminus Y', X \cup Y')$, is a double clique cutset of G. □

Lemma 2.8. *No minimal β-imperfect claw-free graph admits a clique cutset.*

Proof. Let G be a minimally β-imperfect claw-free graph that admits a clique cutset. By minimality, G is connected. We may assume that G is not an even hole (since an even hole admits no clique cutset), so G is even-hole-free (and, in particular, C_4-free) by minimality and has no simplicial vertex by Lemma 2.4. But now, by Lemma 2.7, G admits a double clique cutset, contrary to Lemma 2.6. □

3 Structure of (claw, even hole)-free graphs

In this section we derive from a result of Boncompagni, Penev and Vušković [1] a decomposition theorem for (claw, even hole)-free graphs. We first need the following terminology.

A *wheel* consists of a hole, called the *rim*, together with an additional vertex, called the *center*, that has at least 3 neighbors in the hole. If the center is complete to the hole, then we say that the wheel is a *universal wheel*, and if its neighborhood in the hole consists precisely of three consecutive vertices of the hole, then we say that the wheel is a *twin wheel*. A wheel that is neither a universal wheel nor a twin wheel is called a *proper wheel*. A wheel whose center has an even number of neighbors in the hole is an *even wheel*.

A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$. A *pyramid* is any subdivision of the complete graph K_4 in which one triangle remains unsubdivided, and

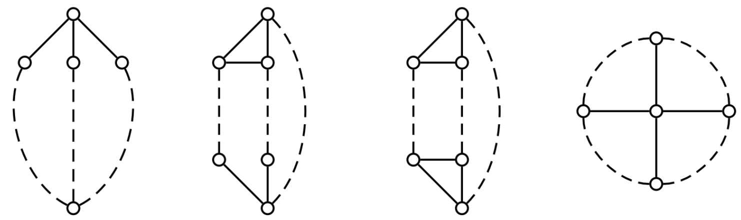

Figure 3: From left-to right: a theta, pyramid, prism and wheel. Dashed lines denote paths of length at least 1.

of the remaining three edges, at least two edges are subdivided at least once. A *prism* is any subdivision of $\overline{C_6}$ in which the two triangles remain unsubdivided. A *three-path-configuration* (or *3PC* for short) is any theta, pyramid, or prism. See Figure 3 for depictions of a theta, pyramid, prism and a wheel.

A *signing* of a graph G is an assignment of 0,1 weights to each edge of G. The *weight* of an induced subgraph of G is the sum of the weights of its edges. A graph is *odd-signable* if it admits a signing in which every triangle and every hole has odd weight. Even-hole-free graphs are odd-signable; assigning weight 1 to every edge gives such a signing.

Theorem 3.1 (Conforti, Cornuéjols, Kapoor and Vušković [3])**.** *A graph is odd-signable if and only if it contains no theta, prism, or even wheel.*

The complement of a graph G is denoted by $\overline{G}$. A *component* of G is a maximal connected induced subgraph of G. A graph is *anticonnected* if its complement is connected. An *anticomponent* of G is a maximal anticonnected induced subgraph of G. A component or anticomponent is *trivial* if it has only one vertex, and *nontrivial* otherwise. Since anticomponents of G are components of $\overline{G}$, between any two anticomponents of G there is every possible edge. Therefore the set of all vertices belonging to trivial anticomponents is a clique.

Lemma 3.2. *If a graph is (claw, even hole)-free, then it is (3PC, proper wheel)-free.*

Proof. Let G be a (claw, even hole)-free graph. Since G is even-hole-free, G is odd-signable, and therefore it follows from Theorem 3.1 that G contains no even wheel, no theta and no prism. A pyramid contains a claw, so G contains no pyramid, and therefore G contains no 3PC. It remains to show that G contains no proper wheel.

Towards a contradiction, suppose G contains a proper wheel W with rim H and center x. Let C be any component of $G[N(x) \cap V(H)]$. If C consists of a single vertex, say c, then x is anticomplete to $N_H(c)$, and hence $N_H[c] \cup \{x\}$ induces a claw, a contradiction. So $|V(C)| \geq 2$. If $C = H$, then W is a universal wheel, a contradiction. So $C \neq H$, and therefore C is a path. If $|V(C)| \geq 5$, then any three pairwise nonadjacent vertices of C together with x induce a claw, a contradiction. So each component of $G[N_H(x)]$ is a path on at least 2 and at most 4 vertices.

Suppose $G[N_H(x)]$ contains only one component C. By definition, x has at least 3 neighbors in H, and W is not an even wheel, so C is a path on 3 vertices. But then W is a twin wheel, and hence not a proper wheel, a contradiction. So $G[N_H(x)]$ contains at least two components. Suppose one of them, say C, contains two nonadjacent vertices u and v. Then the vertices u, v, x together with one vertex from any other component of $G[N_H(x)]$ besides C induce a claw, a contradiction. It follows that each component of $G[N_H(x)]$ has exactly 2 vertices. But then W is an even wheel, a contradiction. □

A *ring* is any graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets $Y_1, \dots, Y_k$ such that for all $i \in \{1, \dots, k\}$ the following hold (where, throughout this paper, subscripts are to be taken modulo k):

- Y_i is a clique;
- Y_i is anticomplete to $V(R) \setminus (Y_{i-1} \cup Y_i \cup Y_{i+1})$;
- some vertex of Y_i is complete to $Y_{i-1} \cup Y_{i+1}$; and
- for all distinct $y, y' \in Y_i$, $N_R[y] \subseteq N_R[y']$ or $N_R[y'] \subseteq N_R[y]$.

Under these circumstances we say that R is of *length* k and that R is a *k-ring*. Furthermore, R is *even* or *odd* according to the parity of k, and is *long* if $k \geq 5$. We sometimes refer to the sets $Y_1, \dots, Y_k$ as the *bags* of R, and to $(Y_1, \dots, Y_k)$ as a *ring partition* of R.

Lemma 3.3 (Boncompagni, Penev and Vušković [1]). *If R is a ring of length k, then every hole in R is of length k.*

A *hyperhole* is any ring $R = (Y_1, \dots, Y_k)$ such that, for each $i \in \{1, \dots, k\}$, Y_i is complete to $Y_{i-1} \cup Y_{i+1}$; and R is a *k-hyperhole*, for an integer $k \geq 4$, if R is a k-ring.

Theorem 3.4 (Boncompagni, Penev and Vušković [1]). *If G is a (3PC, proper wheel)-free graph, then one of the following holds.*

(i) G has exactly one nontrivial anticomponent, and this anticomponent is a long ring;

(ii) G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;

(iii) $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph;

(iv) G admits a clique cutset.

We also use the following well known decomposition theorem for chordal graphs.

Theorem 3.5 (Dirac [5]). *If G is a chordal graph, then either G is a complete graph or G admits a clique cutset.*

By specializing Theorem 3.4 to (claw, even hole)-free graphs, we obtain the following decomposition theorem.

Lemma 3.6. *If G is a (claw, even hole)-free graph, then G is a complete graph or an odd ring, or G contains a universal vertex, or G admits a clique cutset.*

Proof. Let G be a (claw, even hole)-free graph. By Theorem 3.5, we may assume that G contains a hole, and since G is even-hole-free, G contains an odd hole, and hence a long hole. By Lemma 3.2, G is (3PC, proper wheel)-free, and hence G satisfies one of (i)-(iv) in the statement of Theorem 3.4. Since G contains a long hole, (ii) does not hold; and if (iv) holds, i.e., if G has a clique cutset, then we are done; so we may assume (iv) does not hold. Therefore (i) or (iii) holds. That is:

- G has exactly one nontrivial anticomponent, and this anticomponent is a long ring; or
- $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph.

In the first case, G contains a universal vertex (if some anticomponent of G is trivial), or G is an odd ring (if no anticomponent of G is trivial), and we are done. So we may assume that $\alpha(G) = 2$ and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph. Since $\alpha(G) = 2$, H is of length 5, and therefore H belongs to an anticomponent F of G that is a 5-hyperhole. If $G \setminus F$ contains two nonadjacent vertices, then they together with two nonadjacent vertices of H induce a C_4, a contradiction. So $G \setminus F$ is a clique, and hence G contains a universal vertex if $G \setminus F$ is nonempty, and G is a 5-hyperhole (and therefore an odd ring) otherwise. □

4 β-perfect rings

In this section we give a forbidden induced subgraph characterization for the class of β-perfect rings. We first state the following results on the chromatic number of hyperholes and rings.

Theorem 4.1 (Narayanan and Shende [12]). *If H is a hyperhole, then*

$$\chi(H) = \max\left\{\omega(H), \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil\right\}.$$

Observe that if H is an odd hyperhole of length k, then $\alpha(H) = (k-1)/2$, and hence, by Theorem 4.1, $\chi(H) = \max\left\{\omega(G), \left\lceil \frac{2|V(G)|}{k-1} \right\rceil\right\}$.

Theorem 4.2 (Maffray, Penev and Vušković [11]). *Let $k \geq 4$ be an integer and let R be a k-ring. Then $\chi(R) = \max\{\chi(H) : H$ is a k-hyperhole in $R\}$.*

Throughout the remainder of the paper we may use Theorem 4.2 implicitly, i.e., we may write "let H be a hyperhole contained in R such that $\chi(H) = \chi(R)$" without reference to Theorem 4.2.

4.1 Small rings

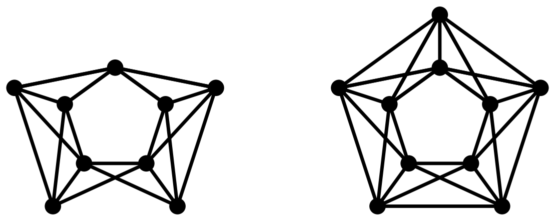

Figure 4: The two minimal β-imperfect rings of length 5.

Let R_5 denote the graph on the left of Figure 4 and H_5 the graph on the right.

Lemma 4.3 (Horsfield and Vušković [7]). *The following hold:*

- *H_5 is minimally β-imperfect.*
- *A 5-hyperhole $H = (X_1, \dots, X_5)$ is β-perfect if and only if $|X_i| = 1$ for some $i \in \{1, \dots, 5\}$.*

Lemma 4.4. *R_5 is minimally β-imperfect.*

Proof. Let $(Y_1, \dots, Y_5)$ be a ring partition of R_5, say with $|Y_1| = 1$. Let H be a hyperhole contained in R_5 such that $\chi(H) = \chi(R_5)$. Since Y_3 is not complete to Y_4 we see that $Y_3 \cup Y_4 \not\subseteq V(H)$ and therefore $|V(H)| \leq |V(R_5)| - 1$. Now, by Theorem 4.1,

$$\chi(H) \leq \max\left\{\omega(R_5), \left\lceil \frac{|V(R_5)| - 1}{2} \right\rceil\right\} \leq 4,$$

so $\chi(R_5) \leq 4$. It follows that $\chi(R_5) \leq 4 < 5 = \delta(R_5) + 1 \leq \beta(R_5)$ and therefore $\chi(R_5) < \beta(R_5)$.

It remains to prove that every proper induced subgraph of R_5 is β-perfect. To the contrary, suppose some proper induced subgraph R of R_5 is minimally β-imperfect. Let $y_3 \in Y_3$ and $y_4 \in Y_4$ be nonadjacent vertices of R_5. By Lemma 2.5, R is not chordal, so R contains a vertex from each of $Y_1, \dots, Y_5$; and R is not a hyperhole, for otherwise (since $|Y_1| = 1$) it follows from Lemma 4.3 that R is β-perfect, a contradiction. So R is a ring that is not a hyperhole, and therefore R contains both y_3 and y_4. If $Y_3 \cup Y_4 \not\subseteq V(R)$, then one of y_3, y_4 is a simplicial vertex of R, contrary to Lemma 2.4; so $Y_3 \cup Y_4 \subseteq V(R)$. Let y_1 be the unique vertex of Y_1. By Lemma 2.4, $d_R(y_1) \geq 3$ and therefore $Y_2 \subseteq V(R)$

without loss of generality. Since $Y_2 \cup Y_3$ is a clique of R_5, we now have that $Y_2 \cup Y_3$ is also a clique of R. But now, by Lemma 2.2, $\beta(R) = \delta(R) + 1 = 4 = |Y_2 \cup Y_3| = \omega(R) \leq \chi(R)$ and hence $\beta(R) = \chi(R)$, a contradiction. □

Lemma 4.5. *A 5-ring is β-perfect if and only if it is (H_5, R_5)-free.*

Proof. A β-perfect 5-ring is (H_5,R_5)-free by Lemmas 4.3 and 4.4. To prove the converse, let $R = (Y_1, \ldots, Y_5)$ be a (H_5, R_5)-free 5-ring. We begin by proving the following two claims.

(1) Let $i \in \{1, \ldots, 5\}$, $y_i \in Y_i$ and $y_{i+1} \in Y_{i+1}$. If y_i and y_{i+1} are nonadjacent, then $|N(y_i) \cap Y_{i-1}| = 1$ or $|N(y_{i+1}) \cap Y_{i+2}| = 1$.

Proof of (1): Suppose otherwise. Then, up to symmetry, there exist nonadjacent vertices $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $|N(y_1) \cap Y_5| \geq 2$ and $|N(y_2) \cap Y_3| \geq 2$. For each $i \in \{1, \ldots, 5\}$, let x_i be a vertex of Y_i that is complete to $Y_{i-1} \cup Y_{i+1}$. Since y_1 and y_2 are nonadjacent, $x_1 \neq y_1$ and $x_2 \neq y_2$. Fix $y_3 \in (N(y_2) \cap Y_3) \setminus \{x_3\}$ and $y_5 \in (N(y_1) \cap Y_5) \setminus \{x_5\}$. But now the graph induced by $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, x_5, y_5\}$ is isomorphic to R_5, a contradiction. This completes the proof of (1).

(2) There exists an integer $i \in \{1, \ldots, 5\}$ and a vertex $x_i \in Y_i$ that is complete to $Y_{i-1} \cup Y_{i+1}$ such that $Y_i \setminus \{x_i\}$ is anticomplete to $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$ for some $x_{i-1} \in Y_{i-1}$ and $x_{i+1} \in Y_{i+1}$.

Proof of (2): Suppose otherwise. Then, for every $i \in \{1, \ldots, 5\}$, there exist distinct vertices $x_i, y_i \in Y_i$ such that x_i is complete to $Y_{i-1} \cup Y_{i+1}$ and y_i has at least one neighbor in $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$. Up to symmetry we may assume that y_1 has a neighbor $y'_2 \in Y_2 \setminus \{x_2\}$ and y_4 has a neighbor y'_5 in $Y_5 \setminus \{x_5\}$. By (1), vertices y_1 and y'_5 are adjacent. Since R is H_5-free, y_3 is not adjacent to both y'_2 and y_4, so we may assume without loss of generality that y_3 and y_4 are nonadjacent. Since y_4 has at least two neighbors in Y_5, we see by (1) that $|N(y_3) \cap Y_2| = 1$, and in particular y'_2 and y_3 are nonadjacent. Now by a symmetric argument applied to y'_2 and y_3 it follows that $|N(y_3) \cap Y_4| = 1$, which contradicts our assumption that y_3 has at least one neighbor in $(Y_2 \cup Y_4) \setminus \{x_2, x_4\}$. This completes the proof of (2).

Towards a contradiction, suppose R is not β-perfect. Since every proper induced subgraph of R contains a simplicial vertex, or is a chordal graph, or is a 5-ring (that is also (H_5, R_5)-free, thereby satisfying (1) and (2)), it follows from Lemmas 2.4 and 2.5 that we may assume R is minimally β-imperfect.

(3) Let $i \in \{1, \ldots, 5\}$ and suppose that there exists a vertex $x_i \in Y_i$ that is complete to $Y_{i-1} \cup Y_{i+1}$ such that $Y_i \setminus \{x_i\}$ is anticomplete to $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$ for some $x_{i-1} \in Y_{i-1}$ and $x_{i+1} \in Y_{i+1}$. Then at least one of Y_{i-1} and Y_{i+1} is of size 1.

Proof of (3): Without loss of generality, suppose $i = 1$, and towards a contradiction suppose that $|Y_2| \geq 2$ and $|Y_5| \geq 2$. Fix vertices $y_2 \in Y_2$ and $y_5 \in Y_5$ such that

$N_R[y_2] \subseteq N_R[y_2']$ for every $y_2' \in Y_2$ and $N_R[y_5] \subseteq N_R[y_5']$ for every $y_5' \in Y_5$. Let Y_3' be the subset of Y_3 such that y_2 is complete to Y_3' and anticomplete to $Y_3 \setminus Y_3'$, and similarly let Y_4' be the subset of Y_4 such that y_5 is complete to Y_4' and anticomplete to $Y_4 \setminus Y_4'$. Now, since Y_3' is complete to Y_2 and Y_4' is complete to Y_5, it follows from (1) that Y_3' is complete to Y_4', and therefore $\{x_1\} \cup Y_2 \cup Y_3' \cup Y_4' \cup Y_5$ induces a hyperhole H. Observe that $N_R(y_2) = \{x_1\} \cup (Y_2 \setminus \{y_2\}) \cup Y_3'$ and $N_R(y_5) = \{x_1\} \cup (Y_5 \setminus \{y_5\}) \cup Y_4'$, and therefore, by Lemma 2.2, the sets $Y_2 \cup Y_3'$ and $Y_4' \cup Y_5$ both have size at least $\beta(R) - 1$. It follows that $|V(H)| \geq 2\beta(R) - 1$, and hence $\chi(H) \geq \beta(R)$ by Theorem 4.1. But now $\chi(R) \geq \beta(R)$, a contradiction. This proves (3).

By (2), and without loss of generality, there is a vertex $x_1 \in Y_1$ that is complete to $Y_2 \cup Y_5$ and $Y_1 \setminus \{x_1\}$ is anticomplete to $(Y_2 \cup Y_5) \setminus \{x_2, x_5\}$. By (3), at least one of Y_2 and Y_5 is of size 1; without loss of generality, suppose $|Y_2| = 1$. Now we may apply (3) with $i = 2$ to get that at least one of Y_1 and Y_3 is of size 1. That is, R has two consecutive bags of size 1; so suppose without loss of generality that $|Y_1| = |Y_2| = 1$, and let x_1 and x_2 be the unique vertices from Y_1 and Y_2 respectively. Observe that $d_R(x_1) = 1 + |Y_5|$ and $d_R(x_2) = 1 + |Y_3|$, and therefore it follows from Lemma 2.2 that $|Y_3| \geq \beta(R) - 2$ and $|Y_5| \geq \beta(R) - 2$. Let x_4 be a vertex of Y_4 that is complete to $Y_3 \cup Y_5$, and consider the hyperhole H induced by $Y_1 \cup Y_2 \cup Y_3 \cup \{y_4\} \cup Y_5$. From the above bounds, we get that $|V(H)| \geq 2\beta(R) - 1$, and therefore it follows from Theorem 4.1 that $\chi(H) \geq \beta(R)$. So $\chi(R) \geq \beta(R)$, but this contradicts the fact that R is minimally β-imperfect. □

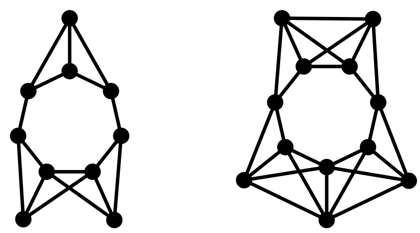

Figure 5: The two minimal β-imperfect rings of length 7.

Let R_7 denote the graph on the left of Figure 5 and H_7 the graph on the right.

Lemma 4.6 (Horsfield and Vušković [7]). *The following hold:*

- *H_7 is minimally β-imperfect.*
- *A 7-hyperhole $H = (X_1, \ldots, X_7)$ is β-perfect if and only if $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, 7\}$.*

Lemma 4.7. *R_7 is minimally β-imperfect.*

Proof. Let $(Y_1, \ldots, Y_7)$ be a ring partition of R_7 and assume without loss of generality that Y_5 is not complete to Y_6. Clearly every proper induced subgraph of R_7 contains a vertex of degree at most 2, and therefore, by Lemma 2.4, every proper induced subgraph of R_7 is β-perfect. So it suffices to show that $\chi(R_7) < \beta(R_7)$. Let H be a hyperhole in R such that $\chi(H) = \chi(R)$. Since Y_5 is not complete to Y_6, we have that $Y_5 \cup Y_6 \not\subseteq V(H)$ and hence $|V(H)| \leq V(R_7) - 1 = 9$. Now by Theorem 4.1 applied to H we see that $\chi(R_7) = \chi(H) \leq 3 < \beta(R_7) \leq 4$. Therefore R_7 is minimally β-imperfect. □

Lemma 4.8. *A 7-ring is β-perfect if and only if it is (H_7, R_7)-free.*

Proof. A β-perfect 7-ring is (H_7, R_7)-free by Lemmas 4.6 and 4.7. We now prove the converse. Let $R = (Y_1, \dots, Y_7)$ be a (H_7, R_7)-free 7-ring. The following fact is an immediate consequence of R being R_7-free.

(1) For all $i \in \{1, \dots, 7\}$, if $|Y_i| \geq 2$, then Y_{i+3} is complete to Y_{i+4}.

We now establish the following.

(2) There exists an integer $i \in \{1, \dots, 7\}$ such that $|Y_i| = |Y_{i+1}| = 1$ or $|Y_i| = |Y_{i+2}| = 1$.

Proof of (2). Suppose otherwise. If each of $Y_1, \dots, Y_7$ has size at least 2, then it follows from (1) that R is a hyperhole each bag of which has size at least 2, and hence it contains H_7, a contradiction. So we may assume without loss of generality that $|Y_1| = 1$. Thus each of Y_2, Y_3, Y_6 and Y_7 contain at least two vertices, and up to symmetry so does Y_4. Now by (1) we see that Y_2 is complete to Y_3, Y_3 is complete to Y_4, and Y_6 is complete to Y_7, and hence R contains H_7, a contradiction. This completes the proof of (2).

Towards a contradiction, suppose that R is not β-perfect. Since every proper induced subgraph of R contains a simplicial vertex or is a 7-ring, by Lemma 2.4 we may assume that R is minimally β-imperfect. In view of (2), we may assume without loss of generality that $|Y_1| = 1$, and $|Y_2| = 1$ or $|Y_3| = 1$.

First suppose that $|Y_2| = 1$. Let y_1 be the unique vertex of Y_1, and observe that $d_R(y_1) = |Y_7|+1$. Thus, by Lemma 2.2, $|Y_7| \geq \beta(R)-2$, and by symmetry $|Y_3| \geq \beta(R)-2$. In particular, since no vertex of R is of degree 2 by Lemma 2.4, it follows that both Y_3 and Y_7 have size at least 2. Now by (1) we see that Y_3 is complete to Y_4 and Y_6 is complete to Y_7. Since $\omega(R) \leq \chi(R) < \beta(R)$, it follows that $|Y_4| = |Y_6| = 1$. But now R is a hyperhole, and it is β-perfect by Lemma 4.6, a contradiction.

So $|Y_3| = 1$. Observe that vertices from Y_2 have degree $|Y_2| + 1$, and hence $|Y_2| \geq \beta(R)-2$ by Lemma 2.2. In particular, since no vertex of R is of degree 2 by Lemma 2.4, we have that $|Y_2| \geq 2$, and hence Y_5 is complete to Y_6 by (1). Fix vertices $y_4 \in Y_4$ and $y_7 \in Y_7$ such that $N_R[y_4] \subseteq N_R[y_4']$ for every $y_4' \in Y_4$ and $N_R[y_7] \subseteq N_R[y_7']$ for every $y_7' \in Y_7$. Let $Y_5' \subseteq Y_5$ and $Y_6' \subseteq Y_6$ be such that y_4 is complete to Y_5' and anticomplete to $Y_5 \setminus Y_5'$, and y_7 is complete to Y_6' and anticomplete to $Y_6 \setminus Y_6'$. Observe that $d_R(y_4) = |Y_4 \cup Y_5'|$ and $d_R(y_7) = |Y_6' \cup Y_7|$, and hence, by Lemma 2.2, $|Y_4 \cup Y_5'| \geq \beta(R)-1$ and $|Y_6' \cup Y_7| \geq \beta(R)-1$. It follows that the graph H induced by $(V(R) \setminus (Y_5 \cup Y_6)) \cup Y_5' \cup Y_6'$ is a hyperhole on at least $3\beta(R) - 2$ vertices. Therefore, by Theorem 4.1, $\chi(H) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction. □

4.2 Big rings

We now turn to odd rings of length at least 9. For the sake of brevity, we call such rings *big*. We first need some terminology from [7], which we slightly adapt so that it may be used in the more general setting of rings.

Let $R = (Y_1, \ldots, Y_k)$ be an odd ring. For $i, j, m \in \{1, \ldots, k\}$, the tuple $(Y_i, \ldots, Y_j)$ is a *sequence of m bags of R* if for $\ell \in \{1, \ldots, m\}$, the ℓ-th element of the sequence is $Y_{i+\ell-1}$ (and in particular the m-th element of the sequence is the bag Y_j). A *sector of* R is a sequence $(Y_i, \ldots, Y_j)$ of at least 2 bags such that $|Y_i| = |Y_j| = 1$, and all the other bags in the sequence have size at least 2, and Y_s is complete to Y_{s+1} for each $s \in \{i, \ldots, j-1\}$. We say that Y_i and Y_j are the *end bags* of the sector, and all the other bags are called the *interior bags* of the sector. The *length* of a sector is the number of its interior bags. A sector is an *n-sector*, for an integer $n \geq 0$, if it is of length n. A sector is *safe* if it has length 1 or length at least 3. A *super-sector of R* is a sequence $(Y_i, \ldots, Y_j)$ of at least 5 bags such that $|Y_i| = |Y_{i+1}| = |Y_{j-1}| = |Y_j| = 1$, and for each $h \in \{i+1, \ldots, j-2\}$, $(|Y_h|, |Y_{h+1}|) \neq (1, 1)$, and for each $s \in \{i, \ldots, j-1\}$, Y_s is complete to Y_{s+1}. If $(Y_i, \ldots, Y_j)$ is a super-sector of R then we say that R *contains* a super-sector. We say that a super-sector $(Y_i, \ldots, Y_j)$ *contains an n-sector* if some subsequence of $(Y_{i+1}, \ldots, Y_{j-1})$ is an n-sector of R (note that although (Y_i, Y_{i+1}) and (Y_{j-1}, Y_j) are 0-sectors of R, they are, by definition of "contains an n-sector", not contained in the super-sector $(Y_i, \ldots, Y_j)$).

A k-hyperhole $H = (X_1, \ldots, X_k)$ is *trivial* if at least one of the following holds:

(i) for some $i \in \{1, \ldots, k\}$, $|X_i| = |X_{i+1}| = |X_{i+2}| = 1$;

(ii) H contains a super-sector that contains only 2-sectors;

(iii) H contains exactly one 0-sector, and all its other sectors are of length 2.

A k-hyperhole H is *nontrivial* if it is not trivial. See Figure 6 for examples of trivial hyperholes.

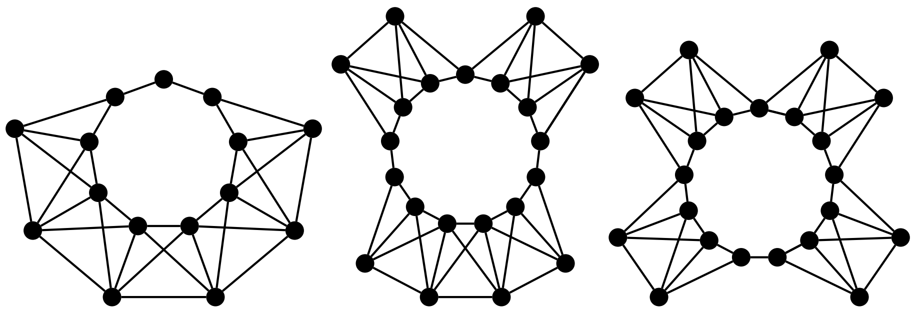

Figure 6: From left to right: hyperholes satisfying parts (i), (ii), and (iii) of the definition of a trivial hyperhole.

Lemma 4.9 (Horsfield and Vušković [7]). *If H is a trivial odd hyperhole, then H is β-perfect.*

A *base hyperhole* is any odd hyperhole $H = (X_1, \ldots, X_k)$ such that, for all $i \in \{1, \ldots, k\}$, $|X_i| \leq 2$, $(|X_i|, |X_{i+1}|, |X_{i+2}|) \neq (1, 1, 1)$, and $(|X_i|, |X_{i+1}|) \neq (2, 2)$. It follows that every sector of H is of length 0 or 1, and therefore every proper induced subgraph of a base hyperhole is either chordal or a trivial hyperhole. Note that if H is a base hyperhole, then $\omega(H) = 3$ and $\beta(H) = 4$. We say that a base hyperhole H is *good* if it has exactly one sector of length 0, and *bad* otherwise. Note that, up to isomorphism, there is only one good base hyperhole of length k. Also observe that, since k is odd, every base hyperhole must have a sector of length 0, and hence bad base hyperholes have at least two sectors of length 0.

Lemma 4.10 (Horsfield and Vušković [7]). *Let $H = (X_1, \ldots, X_k)$ be a base hyperhole. Then H is β-perfect if and only if H is good. Furthermore, if H is bad, then H is minimally β-imperfect.*

The main result of [7], which we use later, is the following.

Theorem 4.11 (Theorem 3.15 in [7]). *A hyperhole is β-perfect if and only if it is (even hole, bad base hyperhole, H_5, H_7)-free.*

If $R = (Y_1, \ldots, Y_k)$ is a ring, then a *triad of* R is any triple (Y_{i-1}, Y_i, Y_{i+1}) such that $i \in \{1, \ldots, k\}$ and $|Y_{i-1}| = |Y_i| = |Y_{i+1}| = 1$. A *bad ring* is any big ring $R = (Y_1, \ldots, Y_k)$ that satisfies the following:

- for every $i \in \{1, \ldots, k\}$, $|Y_i| \leq 2$;
- for every $i \in \{1, \ldots, k\}$, if $|Y_i| = |Y_{i+1}| = 2$, then Y_i is not complete to Y_{i+1} and $|Y_{i-2}| = |Y_{i-1}| = |Y_{i+2}| = |Y_{i+3}| = 1$;
- R has no triad; and
- there exists at least one integer $i \in \{1, \ldots, k\}$ such that $|Y_i| = |Y_{i+1}| = 2$.

Lemma 4.12. *If R is a bad ring, then R is minimally β-imperfect.*

Proof. Let $R = (Y_1, \ldots, Y_k)$ be a bad ring. Clearly every proper induced subgraph of R contains a vertex of degree at most 2, and therefore, by Lemma 2.4, every proper induced subgraph of R is β-perfect. So it remains to prove that $\chi(R) < \beta(R)$. The minimum degree of R is 3, so $\beta(R) = 4$. Let $H = (X_1, \ldots, X_k)$ be a hyperhole contained in R such that $\chi(H) = \chi(R)$ (note that H exists by Theorem 4.2). Then H is a proper induced subgraph of R, and hence, by our earlier observation, H contains a vertex of degree 2. So without loss of generality X_1, X_2 and X_3 each consist of exactly one vertex, say x_1, x_2 and x_3 respectively. The graph $H \setminus \{x_2\}$ is a chordal graph with clique number at most 3, and hence there exists a 3-coloring of $H \setminus \{x_2\}$. Such a coloring can be extended to a 3-coloring of H by assigning to x_2 one of the three colors that has not been assigned to either of x_1 and x_3. Therefore $\chi(R) = \chi(H) = 3 < \beta(R) = 4$, so $\chi(R) < \beta(R)$, and this completes the proof that R is minimally β-imperfect. □

For the remainder of the paper, whenever we speak of a ring $R = (Y_1, \ldots, Y_k)$ that contains a hyperhole $H = (X_1, \ldots, X_k)$, we implicitly assume that $X_i \subseteq Y_i$ for each $i \in \{1, \ldots, k\}$. Moreover, if H is a base hyperhole, we assume in addition that $|X_3| = |X_4| = 1$.

We use the following notation: if $i \in \{1, \ldots, k\}$, then Y_i^1 denotes a set consisting of any one vertex from Y_i that is complete to $Y_{i-1} \cup Y_{i+1}$, and Y_i^2 denotes any set obtained from Y_i^1 by adding a single vertex from $Y_i \setminus Y_i^1$ (provided $Y_i \setminus Y_i^1 \neq \emptyset$).

Lemma 4.13. *Let $R = (Y_1, \ldots, Y_k)$ be a ring that contains a base hyperhole $H = (X_1, \ldots, X_k)$. If R contains no bad base hyperhole and no bad ring, then the following hold:*

- $|Y_i| = 1$ *for all even* $i \in \{8, \ldots, k\}$*;*
- $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$*;*
- $Y_1 \cup Y_2$*,* $Y_3 \cup Y_4$ *and* $Y_5 \cup Y_6$ *are cliques.*

Proof. Since R contains a base hyperhole but no bad base hyperhole, R contains a good base hyperhole, and hence $|Y_i| \geq 2$ for $i = 2$ and for each odd $i \in \{5, \ldots, k\}$.

Suppose for some even $i \in \{8, \ldots, k\}$ that $|Y_i| \geq 2$. But now

$$R[(V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})) \cup Y_{i-1}^1 \cup Y_i^2 \cup Y_{i+1}^1]$$

is a bad base hyperhole with 0-sectors (X_3, X_4), (X_{i-2}, Y_{i-1}^1) and (Y_{i+1}^1, X_{i+2}), a contradiction. This proves that the first bullet holds.

Suppose $|Y_1| \geq 2$ and $|Y_4| \geq 2$. But now

$$R[Y_k^1 \cup Y_1^2 \cup Y_2^1 \cup X_3 \cup Y_4^2 \cup Y_5^1 \cup X_6 \cup \cdots \cup X_{k-1}]$$

is a bad base hyperhole with 0-sectors (Y_2^1, X_3), (Y_5^1, X_6), and (X_{k-1}, Y_k^1), a contradiction. So $\min(|Y_1|, |Y_4|) = 1$, and by a symmetric argument we get that $\min(|Y_3|, |Y_6|) = 1$. This proves that the second bullet holds.

Suppose $Y_1 \cup Y_2$ is not a clique, and fix nonadjacent vertices $v_1 \in Y_1$ and $v_2 \in Y_2$. But now

$$R[Y_k^1 \cup (Y_1^1 \cup \{v_1\}) \cup (Y_2^1 \cup \{v_2\}) \cup X_3 \cup \cdots \cup X_{k-1}]$$

is a bad ring, a contradiction. So $Y_1 \cup Y_2$ is a clique, and by symmetry so is $Y_5 \cup Y_6$. If $Y_3 \cup Y_4$ is not a clique, then there exist nonadjacent vertices $v_3 \in Y_3$ and $v_4 \in Y_4$, and

$$R[X_1 \cup Y_2^1 \cup (Y_3^1 \cup \{v_3\}) \cup (Y_4^1 \cup \{v_4\}) \cup Y_5^1 \cup X_6 \cup \cdots \cup X_k]$$

is a bad ring, a contradiction. So $Y_3 \cup Y_4$ is a clique, and this completes the proof that the third bullet holds. □

Lemma 4.14. *Let $R = (Y_1, \ldots, Y_k)$ be an odd ring. Suppose that R is minimally β-imperfect. Then R has no triad, and for all $i \in \{1, \ldots, k\}$, if $|Y_i| = |Y_{i+1}| = 1$, then $|Y_{i+2}| = \beta(R) - 2$ and exactly one vertex of Y_{i+3} is complete to Y_{i+2}.*

Proof. If (Y_{i-1}, Y_i, Y_{i+1}) is a triad of R, then the unique vertex in Y_i is a simplicial extreme, contrary to Lemma 2.4. So R has no triad. Fix $i \in \{1, \ldots, k\}$, and suppose $|Y_i| = |Y_{i+1}| = 1$. Let v be the unique vertex in Y_{i+1}, and observe that $d_R(v) = 1+|Y_{i+2}|$. It now follows from Lemma 2.2 that $|Y_{i+2}| \geq \beta(R)-2$. Since $\omega(R) \leq \chi(R) < \beta(R)$, every clique of R has size at most $\beta(R)-1$, and since $Y_{i+2} \cup Y^1_{i+3}$ is a clique, $|Y_{i+2}| \leq \beta(R)-2$. Thus $|Y_{i+2}| = \beta(R)-2$. If there are two vertices in Y_{i+3} that are complete to Y_{i+2}, then they together with the set Y_{i+2} form a clique of size $\beta(R)$, a contradiction; so exactly one vertex of Y_{i+3} is complete to Y_{i+2}. □

Lemma 4.15. *Let $R = (Y_1, \ldots, Y_k)$ be a ring that is not β-perfect. Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \ldots, k\}$. Then $F = (F_1, \ldots, F_k)$ is a ring.*

Proof. If some F_i is empty, then F is chordal, and hence β-perfect by Lemma 2.5, a contradiction. So the sets $F_1, \ldots, F_k$ are all nonempty. Since R is a ring, each F_i is a clique that is anticomplete to $V(F) \setminus (F_1 \cup \cdots \cup F_k)$. Also as a result of R being a ring, the following holds: for each $i \in \{1, \ldots, k\}$ and for all $u, v \in F_i$, $N_F[u] \subseteq N_F[v]$ or $N_F[v] \subseteq N_F[u]$. It remains to prove for each $i \in \{1, \ldots, k\}$ that some vertex of F_i is complete to $F_{i-1} \cup F_{i+1}$.

Fix $i \in \{1, \ldots, k\}$ and let u be a vertex from F_i with $|N(u) \cap (F_{i-1} \cup F_{i+1})|$ minimum. It follows from R being a ring that $N_F(u) \cap (F_{i-1} \cup F_{i+1}) \subseteq N_F(v) \cap (F_{i-1} \cup F_{i+1})$ for all $v \in F_i$. By Lemma 2.4, u is not simplicial in F, and hence it has two nonadjacent neighbors $x, y \in V(F)$. Suppose that u is anticomplete to F_{i-1}. Up to symmetry, $x \in F_i$ and $y \in F_{i+1}$. But now $y \in N(u) \setminus N(x)$, contradicting the fact that $N(u) \cap (F_{i-1} \cup F_{i+1}) \subseteq N(v) \cap (F_{i-1} \cup F_{i+1})$ for all $v \in F_i$. So u has a neighbor in F_{i-1}, and by symmetry u has a neighbor in F_{i+1}. It now follows from our choice of u that every vertex in F_i has at least one neighbor in F_{i-1} and at least one neighbor in F_{i+1}.

Suppose for some $i \in \{1, \ldots, k\}$ that there is no vertex in F_i that is complete to $F_{i-1} \cup F_{i+1}$. Let $u \in F_i$ be such that $|N(u) \cap (F_{i-1} \cup F_{i+1})|$ is maximum, and fix $x \in (F_{i-1} \cup F_{i+1}) \setminus N(u)$. Without loss of generality assume $x \in F_{i+1}$. But now x has a neighbor $v \in F_i$ different from u, and it follows from F being an induced subgraph of a ring that v is complete to $N(u) \cap (F_{i-1} \cup F_{i+1})$, and hence $N(u) \cap (F_{i-1} \cup F_{i+1}) \subsetneq N(v) \cap (F_{i-1} \cup F_{i+1})$, contrary to our choice of u. Therefore, by symmetry, some vertex of F_i is complete to $F_{i-1} \cup F_{i+1}$ for every $i \in \{1, \ldots, k\}$. This completes the proof that $F = (F_1, \ldots, F_k)$ is a ring. □

Lemma 4.16. *Let $R = (Y_1, \ldots, Y_k)$ be a big ring that contains a base hyperhole $H = (X_1, \ldots, X_k)$. Then R is β-perfect if and only if R contains no bad base hyperhole and no bad ring.*

Proof. If R is β-perfect, then R contains no bad base hyperhole and no bad ring by Lemmas 4.10 and 4.12 respectively.

For the converse, suppose, towards a contradiction, that R contains no bad base hyperhole and no bad ring but R is not β-perfect. By Lemma 4.13, the following three claims hold.

(1) $|Y_i| = 1$ *for all even* $i \in \{8, \dots, k\}$.

(2) $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$.

(3) $Y_1 \cup Y_2$*,* $Y_3 \cup Y_4$ *and* $Y_5 \cup Y_6$ *are cliques.*

Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \dots, k\}$. By Lemma 4.15, $F = (F_1, \dots, F_k)$ is a ring.
In view of (2), there are, up to symmetry, three cases as follows:

- $|Y_1| = |Y_4| = |Y_3| = |Y_6| = 1$; or
- $|Y_1| \geq 2$ and $|Y_4| = 1$; or
- $|Y_1| = 1$ and $|Y_4| \geq 2$.

In the first case, it follows from (1), (2) and (3) that R is a hyperhole, and R clearly contains no even hole, bad base hyperhole, H_5 or H_7; but then, by Theorem 4.11, R is β-perfect, a contradiction. So the first case does not hold.

(4) F *contains a base hyperhole.*

Proof of (4): By (1), $|F_i| = 1$ for each even $i \in \{8, \dots, k\}$. Suppose $|F_1| = |F_6| = 1$. By Lemma 4.14, F has no triad, and hence $|F_i| \geq 2$ for each odd $i \in \{7, \dots, k\}$. Suppose $|F_2| = 1$. By Lemma 4.14, $|F_3| \geq 2$, and then by the same Lemma (together with (3)), $|F_4| = 1$, and therefore $|F_5| \geq 2$. Now clearly F contains a base hyperhole. So $|F_2| \geq 2$, and by symmetry $|F_5| \geq 2$, and again clearly F contains a base hyperhole. So we may assume it is not the case that $|F_1| = |F_6| = 1$.
Suppose $|F_1| \geq 2$. Then $|Y_1| \geq 2$, and hence by (2), $|Y_4| = 1$, and therefore $|F_4| = 1$. If $|F_3| = 1$, then (since (F_2, F_3, F_4) is not a triad by Lemma 4.14) $|F_2| \geq 2$; but, by (3), F_1 is complete to F_2, and the four bags F_1, F_2, F_3, F_4 contradict Lemma 4.14. So $|F_3| \geq 2$, and hence $|Y_6| = 1$ by (2). Since $|F_4| = |F_6| = 1$, it follows from Lemma 4.14 that $|F_5| \geq 2$. It is now easily seen that F contains a base hyperhole. Thus, if $|F_1| \geq 2$, then F contains a base hyperhole, and a symmetric argument shows that if $|F_6| \geq 2$, then F contains a base hyperhole. This proves (4).

By (4), F contains a base hyperhole, and since R contains no bad base hyperhole or bad ring, neither does F. So all our assumptions about R also hold for F, and therefore we may assume that $R = F$; i.e., we may assume R is minimally β-perfect. Recall from above that there are two cases: $|Y_1| \geq 2$ and $|Y_4| = 1$; or $|Y_1| = 1$ and $|Y_4| \geq 2$.
Suppose $|Y_1| \geq 2$ and $|Y_4| = 1$. If $|Y_3| = 1$, then by Lemma 4.14, Y_1 is not complete to Y_2, contradicting (3). So $|Y_3| \geq 2$, and therefore, by (2), $|Y_6| = 1$. So for any vertex $v \in Y_5$, $d_R(y_5) = |Y_5| + 1$, and hence $|Y_5| \geq \beta(R) - 2$ by Lemma 2.2. By a similar argument, and since $|Y_6| = 1$ and (by (1)) $|Y_i| = 1$ for each even $i \in \{8, \dots, k\}$, we have that $|Y_j| \geq \beta(R) - 2$ for each odd $j \in \{7, \dots, k-2\}$. So among $Y_4, \dots, Y_{k-1}$, there are $\lfloor \frac{k-4}{2} \rfloor$ bags of size at least $\beta(R) - 2$, and the remaining $\lceil \frac{k-4}{2} \rceil$ bags are of size 1.

Fix $y_3 \in Y_3$ with $|N_R(y_3) \cap Y_2|$ minimum and $y_k \in Y_k$ with $|N_R(y_k) \cap Y_1|$ minimum; and set $Z_2 = N_R(y_3) \cap Y_2$ and $Z_1 = N_R(y_k) \cap Y_1$. By assumption, $|Y_4| = 1$, and by (1), $|Y_{k-1}| = 1$, and therefore $d_R(y_3) = 1 + (|Y_3| - 1) + |Z_2| = |Y_3| + |Z_2|$, and similarly $d_R(y_k) = |Y_k| + |Z_1|$. By Lemma 2.2, $\delta(R) = \beta(R) - 1$, and hence $|Y_3| + |Z_2| \geq \beta(R) - 1$ and $|Y_k| + |Z_1| \geq \beta(R) - 1$. Now $Z = R[Z_1 \cup Z_2 \cup Y_3 \cup Y_4 \cup \cdots \cup Y_k]$ is a hyperhole, and by adding together the above bounds, we get that

$$\begin{aligned} |V(Z)| &\geq 2\,(\beta(R) - 1) + \left\lfloor \frac{k-4}{2} \right\rfloor (\beta(R) - 2) + \left\lceil \frac{k-4}{2} \right\rceil \\ &= 2\,(\beta(R) - 1) + \frac{k-5}{2}\,(\beta(R) - 2) + \frac{k-3}{2} \\ &= \frac{k-1}{2}\beta(R) - \frac{k-1}{2} + 1. \end{aligned}$$

Thus, by Theorem 4.1,

$$\chi(Z) \geq \left\lceil \frac{(k-1)\beta(R) - (k-1) + 2}{k-1} \right\rceil = \left\lceil \beta(R) - 1 + \frac{2}{k-1} \right\rceil,$$

and since $\frac{2}{k-1} > 0$ for all $k > 1$, we get that $\chi(Z) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction.

So $|Y_1| = 1$ and $|Y_4| \geq 2$. If $|Y_6| \geq 2$, then by (2), $|Y_3| = 1$, and hence (after relabeling the indices of $Y_1, \ldots, Y_k$) we may apply the earlier argument which handles the case where $|Y_1| \geq 2$ and $|Y_4| = 1$. So we may assume $|Y_6| = 1$; and it follows from this assumption together with (1) that for each odd $i \in \{7, 9, \ldots, k\}$, $|Y_{i-1}| = |Y_{i+1}| = 1$. By Lemma 2.2, $\delta(R) = \beta(R) - 1$, and therefore $|Y_i| \geq \beta(R) - 2$ for each odd $i \in \{7, 9, \ldots, k\}$. So, among the bags $Y_6, \ldots, Y_k, Y_1$, there are $\lfloor \frac{k-4}{2} \rfloor = \frac{k-5}{2}$ bags of size at least $\beta(R) - 2$, and the remaining $\lceil \frac{k-4}{2} \rceil = \frac{k-3}{2}$ bags are of size 1.

Fix $y_2 \in Y_2$ with $|N_R(y_2) \cap Y_3|$ minimum and $y_5 \in Y_5$ with $|N_R(y_5) \cap Y_4|$ minimum. Set $Z_3 = N_R(y_2) \cap Y_3$ and $Z_4 = N_R(y_5) \cap Y_4$. Since $d_R(y_2) = |Y_2| + |Z_3|$ and $d_R(y_5) = |Y_5| + |Z_4|$, it follows from Lemma 2.2 that $|Y_2| + |Z_3| \geq \beta(R) - 1$ and $|Y_5| + |Z_4| \geq \beta(R) - 1$. Since, by (3), Y_3 is complete to Y_4, it follows that $Z = R[Y_1 \cup Y_2 \cup Z_3 \cup Z_4 \cup Y_5 \cup \cdots \cup Y_k]$ is a hyperhole, and by adding together the above bounds we get that

$$\begin{aligned} |V(Z)| &\geq 2(\beta(R) - 1) + \frac{k-5}{2}(\beta(R) - 2) + \frac{k-3}{2} \\ &= \frac{k-1}{2}\beta(R) - \frac{k-1}{2} + 1. \end{aligned}$$

Thus, by Theorem 4.1,

$$\chi(Z) \geq \left\lceil \frac{(k-1)\beta(R) - (k-1) + 2}{k-1} \right\rceil = \left\lceil \beta(R) - 1 + \frac{2}{k-1} \right\rceil,$$

and since $\frac{2}{k-1} > 0$ for all $k > 1$, we get that $\chi(Z) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction. □

A super-sector is a *2-super-sector* if it contains only 2-sectors.

Lemma 4.17. *Let R be an odd ring. If R has a triad or a 2-super-sector, then R is β-perfect.*

Proof. Towards a contradiction, suppose R has a triad or a 2-super-sector but is not β-perfect. Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \ldots, k\}$. By Lemma 2.5, F is not chordal, and hence $F_1, \ldots, F_k$ are all nonempty. By Lemma 4.15, $F = (F_1, \ldots, F_k)$ is a ring. It follows that if (Y_{i-1}, Y_i, Y_{i+1}) is a triad of R, then F has a vertex of degree 2, contrary to Lemma 2.4; thus, R has no triad. So R has a 2-super-sector, say $S = (Y_\ell, \ldots, Y_r)$.

We now show for each 2-sector $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ contained in S that $|F_{s+1}| = \beta(F) - 2$ and $|F_{s+2}| = 1$. Since $Y_\ell, Y_{\ell+1}$ are of size 1, so are $F_\ell, F_{\ell+1}$, and hence it follows from Lemma 4.14 that $|F_{\ell+2}| = \beta(F) - 2$ and exactly one vertex in $F_{\ell+3}$ is complete to $F_{\ell+2}$. More specifically, since Y_i is complete to Y_{i+1} for each $i \in \{\ell, \ell+1, \ldots, r-1\}$ (and hence every vertex in $F_{\ell+3}$ is complete to $F_{\ell+2}$), we get that $|F_{\ell+3}| = 1$. So our claim holds for the 2-sector $(Y_{\ell+1}, Y_{\ell+2}, Y_{\ell+3}, Y_{\ell+4})$.

Since we now have that $|F_{\ell+3}| = |F_{\ell+4}| = 1$, we may repeat this argument for the 2-sector $(Y_{\ell+4}, Y_{\ell+5}, Y_{\ell+6}, Y_{\ell+7})$, if it exists, to get that $|F_{\ell+5}| = \beta(F)-2$ and $|F_{\ell+6}| = 1$; and then for the 2-sector $(Y_{\ell+7}, Y_{\ell+8}, Y_{\ell+9}, Y_{\ell+10})$, if it exists, to get that $|F_{\ell+8}| = \beta(F)-2$ and $|F_{\ell+9}| = 1$; and so on, until we get that $|F_{r-3}| = \beta(F) - 2$ and $|F_{r-2}| = 1$. But now the unique vertex of F_{r-1} is of degree 2 in F, contrary to Lemma 2.4. □

Lemma 4.18. *Let $R = (Y_1, \ldots, Y_k)$ be an odd ring that has no triad, no 2-super-sector, and contains no bad ring. Then some induced subgraph H of R is a hyperhole that has no triad and no 2-super-sector.*

Proof. Towards a contradiction, suppose every hyperhole in R has a triad or a 2-super-sector. For any hyperhole $H = (X_1, \ldots, X_k)$ in R, let $t(H)$ be the number of triads of H; i.e., $t(H)$ denotes the size of the set

$$\{i \in \{1, \ldots, k\} : (X_{i-1}, X_i, X_{i+1}) \text{ is a triad of } H\};$$

and let $s(H)$ be the number of 2-super-sectors of H; i.e., $s(H)$ denotes the size of the set

$$\{(\ell, r) : \ell, r \in \{1, \ldots, k\} \text{ and } (X_\ell, \ldots, X_r) \text{ is a 2-super-sector of } H\}.$$

Fix a hyperhole $H = (X_1, \ldots, X_k)$ in R that minimizes $t(H)$, and subject to that, also minimizes $s(H)$. For each $i \in \{1, \ldots, k\}$, if $|X_i| = 1$, then we assume that the unique vertex in X_i is complete to $Y_{i-1} \cup Y_{i+1}$ (such a vertex exists by the definition of a ring).

(1) If (X_{i-1}, X_i, X_{i+1}) is a triad of H, then $X_i = Y_i$. Consequently, there exist no five bags $X_j, X_{j+1}, \ldots, X_{j+4}$ of H all of size one.

Proof of (1): Suppose that (X_{i-1}, X_i, X_{i+1}) is a triad but $X_i \neq Y_i$. But now for any $y \in Y_i \setminus X_i$ the graph $H \cup \{y\}$ is a hyperhole with $t(H \cup \{y\}) < t(H)$, a contradiction. We now prove the second statement of the claim. Suppose, without loss of generality, that $X_1, X_2, \ldots, X_5$ are all of size 1. It follows from the first statement of the claim applied to each of the triads (X_1, X_2, X_3), (X_2, X_3, X_4) and (X_3, X_4, X_5) that $|Y_2| = |Y_3| = |Y_4| = 1$. But then (Y_2, Y_3, Y_4) is a triad of R, a contradiction. This proves (1).

(2) Let (X_{i-1}, X_i, X_{i+1}) be a triad of H. If $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, then:

- $|X_{i-2}| \geq 2$*;*
- $|X_{i-4}| = |X_{i-3}| = |Y_{i-3}| = 1$*; and*
- *no vertex in $Y_{i-1} \setminus X_{i-1}$ has at least two neighbors in Y_{i-2}.*

Similarly, if $Y_{i+1} \setminus X_{i+1} \neq \emptyset$, then:

- $|X_{i+2}| \geq 2$*;*
- $|Y_{i+3}| = |X_{i+3}| = |X_{i+4}| = 1$*; and*
- *no vertex in $Y_{i+1} \setminus X_{i+1}$ has at least two neighbors in Y_{i+2}.*

Proof of (2): If $|X_{i-4}| \geq 2$ or $|X_{i-3}| \geq 2$, then the graph $H' = (H \setminus (X_{i-2} \cup X_{i-1})) \cup Y^1_{i-2} \cup Y^2_{i-1}$ is a hyperhole with $t(H') < t(H)$, a contradiction. So $|X_{i-4}| = |X_{i-3}| = 1$. If $|X_{i-2}| = 1$, then (X_{i-2}, X_{i-1}, X_i) is a triad and $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, contrary to (1); so $|X_{i-2}| \geq 2$. If $|Y_{i-3}| \geq 2$, then the graph $H' = (H \setminus (X_{i-3} \cup X_{i-2} \cup X_{i-1})) \cup Y^2_{i-3} \cup Y^1_{i-2} \cup Y^2_{i-1}$ is a hyperhole with $t(H') < t(H)$, a contradiction; so $|Y_{i-3}| = 1$. Suppose that some vertex $y \in Y_{i-1} \setminus X_{i-1}$ has two neighbors $y', y'' \in Y_{i-2}$. Then the graph $H' = (H \setminus X_{i-2}) \cup \{y, y', y''\}$ is a hyperhole with $t(H') < t(H)$, a contradiction. The analogous statements when $Y_{i+1} \setminus X_{i+1} \neq \emptyset$ follow by symmetry. This proves (2).

(3) We may assume that no super-sector of H contains a sector of length at least 3.

Proof of (3): Suppose that some super-sector $S = (X_\ell, \ldots, X_r)$ of H contains a sector $T = (X_s, \ldots, X_t)$ of length at least 3. For each $i \in \{\ell, \ldots, r\}$ with $|X_i| = 1$, let $X'_i = X_i$. Let $Q = (X_a, \ldots, X_b)$ be any sector contained in the super-sector S. Suppose that Q is not a 2-sector. If the length of Q is odd, then for each $j \in \{a+1, \ldots, b-1\}$ let $X'_j = X^2_j$ if $j - a$ is odd, and let $X'_j = X^1_j$ if $j - a$ is even. If the length of Q is even, then let $X'_{b-1} = X^2_{b-1}$, let $X'_{b-2} = X^1_{b-2}$, and for each $j \in \{a+1, \ldots, b-3\}$ let $X'_j = X^2_j$ if $j - a$ is odd, and let $X'_j = X^1_j$ if $j - a$ is even. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S, let $X'_s = X_s$, $X'_{s+1} = X_{s+1}$, $X'_{s+2} = X_{s+2}$, and $X'_{s+3} = X_{s+3}$. Now the graph H' induced by $X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \cdots \cup X_r))$ is a hyperhole with $t(H') = t(H)$, $s(H') = s(H)$, and with one fewer super-sector that contains a sector of length at least 3. By repeating this process, we obtain a hyperhole with the same number of triads as H and the same number of 2-super-sectors as H but with no super-sector that contains a sector of length at least 3. Thus we may assume that H has no super-sector that contains a sector of length at least 3. This proves (3).

By (3), we may assume that each sector contained in a super-sector of H is of length 1 or 2.

(4) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. Then, for all $i \in \{\ell+1, \dots, r-1\}$, if $|X_i| = 1$ then $|Y_i| = 1$.

Proof of (4): Observe that $|X_{\ell+1}| = |X_{r-1}| = 1$. Suppose that $|Y_{\ell+1}| \neq 1$. For each $i \in \{\ell, \dots, r\} \setminus \{\ell+1\}$ such that $|X_i| = 1$, let $X'_i = X_i$. Let $X'_{\ell+1} = Y^2_{\ell+1}$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \dots, X_r)$, let $X'_{s+1} = X^1_{s+1}$, and let $X'_{s+2} = X^2_{s+2}$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. So $|Y_{\ell+1}| = 1$, and by a symmetric argument we get that $|Y_{r-1}| = 1$.

Now suppose that $|X_i| = 1$ but $|Y_i| \neq 1$ for some $i \in \{\ell+2, \dots, r-2\}$. For each $j \in \{\ell, \dots, r\} \setminus \{i\}$ such that $|X_j| = 1$, let $X'_j = X_j$. Let $X'_i = Y^2_i$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in the subsequence $(X_\ell, \dots, X_i)$, let $X'_{s+1} = X^2_{s+1}$ and let $X'_{s+2} = X^1_{s+2}$; and for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in the subsequence $(X_i, \dots, X_r)$, let $X'_{s+1} = X^1_{s+1}$ and let $X'_{s+2} = X^2_{s+2}$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. Thus, for every $i \in \{\ell+2, \dots, r-2\}$, if $|X_i| = 1$ then $|Y_i| = 1$. This proves (4).

(5) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. If $|Y_\ell| \geq 2$, then $|X_{\ell-3}| = |X_{\ell-2}| = |Y_{\ell-2}| = 1$. Similarly, if $|Y_r| \geq 2$, then $|Y_{r+2}| = |X_{r+2}| = |X_{r+3}| = 1$.

Proof of (5): We prove the statement when $|Y_\ell| \geq 2$, and the case where $|Y_r| \geq 2$ follows by symmetry. Suppose that $|Y_\ell| \geq 2$. Let $X'_\ell = Y^2_\ell$ and $X'_r = X_r$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \dots, X_r)$, let $X'_s = X_s$, $X'_{s+1} = X^1_{s+1}$, $X'_{s+2} = X^2_{s+2}$, and $X'_{s+3} = X_{s+3}$. Let $X'_{\ell-1} = X^1_{\ell-1}$. If $|X_{\ell-3}| \geq 2$, then let $X'_{\ell-2} = X^1_{\ell-2}$; now $X'_{\ell-2} \cup X'_{\ell-1} \cup X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_{\ell-2} \cup X_{\ell-1} \cup X_\ell \cup \dots \cup X_r))$ induces a hyperhole H' with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. So $|X_{\ell-3}| = 1$. If $|X_{\ell-2}| \geq 2$, then $X'_{\ell-1} \cup X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_{\ell-1} \cup X_\ell \cup \dots \cup X_r)$ again induces a hyperhole H' with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. A similar contradiction arises if $|Y_{\ell-2}| \geq 2$. Thus, if $|Y_\ell| \geq 2$, then $|X_{\ell-3}| = |X_{\ell-2}| = |Y_{\ell-2}| = 1$. This proves (5).

(6) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. Then we may assume that no vertex in $Y_\ell \setminus X_\ell$ has at least two neighbors in $Y_{\ell-1}$. Similarly, we may assume that no vertex in $Y_r \setminus X_r$ has at least two neighbors in Y_{r+1}.

Proof of (6): On the contrary, suppose that some vertex $y \in Y_\ell \setminus X_\ell$ has two neighbors $u, v \in Y_{\ell-1}$. It follows that $|Y_\ell| \geq 2$, and hence $|X_{\ell-3}| = |X_{\ell-2}| = 1$ by (5). Now consider the graph $H' = R[\{u, v, y\} \cup (V(H) \setminus X_{\ell-1})]$; it is a hyperhole, with $t(H') = t(H)$ and $s(H') = s(H)$, of which $(X_{\ell-3}, X_{\ell-2}, \{u, v\}, X_\ell \cup \{y\}, X_{\ell+1}, \dots, X_r)$ is a 2-super-sector. If some vertex in $Y_{\ell-3} \setminus X_{\ell-3}$ has at least two neighbors in $Y_{\ell-4}$ or if some vertex in

$Y_r \setminus X_r$ has at least two neighbors in Y_{r+1}, then we may repeat this process. Since this process must terminate, we obtain in the end a hyperhole H^+ (say with bags $X'_1, \dots, X'_k$ satisfying $X'_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$) with $t(H^+) = t(H)$, $s(H^+) = s(H)$, and having a 2-super-sector $(X'_{\ell'}, \dots, X'_{r'})$ where $\{\ell, \dots, r\} \subseteq \{\ell', \dots, r'\}$, and no vertex in $Y_{\ell'} \setminus X'_{\ell'}$ has at least two neighbors in $Y_{\ell'-1}$, and no vertex in $Y_{r'} \setminus X'_{r'}$ has at least two neighbors in $Y_{r'+1}$. This proves (6).

(7) Let $S = (X_\ell, \dots, X_r)$ be a super-sector of H. If S contains a 2-sector, then we may assume that S is a 2-super-sector.

Proof of (7): Suppose that S contains at least one 2-sector. If S contains only 2-sectors, then we are done; thus, we may assume that S contains a sector of some other length, and by (3), each such sector is of length 1. Let (X_{t-1}, X_t, X_{t+1}) be a 1-sector contained in S. For each $i \in \{\ell, \dots, r\}$ with $|X_i| = 1$, let $X'_i = X_i$. For each 1-sector (X_{s-1}, X_s, X_{s+1}), let $X'_s = X_s$, and for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$, let $X'_{s+1} = X^2_{s+1}$ and $X'_{s+2} = X^1_{s+2}$ if $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ appears in the subsequence $(X_\ell, \dots, X_{t-1})$, and otherwise, let $X'_{s+1} = X^1_{s+1}$ and $X'_{s+2} = X^1_{s+2}$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$, $s(H') = s(H)$, and $(X'_\ell, \dots, X'_r)$ is not a super-sector of H'. After repeating this procedure for each super-sector that contains a 2-sector, we obtain a hyperhole in which every super-sector containing a 2-sector is a 2-super-sector. This proves (7).

(8) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. If $|Y_\ell| \geq 2$, then $|X_{\ell-1}| \geq 2$. Similarly, if $|Y_r| \geq 2$, then $|X_{r+1}| \geq 2$.

Proof of (8): Suppose that $|Y_\ell| \geq 2$ but $|X_{\ell-1}| = 1$. Let $X'_\ell = Y^2_\ell$, for each $i \in \{\ell+2, \ell+5, \ell+8, \dots, r-3\}$, let $X'_i = X^1_i$, and for every other $i \in \{\ell, \dots, r\}$ let $X'_i = X_i$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. Thus, if $|Y_\ell| \geq 2$, then $|X_{\ell-1}| \geq 2$, and by symmetry, if $|Y_r| \geq 2$, then $|X_{r+1}| \geq 2$. This proves (8).

(9) We may assume that every sector of length at least 2 in H is contained in a super-sector of H.

Proof of (9): Let $S = (X_a, \dots, X_b)$ be a sector of length at least 2 in H, and say without loss of generality that $a = 1$. Suppose that H contains two 0-sectors (X_i, X_{i+1}) and (X_j, X_{j+1}) such that $i, i+1, j, j+1$ are all distinct. Assume i and j are chosen so that i is minimum (possibly $i = b$) and j is maximum (possibly $j = k$). By the definition of a sector we have that $\{i, i+1, j, j+1\} \cap \{2, \dots, b-1\} = \emptyset$, and by our choice of i and j, there is no 0-sector in $(X_{j+1}, \dots, X_1, \dots, X_b, \dots, X_i)$. Now $(X_j, X_{j+1}, \dots, X_1, \dots, X_b, \dots, X_i, X_{i+1})$ is a super-sector of H that contains S, as required.

Thus, it remains to show that H has at least two 0-sectors that are formed by four distinct bags of H. If H has a super-sector $(X_\ell, \dots, X_r)$, then $(X_\ell, X_{\ell+1})$ and (X_{r-1}, X_r)

are two such 0-sectors; thus, we may assume that H has no super-sector. If H has no triad, then H is as desired (i.e., H is an induced subgraph of R, and is a hyperhole, and has no triad and no 2-super-sector), and we are done; so suppose (X_{i-1}, X_i, X_{i+1}) is a triad of H. Since R has no triad, it follows from (1) that $|Y_{i-1}| \geq 2$ or $|Y_{i+1}| \geq 2$. Suppose without loss of generality that $|Y_{i-1}| \geq 2$. Then, by (2), $|X_{i-4}| = |X_{i-3}| = 1$, and hence (X_{i-4}, X_{i-3}) and (X_{i-1}, X_i) are two 0-sectors of H, as required. This proves (9).

A 2-super-sector $(X_\ell, \ldots, X_r)$ of H is of *type 1* if Y_{s+1} is complete to Y_{s+2} for every 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \ldots, X_r)$, and is of *type 2* otherwise. Thus, if a 2-super-sector $(X_\ell, \ldots, X_r)$ of H is of type 2, then it contains at least one 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ such that Y_{s+1} is not complete to Y_{s+2}.

Suppose $(X_\ell, \ldots, X_r)$ is a 2-super-sector of type 1 of H. By (4), for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \ldots, X_r)$, we have that $|Y_s| = |Y_{s+3}| = 1$, and by the definition of type 1, Y_{s+1} is complete to Y_{s+2}. Since R has no 2-super-sector (so in particular $(Y_\ell, \ldots, Y_r)$ is not a 2-super-sector of R), it follows that $|Y_\ell| \geq 2$ or $|Y_r| \geq 2$ (or both). We say a 2-super-sector $(X_\ell, \ldots, X_r)$ of type 1 of H is *left* if $|Y_\ell| \geq 2$, and *right* otherwise; so $(X_\ell, \ldots, X_r)$ is left or right, but not both, and if it is right, then $|Y_\ell| = 1$ and $|Y_r| \geq 2$.

(10) Let $S = (X_\ell, \ldots, X_r)$ be a 2-super-sector of type 2 of H. Then we may assume that S contains only one 2-sector.

Proof of (10): Suppose $(X_\ell, \ldots, X_r)$ contains at least two 2-sectors. We show that we may modify some of the bags among $X_\ell, \ldots, X_r$ so that H contains one fewer 2-super-sector of type 2 that contains at least two 2-sectors but with $t(H)$ and $s(H)$ unchanged. We consider two cases; first, the case where there is some 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S such that Y_{s+1} is complete to Y_{s+2}. Let $(X_a, \ldots, X_b)$ be a maximal subsequence of $(X_\ell, \ldots, X_r)$ such that:

- $(X_a, X_{a+1}, X_{a+2}, X_{a+3})$ and $(X_{b-3}, X_{b-2}, X_{b-1}, X_b)$ are 2-sectors (possibly not distinct); and
- every 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in $(X_a, \ldots, X_b)$ is such that Y_{t+1} is complete to Y_{t+2}.

It follows from the existence of $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ that there exists such a subsequence. Note that, by maximality, if $(X_{a-3}, X_{a-2}, X_{a-1}, X_a)$ is a 2-sector contained in S, then Y_{a-2} is not complete to Y_{a-1}, and similarly, if $(X_b, X_{b+1}, X_{b+2}, X_{b+3})$ is a 2-sector contained in S, then Y_{b+1} is not complete to Y_{b+2}. Now, for each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in the subsequence $(X_\ell, \ldots, X_{a-1}, X_a)$, set $X_{t+2} = Y^1_{t+2}$, and for each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in the subsequence $(X_b, X_{b+1}, \ldots, X_r)$, let $X_{t+1} = Y^1_{t+1}$. Observe that $(X_{a-1}, X_a, \ldots, X_b, X_{b+1})$ is now a 2-super-sector of type 1 of H, and that no other subsequence of $(X_\ell, \ldots, X_r)$ besides $(X_{a-1}, X_a, \ldots, X_b, X_{b+1})$ is a 2-super-sector of H. Thus, $s(H)$ remains unchanged. It is clear that $t(H)$ also remains unchanged, and H now has one fewer 2-super-sector of type 2 that contains at least two 2-sectors.

Now for the other case, i.e., for every 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S, we have that Y_{s+1} is not complete to Y_{s+2}. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \dots, X_r)$, besides $(X_{\ell+1}, X_{\ell+2}, X_{\ell+3}, X_{\ell+4})$, set $X_{s+1} = Y^1_{s+1}$. Observe that $(X_\ell, X_{\ell+1}, \dots, X_{\ell+5})$ is now a 2-super-sector of type 2 of H, which contains only one 2-sector (namely, $(X_{\ell+1}, \dots, X_{\ell+4})$), and that no other subsequence of $(X_\ell, \dots, X_r)$ besides $(X_\ell, X_{\ell+1}, \dots, X_{\ell+5})$ is a 2-super-sector of H. Thus, $s(H)$ remains unchanged. It is clear that $t(H)$ also remains unchanged, and H now has one fewer 2-super-sector of type 2 that contains at least two 2-sectors. This proves (10).

In view of (10), from here on we assume that every 2-super-sector of type 2 of H contains only one 2-sector.

Suppose (X_{i-1}, X_i, X_{i+1}) is a triad of H. Since R has no triad, it follows from (1) that $Y_{i-1} \setminus X_{i-1} \neq \emptyset$ or $Y_{i+1} \setminus X_{i+1} \neq \emptyset$ (or possibly both). We say (X_{i-1}, X_i, X_{i+1}) is *left* if $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, and *right* otherwise. Thus each triad of H is left or right, but not both left and right.

We now construct an induced subgraph Z of R in the following way.

Step 1. Let $Z_i = X_i$ for each $i \in \{1, \dots, k\}$.

Step 2. For each left triad (X_{i-1}, X_i, X_{i+1}) of H, set $Z_{i-1} = Y^2_{i-1}$, and for each right triad (X_{i-1}, X_i, X_{i+1}) of H, set $Z_{i+1} = Y^2_{i+1}$.

Step 3. For each left 2-super-sector $S = (X_\ell, \dots, X_r)$ of type 1 of H, do the following:

- let $Z_\ell = Y^2_\ell$; and
- for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in S, let $Z_{s+1} = X^1_{s+1}$.

For each right 2-super-sector $S = (X_\ell, \dots, X_r)$ of type 1 of H, do the following:

- let $Z_r = Y^2_r$; and
- for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in S, let $Z_{s+2} = X^1_{s+2}$.

For each 2-super-sector $S = (X_\ell, \dots, X_r)$ of type 2 of H, do the following:

- let $y \in Y_{\ell+2}$ and $y' \in Y_{\ell+3}$ be nonadjacent vertices, and let $Z_{\ell+2} = Y^1_{\ell+2} \cup \{y\}$ and $Z_{\ell+3} = Y^1_{\ell+3} \cup \{y'\}$.

Step 4. For each $i \in \{1, \dots, k\}$ such that Z_i was not modified in Step 2 or 3, let $Z_i = X^1_i$ if $|X_i| = 1$, and otherwise let $Z_i = X^2_i$.

Let $Z_1, \dots, Z_k$ be as they are when the above algorithm terminates, and let $Z = R[Z_1 \cup \dots \cup Z_k]$.

Observation 1: $Z_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$.

Observation 2: $|Z_i| \leq 2$ for each $i \in \{1, \dots, k\}$.

Observation 3: If $|X_i| = 1$ and $|Z_i| = 2$, then the algorithm set Z_i to be of size 2 in Step 2 or Step 3, and therefore at least one of the following holds:

- (X_i, X_{i+1}, X_{i+2}) is a left triad of H;
- (X_{i-2}, X_{i-1}, X_i) is a right triad of H;
- $(X_i, X_{i+1}, \dots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \dots, k\}$;
- $(X_\ell, \dots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1 of H for some $\ell \in \{1, \dots, k\}$.

Observation 4: If $|X_i| \geq 2$ and $|Z_i| = 1$, then the algorithm set Z_i to be of size 1 in Step 3 as a result of:

- $(X_{i-1}, X_i, X_{i+1}, X_{i+2})$ being a 2-sector contained in a left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1; or
- $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ being a 2-sector contained in a right 2-super-sector of type 1.

We use the above observations repeatedly throughout the remainder of the proof.

(11) For each left triad (X_{i-1}, X_i, X_{i+1}) of H, the following hold:

(a) $|Z_{i-4}| = |Z_{i-3}| = 1$*;*

(b) $|Z_{i-2}| = |Z_{i-1}| = 2$*; and*

(c) Z_{i-2} *is not complete to* Z_{i-1}.

Similarly, for each right triad (X_{i-1}, X_i, X_{i+1}) of H, the following hold:

(d) $|Z_{i+3}| = |Z_{i+4}| = 1$*;*

(e) $|Z_{i+1}| = |Z_{i+2}| = 2$*; and*

(f) Z_{i+1} *is not complete to* Z_{i+2}.

Proof of (11): We prove the three statements about left triads, and the analogous statements about right triads follow from a symmetric argument; so let (X_{i-1}, X_i, X_{i+1}) be a left triad of H. By (2), $|Y_{i-3}| = 1$, and therefore $|Z_{i-3}| = 1$ by Observation 1. Thus, in order to prove (a), it remains to show that $|Z_{i-4}| = 1$. To the contrary, suppose $|Z_{i-4}| \geq 2$. By (2), $|X_{i-4}| = 1$, and therefore it follows from Observation 3 that one of the following holds:

- $(X_{i-4}, X_{i-3}, X_{i-2})$ is a left triad of H;
- $(X_{i-6}, X_{i-5}, X_{i-4})$ is a right triad of H;
- $S = (X_{i-4}, X_{i-3}, \dots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \dots, k\}$;
- $S = (X_\ell, \dots, X_{i-5}, X_{i-4})$ is a right 2-super-sector of type 1 of H for some $\ell \in \{1, \dots, k\}$.

Suppose the first bullet holds. Then the five bags $X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1}, X_i$ are all of size 1, contrary to (1). Thus, the first bullet does not hold. Suppose the second bullet holds. Then, by (2) applied to $(X_{i-6}, X_{i-5}, X_{i-4})$, we get that $|X_{i-3}| \geq 2$, contrary to the already established fact that $|Y_{i-3}| = 1$. So the second bullet does not hold. Suppose the third bullet holds. Then $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector, and in particular $|X_{i-1}| \geq 2$. But this contradicts the fact that (X_{i-1}, X_i, X_{i+1}) is a triad. So the third bullet does not hold. Finally, if the fourth bullet holds, then $|Y_{i-4}| \geq 2$, and hence, by (8), $|X_{i-3}| \geq 2$, contrary to the fact that $|Y_{i-3}| = 1$. Thus, we conclude that $|Z_{i-4}| = 1$, and this completes the proof of (a).

We now prove (b). By Observation 2, it suffices to show that $|Z_{i-2}| \neq 1$ and $|Z_{i-1}| \neq 1$. Suppose $|Z_{i-2}| = 1$. Since, by (2), $|X_{i-2}| \geq 2$, it follows from Observation 4 that $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ or $(X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1})$ is a 2-sector in a 2-super-sector of H. If $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector, then $|X_{i-1}| \geq 2$, contrary to (X_{i-1}, X_i, X_{i+1}) being a triad; and if $(X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1})$ is a 2-sector, then $|X_{i-3}| \geq 2$, and hence $|Y_{i-3}| \geq 2$, contrary to (2). Thus $|Z_{i-2}| \neq 1$. Suppose $|Z_{i-1}| = 1$. Since (X_{i-1}, X_i, X_{i+1}) is a left triad of H, the set Z_{i-1} was set to be of size 2 in Step 2. Since on termination $|Z_{i-1}| = 1$, it follows that Z_{i-1} was modified in Step 3, and hence $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ or $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector in a 2-super-sector of H. But then, in both cases, $|X_{i-1}| \geq 2$, a contradiction. Therefore $|Z_{i-1}| \neq 1$, and this completes the proof of (b).

By (b), $|Z_{i-2}| = |Z_{i-1}| = 2$. Since $|X_{i-1}| = 1$, one vertex from Z_{i-1}, say y, belongs to $Y_{i-1} \setminus X_{i-1}$. By (2) applied to the triad (X_{i-1}, X_i, X_{i+1}), it follows that y has exactly one neighbor in Y_{i-2}, and hence y is not complete to Z_{i-2}. Therefore Z_{i-1} is not complete to Z_{i-2}, and this completes the proof of (c). This proves (11).

(12) For each left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, the following hold:

- *(a) $|Z_{\ell-3}| = |Z_{\ell-2}| = |Z_{r-1}| = |Z_r| = 1$;*
- *(b) $|Z_{\ell-1}| = |Z_\ell| = 2$, and $Z_{\ell-1}$ is not complete to Z_ℓ; and*
- *(c) for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \dots, X_r)$, we have that $|Z_s| = |Z_{s+1}| = |Z_{s+3}| = 1$ and $|Z_{s+2}| = 2$.*

Similarly, for each right 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, the following hold:

- *(d) $|Z_\ell| = |Z_{\ell+1}| = |Z_{r+2}| = |Z_{r+3}| = 1$;*
- *(e) $|Z_r| = |Z_{r+1}| = 2$, and Z_r is not complete to Z_{r+1}; and*
- *(f) for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \dots, X_r)$, we have that $|Z_s| = |Z_{s+2}| = |Z_{s+3}| = 1$ and $|Z_{s+1}| = 2$.*

Proof of (12): We prove (a), (b) and (c) for a left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, and the analogous statements (d), (e) and (f) for right 2-super-sectors of type 1 follow from a symmetric argument. We first prove (a). By Observation 2, each of $Z_{\ell-3}, Z_{\ell-2}, Z_{r-1}, Z_r$ has size at most 2. By (4), $|Y_{r-1}| = 1$, and by (5), $|Y_{\ell-2}| = 1$, and therefore $|Z_{\ell-2}| = |Z_{r-1}| = 1$ by Observation 1. By (5), we also get that $|X_{\ell-3}| = 1$.

Suppose $|Z_{\ell-3}| = 2$. Then, by Observation 3, $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1})$ is a left triad of H, or $(X_{\ell-5}, X_{\ell-4}, X_{\ell-3})$ is a right triad of H, or there exists $r' \in \{1, \ldots, k\}$ such that $(X_{\ell-3}, X_{\ell-2}, \ldots, X_{r'})$ is a left 2-super-sector of type 1 of H, or there exists $\ell' \in \{1, \ldots, k\}$ such that $(X_{\ell'}, \ldots, X_{\ell-4}, X_{\ell-3})$ is a right 2-super-sector of type 1 of H. In the first case, $|X_{\ell-1}| = 1$, contrary to the fact that, by (8), $|X_{\ell-1}| \geq 2$. In the second case, by (2) applied to the triad $(X_{\ell-5}, X_{\ell-4}, X_{\ell-3})$, we get that $|X_{\ell-2}| \geq 2$, contrary to the fact that $|Y_{\ell-2}| = 1$. In the third case, $(X_{\ell-2}, X_{\ell-1}, X_{\ell}, X_{\ell+1})$ is a 2-sector, and in particular, $|X_{\ell}| \geq 2$, contrary to $(X_{\ell}, \ldots, X_r)$ being a 2-super-sector. In the fourth case, it follows from $(X_{\ell'}, \ldots, X_{\ell-4}, X_{\ell-3})$ being a right 2-super-sector of type 1 that $|Y_{\ell-3}| \geq 2$, and hence, by (8), $|Y_{\ell-2}| \geq 2$, contrary to the already established fact that $|Y_{\ell-2}| = 1$. In each of the four cases we obtained a contradiction, and therefore we conclude that $|Z_{\ell-3}| = 1$. Suppose $|Z_r| \geq 2$. Then, by Observation 3, (X_r, X_{r+1}, X_{r+2}) is a left triad of H, or (X_{r-2}, X_{r-1}, X_r) is a right triad of H, or there exists $r' \in \{1, \ldots, k\}$ such that $(X_r, X_{r+1}, \ldots, X_{r'})$ is a left 2-super-sector of type 1 of H, or there exists $\ell' \in \{1, \ldots, k\}$ such that $(X_{\ell'}, \ldots, X_{r-1}, X_r)$ is a right 2-super-sector of type 1 of H. In the first case, by (2) applied to (X_r, X_{r+1}, X_{r+2}), we get that $|X_{r-1}| \geq 2$, contrary to the fact that $|X_{r-1}| = 1$. In the second case, $|X_{r-2}| = 1$, contrary to $(X_{\ell}, \ldots, X_r)$ being a 2-super-sector. Suppose the third case holds, i.e., $(X_r, X_{r+1}, \ldots, X_{r'})$ is a left 2-super-sector of type 1 of H. Then $|Y_r| \geq 2$, and hence by (8) applied to the 2-super-sector $(X_{\ell}, \ldots, X_r)$, we get that $|X_{r+1}| \geq 2$, contrary to $(X_r, X_{r+1}, \ldots, X_{r'})$ being a 2-super-sector. In the fourth case, clearly we must have that $\ell = \ell'$; but then $(X_{\ell}, \ldots, X_r)$ is both left and right, a contradiction. So $|Z_r| = 1$, and this completes the proof of (a).

We now prove (b). By Observation 2, it suffices to show that $|Z_{\ell-1}| \neq 1$ and $|Z_{\ell}| \neq 1$. Suppose $|Z_{\ell-1}| = 1$. Since $|Y_{\ell}| \geq 2$, it follows from (8) that $|X_{\ell-1}| \geq 2$, and hence it follows from Observation 4 that $(X_{\ell-2}, X_{\ell-1}, X_{\ell}, X_{\ell+1})$ or $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1}, X_{\ell})$ is a 2-sector contained in a 2-super-sector of H. If $(X_{\ell-2}, X_{\ell-1}, X_{\ell}, X_{\ell+1})$ is a 2-sector, then $|X_{\ell}| \geq 2$, a contradiction; and if $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1}, X_{\ell})$ is a 2-sector, then $|Y_{\ell-2}| \geq 2$, contrary to (5). So $|Z_{\ell-1}| = 2$. By Step 3 of the algorithm, $|Z_{\ell}| = 2$. Finally, since $|X_{\ell}| = 1$, one vertex from Z_{ℓ}, say y, belongs to $Y_{\ell} \setminus X_{\ell}$. By (6) applied to the 2-super-sector $(X_{\ell}, \ldots, X_r)$, y has exactly one neighbor in $Y_{\ell-1}$, and hence y is not complete to $Z_{\ell-1}$. Therefore $Z_{\ell-1}$ is not complete to Z_{ℓ}, and this completes the proof of (b).

Finally, we prove (c); let $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ be a 2-sector contained in $(X_{\ell}, \ldots, X_r)$. By (4), $|Y_s| = |Y_{s+3}| = 1$, and hence $|Z_s| = |Z_{s+3}| = 1$ by Observation 1. By Step 3 of the algorithm, $|Z_{s+1}| = 1$, and $|Z_{s+2}| = 2$, and this completes the proof of (c). This proves (12).

Recall that, by (10), each 2-super-sector $(X_{\ell}, \ldots, X_r)$ of type 2 of H contains only one 2-sector, and hence $(X_{\ell}, \ldots, X_r) = (X_{\ell}, X_{\ell+1}, \ldots, X_{\ell+5})$.

(13) For each 2-super-sector $(X_{\ell}, \ldots, X_{\ell+5})$ of type 2 of H, the following hold:

(a) $|Z_{\ell}| = |Z_{\ell+1}| = |Z_{\ell+4}| = |Z_{\ell+5}| = 1$;

(b) $|Z_{\ell+2}| = |Z_{\ell+3}| = 2$, and $Z_{\ell+2}$ is not complete to $Z_{\ell+3}$.

Proof of (13): Let $(X_{\ell}, \ldots, X_r)$ be a 2-super-sector of type 2 of H. We first prove (a).

By (4), $|Y_{\ell+1}| = |Y_{\ell+4}| = 1$, and therefore $|Z_{\ell+1}| = |Z_{\ell+4}| = 1$ by Observation 1; so it remains to prove that $|Z_\ell| = |Z_{\ell+5}| = 1$. Suppose $|Z_\ell| = 2$. By Observation 3, $(X_\ell, X_{\ell+1}, X_{\ell+2})$ is a left triad of H, or $(X_{\ell-2}, X_{\ell-1}, X_\ell)$ is a right triad of H, or $(X_\ell, X_{\ell+1}, \ldots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \ldots, k\}$, or $(X_{\ell'}, \ldots, X_{\ell-1}, X_\ell)$ is a right 2-super-sector of type 1 of H for some $\ell' \in \{1, \ldots, k\}$. In the first case, $|X_{\ell+2}| = 1$, a contradiction. In the second case, it follows from (2) applied to the triad $(X_{\ell-2}, X_{\ell-1}, X_\ell)$ that $|X_{\ell+1}| \geq 2$, a contradiction. In the third case, clearly we must have that $(X_\ell, X_{\ell+1}, \ldots, X_{r'}) = (X_\ell, \ldots, X_{\ell+5})$, and hence this 2-super-sector is both of type 1 and type 2, a contradiction. In the fourth case, by (8) applied to $(X_{\ell'}, \ldots, X_{\ell-1}, X_\ell)$, we get that $|X_{\ell+1}| \geq 2$, a contradiction. We therefore conclude that $|Z_\ell| = 1$, and by a symmetric argument we get that $|Z_{\ell+5}| = 1$. This completes the proof of (a).

We now prove (b). It follows from Observation 2 and Step 3 that $|Z_{\ell+2}| = |Z_{\ell+3}| = 2$. That $Z_{\ell+2}$ is not complete to $Z_{\ell+3}$ follows from the choice of vertices y and y' in Step 3. Therefore (b) holds, and this completes the proof of (13).

It is clear from construction that Z is a ring of length k with bags $Z_1, \ldots, Z_k$. We now show that Z is a bad ring by checking that $Z = (Z_1, \ldots, Z_k)$ satisfies the following conditions from the definition of a bad ring.

- For every $i \in \{1, \ldots, k\}$, $|Z_i| \leq 2$.
- For every $i \in \{1, \ldots, k\}$, if $|Z_i| = |Z_{i+1}| = 2$, then Z_i is not complete to Z_{i+1} and $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$.
- Z has no triad.
- There exists at least one integer $i \in \{1, \ldots, k\}$ such that $|Z_i| = |Z_{i+1}| = 2$.

By Observation 2, the first bullet holds.

We now prove that the second bullet holds; so assume $|Z_i| = |Z_{i+1}| = 2$ for some $i \in \{1, \ldots, k\}$. For our first step, suppose $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$. Let $s, t \in \{1, \ldots, k\}$ be such that X_s is the only bag of size 1 in the sequence $(X_s, \ldots, X_i)$ and X_t is the only bag of size 1 in the sequence $(X_{i+1}, \ldots, X_t)$; that two such bags of size 1 exist follows from the fact that H has a triad or a 2-super-sector together with the observation that, since $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$, $s \neq i$ and $t \neq i+1$. Then $(X_s, \ldots, X_i, X_{i+1}, \ldots, X_t)$ is a sector of H, and by (3), it is a 2-sector; thus, $s = i-1$ and $t = i+2$. By (7) and (9), $(X_{s-1}, X_s, X_{s+1}, X_{s+2})$ is contained in a 2-super-sector $(X_\ell, \ldots, X_r)$ of H. Since $|Z_i| = |Z_{i+1}| = 2$, it follows from (12) that $(X_\ell, \ldots, X_r)$ is of type 2, and then from (13) that $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}. Therefore, if $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$, then the second bullet holds. So we may assume that $|X_i| = 1$ or $|X_{i+1}| = 1$.

Suppose $|X_i| = 1$. Then, since $|Z_i| = 2$, it follows from Observation 3 that (X_i, X_{i+1}, X_{i+2}) is a left triad of H, or (X_{i-2}, X_{i-1}, X_i) is a right triad of H, or $(X_i, X_{i+1}, \ldots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \ldots, k\}$, or $(X_\ell, \ldots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1 of H. In the first case, by (1), $|Y_{i+1}| = 1$, and hence $|Z_{i+1}| = 1$, a contradiction. Suppose the second case holds, i.e., suppose that (X_{i-2}, X_{i-1}, X_i) is a right triad of H. It follows from (11) that $|Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}. By (1), $|Y_{i-1}| = 1$, and hence $|Z_{i-1}| = 1$ by Observation 1. If $|Z_{i-2}| = 1$, then we see from the facts just established that the second bullet holds; so we may assume $|Z_{i-2}| = 2$. Then, by Observation 3, (X_{i-2}, X_{i-1}, X_i) is a left triad of H (in which case the triad (X_{i-2}, X_{i-1}, X_i) is both left and right, a contradiction); or $(X_{i-4}, X_{i-3}, X_{i-2})$ is a right triad of H (in which case $X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1}, X_i$ are five consecutive bags of size 1, contrary to (1)); or $(X_{i-2}, X_{i-1}, \dots, X_{r'})$ is a left 2-super-sector of type 1 of H (in which case Y_i is complete to Y_{i+1}, contrary to the already established fact that Z_i is not complete to Z_{i+1}); or $(X_\ell, \dots, X_{i-3}, X_{i-2})$ is a right 2-super-sector of type 1 of H (in which case, by (8), $|X_{i-1}| \geq 2$, contrary to the assumption that (X_{i-2}, X_{i-1}, X_i) is a triad). So $|Z_{i-2}| = 1$, and we now conclude that in the second case (i.e., when (X_{i-2}, X_{i-1}, X_i) is a right triad), the second bullet holds. So we may assume that the second case does not hold. In the third case, i.e., when $(X_i, X_{i+1}, \dots, X_r)$ is a left 2-super-sector of type 1 of H, we get from (4) that $|Y_{i+1}| = 1$, and hence $|Z_{i+1}| = 1$, a contradiction. In the fourth case, i.e., when $(X_\ell, \dots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1, it follows from (12) that $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}, and hence the second bullet holds. Thus, we conclude that $|X_i| \neq 1$, and from a symmetric argument we get that $|X_{i+1}| \neq 1$, a contradiction. This completes the proof that the second bullet holds.

To prove the third bullet, suppose Z has a triad (Z_{i-1}, Z_i, Z_{i+1}). If (X_{i-1}, X_i, X_{i+1}) is a triad of H, then by part (b) or (e) of (11), $|Z_{i-1}| = 2$ or $|Z_{i+1}| = 2$, a contradiction. So at least one of X_{i-1}, X_i, X_{i+1} has size at least 2. Suppose $|X_{i-1}| \geq 2$. Then, by Observation 4, exactly one of $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ and $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$, call it S, is a 2-sector contained in a 2-super-sector $T = (X_\ell, \dots, X_r)$ of H. It follows from (13) that T is not of type 2, and hence T is of type 1. If $S = (X_{i-2}, X_{i-1}, X_i, X_{i+1})$, then by (12), T is a left 2-super-sector of type 1, in which case, by (12)(c), $|Z_i| = 2$, a contradiction. If $S = (X_{i-3}, X_{i-2}, X_{i-1}, X_i)$, then by (12), T is a right 2-super-sector of type 1, in which case, either: $(X_i, X_{i+1}, X_{i+2}, X_{i+3})$ is a 2-sector contained in T, and hence by (12)(f) we have that $|Z_{i+1}| = 2$, a contradiction; or $(X_i, X_{i+1}) = (X_{r-1}, X_r)$, and hence by (12)(e) we again have that $|Z_{i+1}| = 2$, a contradiction. We conclude that $|X_{i-1}| = 1$, and it follows from a symmetric argument that $|X_{i+1}| = 1$. So $|X_i| \geq 2$. Then, by Observation 4, either $(X_{i-1}, X_i, X_{i+1}, X_{i+2})$ is a 2-sector contained in a left 2-super-sector of type 1 of H (in which case, by (12)(c), $|Z_{i+1}| = 2$, a contradiction), or $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ is a 2-sector contained in a right 2-super-sector of type 1 of H (in which case, by (12)(f), $|Z_{i-1}| = 2$, a contradiction). This completes the proof that Z has no triad, and therefore the third bullet holds.

Suppose the fourth bullet does not hold. Then Z is a hyperhole, since each bag of Z is of size 1 or 2 (by the first bullet), no two of which are consecutive and of size 2 (by assumption); therefore clearly Z has no 2-super-sector; and by the third bullet, Z has no triad. But then Z is a hyperhole in R with no triad and no 2-super-sector, contrary to our initial assumption that R contains no such induced subgraph. Thus, the fourth bullet holds.

So Z is a bad ring, contrary to the fact that R contains no bad ring. □

We need the following from [7].

Lemma 4.19 (Lemma 3.11 in [7]). *Every nontrivial odd hyperhole contains a base hyperhole.*

Lemma 4.20. *Let $R = (Y_1, \dots, Y_k)$ be a ring. Suppose R contains a hyperhole $H = (X_1, \dots, X_k)$ that has exactly one 0-sector and all its other sectors are of length 2. If R contains no bad ring and no base hyperhole, then R is a hyperhole that contains exactly one 0-sector and all its other sectors are of length 2.*

Proof. Say (X_3, X_4) is the 0-sector of H, and suppose towards a contradiction that R contains no bad ring and no base hyperhole but R is not a hyperhole that has exactly one 0-sector and all its other sectors are of length 2. Then at least one of the following holds:

- (Y_3, Y_4) is not a 0-sector of R; or
- there is some 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ of H such that $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ is not a 2-sector of R.

Note that the first bullet implies the second, for if $|Y_3| \geq 2$, then (Y_k, Y_1, Y_2, Y_3) is not a 2-sector of R, and if $|Y_4| \geq 2$, then (Y_4, Y_5, Y_6, Y_7) is not a 2-sector of R; so it suffices to consider only the second bullet. Let $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ be a 2-sector of H such that $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ is not a 2-sector of R. So $|Y_s| \geq 2$, or $|Y_{s+3}| \geq 2$, or Y_{s+1} is not complete to Y_{s+3}.

Suppose first that $|Y_s| \geq 2$. Let H' be the subgraph of R induced by $(V(H) \setminus (X_{s-1} \cup X_{s+1})) \cup Y^1_{s-1} \cup Y_s \cup Y^1_{s+1}$. Clearly H' is a hyperhole, and it can easily be checked that H' has no triad, no 2-super-sector, and that it is not the case that H' has exactly one 0-sector and all its other sectors are of length 2. Thus, by the definition of trivial, H' is a nontrivial hyperhole, and therefore it follows from Lemma 4.19 that R contains a base hyperhole, a contradiction. So $|Y_s| = 1$, and by a symmetric argument we get that $|Y_{s+3}| = 1$.

So Y_{s+1} is not complete to Y_{s+2}. Let $y \in Y_{s+1}$ and $y' \in Y_{s+2}$ be nonadjacent vertices of R, and let $Y'_{s+1} = Y^1_{s+1} \cup \{y\}$ and $Y'_{s+2} = Y^1_{s+2} \cup \{y'\}$. For each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ of H in the subsequence $(X_4, \dots, X_s)$, let $Y'_t = Y^1_t$, $Y'_{t+1} = Y_{t+1}$, $Y'_{t+2} = Y^1_{t+2}$, and $Y'_{t+3} = Y^1_{t+3}$. For each 2-sector of H in the subsequence $(X_{s+3}, \dots, X_k, X_1, X_2, X_3)$, let $Y'_t = Y^1_t$, $Y'_{t+1} = Y^1_{t+1}$, $Y'_{t+2} = Y_{t+2}$, and $Y'_{t+3} = Y^1_{t+3}$. Let $Y'_3 = Y^1_3$ and $Y'_4 = Y^1_4$ (this ensures that Y'_4 is defined in the case $(X_s, X_{s+1}, X_{s+2}, X_{s+3}) = (X_4, X_5, X_6, X_7)$ and that Y'_3 is defined in the case $(X_s, X_{s+1}, X_{s+2}, X_{s+3}) = (X_k, X_1, X_2, X_3)$). The graph induced by $Y'_1 \cup \cdots \cup Y'_k$ is a bad ring, a contradiction. □

Lemma 4.21. *Let $R = (Y_1, \dots, Y_k)$ be a big ring. If R is minimally β-imperfect, then R is a bad ring or R contains a base hyperhole.*

Proof. Suppose that R is minimally β-imperfect, is not a bad ring, and contains no base hyperhole. By minimality and by Lemma 4.12, R contains no bad ring. By Lemma 4.17, R has no triad and no 2-super-sector, and hence by Lemma 4.18, R contains a hyperhole

$H = (X_1, \dots, X_k)$ that has no triad and no 2-super-sector. Since R contains no base hyperhole, it follows from Lemma 4.19 that H is trivial, and since H has no triad and no 2-super-sector, it follows from the definition of trivial that H contains exactly one 0-sector and all its other sectors are of length 2. Thus, by Lemma 4.20, R is a hyperhole that has exactly one 0-sector and all its other sectors are of length 2. That is, R is a trivial hyperhole, and hence, by Lemma 4.9, R is β-perfect, a contradiction. □

5 Forbidden induced subgraph characterization

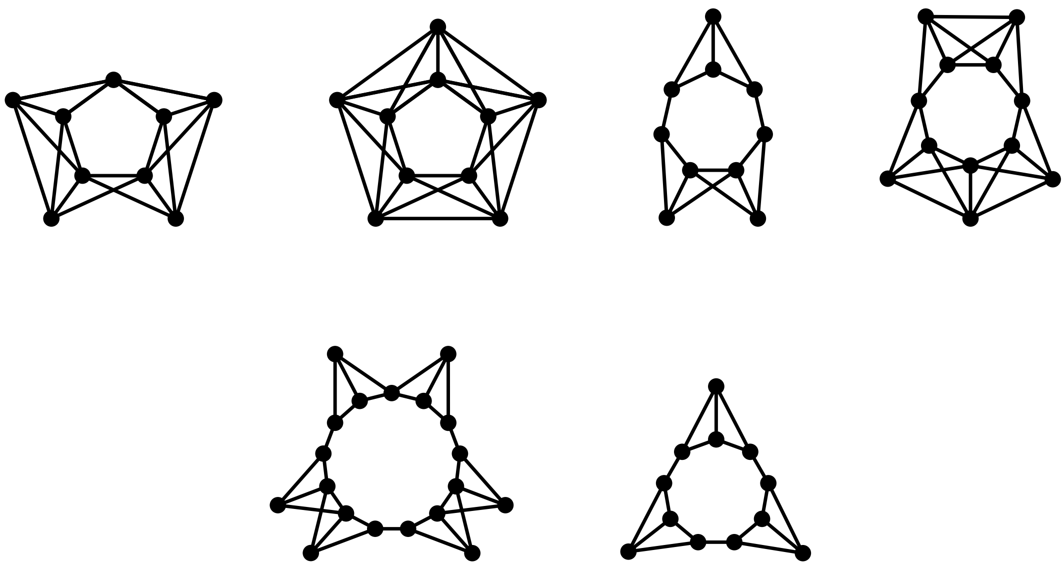

Figure 7: From left-to-right, top-to-bottom: the first four figures depict the graphs R_5, H_5, R_7 and H_7 respectively, and the other two depict an example of a bad ring and a bad base hyperhole respectively.

The following is our main result, a forbidden induced subgraph characterization for the class of claw-free β-perfect graphs (an example of each of the forbidden induced subgraphs (besides an even hole) is given in Figure 7).

Theorem 5.1. *A claw-free graph is β-perfect if and only if it contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7.*

Proof. Let G be a claw-free graph. Suppose G is β-perfect. Then G contains no even hole, and by Lemmas 4.3, 4.4, 4.6, 4.7, 4.10 and 4.12 respectively, G contains no H_5, R_5, H_7, R_7, bad base hyperhole or bad ring.

We now prove the converse; suppose G contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7, and suppose towards a contradiction that G is not β-perfect. Since every induced subgraph of G is claw-free and contains none of the forbidden induced subgraphs mentioned in the statement of the present theorem, we may assume G is minimally β-imperfect. Then, by Lemma 2.8, G has no clique cutset, and hence by Lemma 3.6, G is a complete graph or an odd ring, or G contains a universal vertex. Clearly complete graphs are β-perfect, and it is easily seen that no minimal β-imperfect graph contains a universal vertex (adding to a β-perfect graph a universal vertex yields another β-perfect graph). Thus, G is an odd ring. By Lemma 4.5, G is not a 5-ring, and by Lemma 4.8, G is not a 7-ring. So G is a big ring. By Lemma 4.21, G contains a base hyperhole, and now it follows from Lemma 4.16 that G is β-perfect, a contradiction. Thus, if G is a claw-free graph that contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7, then G is β-perfect. □

6 Recognition algorithm

In this section we present a polynomial-time algorithm that determines whether a given claw-free graph is β-perfect. We first need an algorithm that decides whether a ring is β-perfect, and ours will use the following as a subroutine.

Theorem 6.1 (Theorem 3.16 in [7]). *There is an algorithm with the following specifications:*

Input: A k-hyperhole $H = (X_1, \dots, X_k)$.
Output: Yes if H is β-perfect, and No otherwise.
Running time: $\mathcal{O}(k)$.

Theorem 6.2. *There is an algorithm with the following specifications:*

Input: A ring $R = (Y_1, \dots, Y_k)$.
Output: Yes if R is β-perfect, and No otherwise.
Running time: $\mathcal{O}(n^4)$, where $n = |V(R)|$.

Proof. Consider the following algorithm.

Step 1. If k is even, then return No.

Step 2. For each $i \in \{1, \dots, k\}$, order Y_i as $Y_i = \{y_i^1, \dots, y_i^{|Y_i|}\}$ so that $N_R[y_i^{|Y_i|}] \subseteq \cdots \subseteq N_R[y_i^1] = Y_{i-1} \cup Y_i \cup Y_{i+1}$.

Step 3. If $k = 5$, then return No if both of the following hold:

- $|Y_i| \geq 2$ for each $i \in \{1, \dots, k\}$;
- y_i^2 is adjacent to y_{i+1}^2 for each $i \in \{1, \dots, 4\}$;

and also return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+1}|, |Y_{i+2}|, |Y_{i+3}| \geq 2$;
- there exist nonadjacent vertices $y_{i+1}^j \in Y_{i+1}$ and $y_{i+2}^{j'} \in Y_{i+2}$, and y_i^2 is adjacent to y_{i+1}^j and y_{i+3}^2 is adjacent to $y_{i+2}^{j'}$.

and otherwise, return Yes.

Step 4. If $k = 7$, then return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+1}|, |Y_{i+3}|, |Y_{i+4}|, |Y_{i+5}| \geq 2$;
- y_i^2 is adjacent to y_{i+1}^2, y_{i+3}^2 is adjacent to y_{i+4}^2, and y_{i+4}^2 is adjacent to y_{i+5}^2;

and also return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+3}|, |Y_{i+4}| \geq 2$;
- $y_{i+3}^{|Y_{i+3}|}$ is nonadjacent to $y_{i+4}^{|Y_{i+4}|}$;

and otherwise, return Yes.

Step 5. (At this point, $k \geq 9$.) If R contains a triad or a 2-super-sector, then return Yes. If R is a hyperhole, then apply the algorithm of Theorem 6.1; and if the output of that algorithm is Yes, then return Yes, and otherwise, return No. Now check whether, up to some cyclic permutation of the indices $1, \dots, k$,[1] the following three conditions hold:

- $|Y_i| = 1$ for each even $i \in \{8, \dots, k\}$;
- $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$;
- $Y_1 \cup Y_2$, $Y_3 \cup Y_4$ and $Y_5 \cup Y_6$ are cliques.

If so, then return Yes, and otherwise, return No.

We now prove that this algorithm correctly decides whether a ring $R = (Y_1, \dots, Y_k)$ is β-perfect. First, suppose the algorithm returns Yes but R is not β-perfect. By Step 1, k is odd. Suppose $k = 5$. Then, by Lemma 4.5, R contains a ring, say $F = (X_1, \dots, X_k)$ where $X_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$, that is isomorphic to H_5 or R_5. If $F = H_5$, then clearly the first two bullets in Step 3 hold for any $i \in \{1, \dots, 5\}$, and hence the algorithm returns No, a contradiction. If $F = R_5$, say with X_1 being its unique bag of size 1, then the last two bullets in Step 3 hold for $i = 2$, and hence the algorithm returns No, a contradiction. So $k \neq 5$.

Suppose $k = 7$. Then, by Lemma 4.8, R contains a ring, say $F = (X_1, \dots, X_k)$ where $X_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$, that is isomorphic to H_7 or R_7. If $F = H_7$, say with X_1 and X_4 as its only two bags of size 1, then the first two bullets in Step 4 hold for $i = 2$, and hence the algorithm returns No, a contradiction. If $F = R_7$, say with X_1 not complete to X_2, then the last two bullets in Step 4 hold for $i = 5$, and hence the algorithm returns No, a contradiction. Therefore $k \neq 7$.

So $k \geq 9$. Since R is not β-perfect, it follows from Lemma 4.17 that R has no triad and no 2-super-sector. If R is a hyperhole, then the algorithm returned Yes as a result of the algorithm of Theorem 6.1 returning Yes, in which case R is β-perfect, a contradiction. Therefore the algorithm does not return Yes as a result of R being a hyperhole or having a triad or 2-super-sector. So, possibly after cyclically permuting indices $1, \dots, k$, the bags $Y_1, \dots, Y_k$ satisfy the three bullets in Step 5.

[1] $2, \dots, k, 1$, and $3, \dots, k, 1, 2$, and $k, 1, \dots, k-1$ are examples of cyclic permutations of $1, \dots, k$.

By Theorem 5.1, R contains a bad base hyperhole or bad ring, say $F = (F_1, \ldots, F_k)$. Suppose F is a bad ring. By definition, there exists $i \in \{1, \ldots, k\}$ such that $|F_i|, |F_{i+1}| = 2$, and F_i is not complete to F_{i+1}, and $|F_{i-2}| = |F_{i-1}| = |F_{i+2}| = |F_{i+3}| = 1$. Also by definition, F has no triad, and hence it follows that $|F_{i-3}| = 2$ and $|F_{i+4}| = 2$. If $\{i, i+1\} \subseteq \{7, \ldots, k\}$, then one of $i, i+1$ is even and the bag of F indexed by it has size at least 2, contrary to the first bullet in Step 5. So $\{i, i+1\} \subseteq \{k, 1, \ldots, 7\}$. If $(i, i+1) \in \{(k, 1), (4, 5)\}$, then $|Y_1| \geq 2$ and $|Y_4| \geq 2$, contrary to the second bullet in Step 5; and if $(i, i+1) \in \{(2, 3), (6, 7)\}$, then $|Y_3| \geq 2$ and $|Y_6| \geq 2$, again contradicting the second bullet in Step 5. Thus $(i, i+1) \in \{(1, 2), (3, 4), (5, 6)\}$. But then not all of $Y_1 \cup Y_2$, $Y_3 \cup Y_4$ and $Y_5 \cup Y_6$ are cliques, contrary to the third bullet in Step 5. Therefore F is not a bad ring.

So F is a bad base hyperhole. By definition, F contains (at least) two 0-sectors, say (F_i, F_{i+1}) and (F_j, F_{j+1}). It follows from F having no triad that all of $Y_{i-1}, Y_{i+2}, Y_{j-1}, Y_{j+2}$ have size at least 2. If $(i, i+1) = (6, 7)$ or $(i, i+1) = (k, 1)$, then Y_{k-1} or Y_8 has size at least 2, contrary to the first bullet in Step 5; so $(i, i+1) \notin \{(6, 7), (k, 1)\}$, and by a symmetric argument, $(j, j+1) \notin \{(6, 7), (k, 1)\}$. Suppose $\{i, i+1\} \subseteq \{8, \ldots, k-1\}$. (As a consequence of this assumption, and since k is odd, we have that $k \geq 11$.) Then one of $i-1, i+2$ is even, belongs to $\{10, \ldots, k-1\}$, and the bag of F (and hence also the bag of R) indexed by it has size at least 2, contrary to the first bullet in Step 5. So $\{i, i+1\} \not\subseteq \{8, \ldots, k-1\}$, and by a symmetric argument, $\{j, j+1\} \not\subseteq \{8, \ldots, k-1\}$. Suppose $(i, i+1) = (7, 8)$. Then $|F_6| \geq 2$, and hence by the second bullet in Step 5, $|F_3| = 1$, and by the definition of a base hyperhole, $|F_5| = 1$. Since (F_3, F_4, F_5) is not a triad, $|F_4| \geq 2$, and thus by the second bullet in Step 5, $|F_1| = 1$. Similarly, since (F_1, F_2, F_3) is not a triad, $|F_2| \geq 2$. From these facts about the sizes of bags $F_1, \ldots, F_6$, together with the fact established earlier that $\{(i, i+1), (j, j+1)\} \cap \{(6, 7), (k, 1)\} = \emptyset$, we deduce that there is no 0-sector in the subsequence $(F_k, F_1, \ldots, F_7)$. Since $\{j, j+1\} \not\subseteq \{8, \ldots, k-1\}$, it follows that $(j, j+1) = (k-1, k)$. But $|F_1| = 1$, contrary to the fact that $|F_{j+1}| \geq 2$. So $(i, i+1) \neq (7, 8)$, and by symmetry, $(i, i+1) \neq (k-1, k)$. By analogous argument, $(j, j+1) \neq (7, 8)$ and $(j, j+1) \neq (k-1, k)$. Putting all these things together, we see that $\{i, i+1, j, j+1\} \subseteq \{1, \ldots, 6\}$, and hence up to symmetry (and since F has no triad and no two consecutive bags of size at least 2), there are two cases:

1. $(i, i+1) = (1, 2)$ and $(j, j+1) = (4, 5)$; or
2. $(i, i+1) = (2, 3)$ and $(j, j+1) = (5, 6)$.

In the first case, $|Y_3|, |Y_6| \geq 2$, and in the second, $|Y_1|, |Y_4| \geq 2$; in each case, the second bullet in Step 5 is contradicted. This completes the proof that if the algorithm returns Yes, then R is β-perfect.

We now prove the converse. Towards a contradiction, suppose R is β-perfect but the algorithm returns No. Since R is β-perfect, k is odd, and hence the algorithm does not return No in Step 1. Suppose the algorithm returns No in Step 3. If the algorithm returns No as a result of the first two bullets in Step 3 being satisfied, then clearly R contains H_5, and hence R is not β-perfect by Lemma 4.3. Similarly, if the algorithm returns No as a result of the last two bullets in Step 3 being satisfied, then it is clear that R contains R_5, and hence R is not β-perfect by Lemma 4.4. So the algorithm does not return No in Step 3. In a similar way (but using Lemmas 4.6 and 4.7 instead of Lemmas 4.3 and 4.3 respectively), we can show that the algorithm does not return No in Step 4. So the algorithm returns No in Step 5. By Step 5, R has no triad and no 2-super-sector, and (by the correctness of the algorithm of Theorem 6.1, which is called in Step 5) is not a hyperhole, and there is no cyclic permutation of $1, \dots, k$ for which the three bullets in Step 5 hold. Since R is β-perfect, it follows from Lemmas 4.10 and 4.12 that R contains no bad ring and no bad base hyperhole. If R contains a base hyperhole, then by Lemma 4.13 the three bullets in Step 5 hold, a contradiction. So R contains no base hyperhole. Now, By Lemma 4.18, R contains a hyperhole H that has no triad and no 2-super-sector. It follows from Lemma 4.19 that H is trivial, and since H has no triad and no 2-super-sector, it follows from the definition of trivial that H has exactly one 0-sector and all its other sectors are of length 2. By Lemma 4.20, R is a hyperhole that contains exactly one 0-sector and all its other sectors are of length 2. But this contradicts the fact that R is not a hyperhole, a contradiction. This completes the proof that if the algorithm returns No, then R is not β-perfect.

Finally, we show that this algorithm runs in time $\mathcal{O}(n^4)$, where $n = |V(G)|$. Step 1 takes $\mathcal{O}(1)$ time. Step 2 can be done in $\mathcal{O}(n^2)$ time, as observed in [11]. Step 3 takes $\mathcal{O}(n^2)$ time, and Step 4 takes $\mathcal{O}(n)$ time. For Step 5: checking whether R contains a triad can be done in $\mathcal{O}(n)$ time; checking whether R contains a 2-super-sector can be done in $\mathcal{O}(n^2)$ time; and checking whether R is a hyperhole can be done in $\mathcal{O}(n + m)$ time [1]. Then we check $\mathcal{O}(n)$ times whether the three bullets in Step 5 are satisfied, which can be done in $\mathcal{O}(n)$ time, $\mathcal{O}(1)$ time, and $\mathcal{O}(n^3)$ time respectively. Therefore Step 5 can be done in $\mathcal{O}(n^4)$ time. It follows that the algorithm has running time $\mathcal{O}(n^4)$. □

Our algorithm for deciding whether a claw-free graph is β-perfect involves a process of clique-cutset decomposition, and for that we use the following.

A *clique cutset decomposition tree* of a graph G is a tree T satisfying the following:

- the root of T is G;
- each non-leaf node H of T has a clique cutset C such that $V(H) \setminus C$ admits a partition (A, B) where A is anticomplete to B in H, and the children of H in T are the graphs $G[A \cup C]$ and $G[B \cup C]$, one of which has no clique cutset and is a leaf of T;
- the leaves of T are induced subgraphs of G that have no clique cutset.

Observe that such a clique cutset decomposition tree has $\mathcal{O}(|V(G)|)$ many leaves.

Theorem 6.3 (Tarjan [13]). *There exists an algorithm that computes a clique cutset decomposition tree of any n-vertex m-edge graph G in $\mathcal{O}(nm)$ time.*

Lemma 6.4. *Let G be a graph and let T be a clique cutset decomposition tree of G with leaves $H_1, \dots, H_t$. Let F be an induced subgraph of G that has no clique cutset. Then F is an induced subgraph of one of $H_1, \dots, H_t$.*

Proof. It suffices to show for any node B of T containing F that one of the children $B_1, \dots, B_b$ of B in T also contains F. Suppose otherwise. Then there exist two nonadjacent vertices x, y of F such that, without loss of generality, $x \in V(B_1) \setminus V(B_2)$ and $y \in V(B_2) \setminus V(B_1)$. Now $V(B_1) \cap V(B_2)$ is a clique cutset of B that separates x and y, and therefore $V(F) \cap V(B_1) \cap V(B_2)$ is a clique cutset of F that separates x and y, contrary to the fact that F has no clique cutset. □

Let W_5^4 be the graph consisting of a hole of length five together with an additional vertex that has exactly four neighbours in this hole.

Lemma 6.5 (Boncompagni, Penev and Vušković [1])**.** *Let G be a graph and let T be a clique cutset decomposition tree of G with leaves $H_1, \dots, H_t$. Then the following are equivalent.*

- *G is (3PC, proper wheel)-free.*
- *G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, for all $H_i \in \{H_1, \dots, H_t\}$ and for all anticomponents H of H_i, either H is a long ring, or H contains no long holes, or $\alpha(H) \leq 2$.*

Lemma 6.6. *Let G be a claw-free graph and let T be a clique cutset decomposition tree of G with leaves $H_1, \dots, H_t$. Then the following are equivalent.*

- *G is (C_4, 3PC, proper wheel)-free.*
- *For every $H_i \in \{H_1, \dots, H_t\}$, either H_i is a chordal graph, or H_i contains a long ring R and every vertex in $V(H_i) \setminus V(R)$ is a universal vertex of H_i.*

Proof. Suppose that G is (C_4, 3PC, proper wheel)-free. By Lemma 6.5, G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, for all $H_i \in \{H_1, \dots, H_t\}$ and for all anticomponents H of H_i, either H is a long ring, or H contains no long holes, or $\alpha(H) \leq 2$.

Fix $H_i \in \{H_1, \dots, H_t\}$; we show that H_i is as described in the second bullet. If H_i is even-hole-free, then it follows from Lemma 3.6 that H_i is a complete graph, or is an odd (and therefore a long) ring, or contains a ring R and every vertex of $V(H_i) \setminus V(R)$ is a universal vertex of H_i. So we may assume that H_i contains an even hole, and in particular, since G is C_4-free, H_i contains a hole of length at least six; let H be an anticomponent of H_i containing such a hole. It follows that $\alpha(H) \geq 3$, and therefore, by Lemma 6.5, H is a long ring. Suppose F is an anticomponent of H_i different from H. If F contains at least two vertices, then F contains two nonadjacent vertices, which together with any two nonadjacent vertices from H induce a C_4, a contradiction. So every anticomponent of H_i different from H consists of a single vertex, and therefore every vertex of $V(H_i) \setminus V(H)$ is a universal vertex of H_i.

We now prove the converse. Suppose the second bullet holds. Then each of $H_1, \dots, H_t$ is chordal or consists of a long ring possibly together with some universal vertices and therefore contains no C_4, 3PC or proper wheel. If G contains a C_4, a 3PC or a proper wheel, then by Lemma 6.4 so does one of $H_1, \dots, H_t$, a contradiction. Therefore G is (C_4, 3PC, proper wheel)-free. □

Our main algorithmic result is the following.

Theorem 6.7. *There is an algorithm with the following specifications:*

> Input: A claw-free graph G.
> Output: Yes if G is β-perfect, and No otherwise.
> Running time: $\mathcal{O}(n^5)$.

Proof. Consider the following algorithm.

Step 1. Using the algorithm of Theorem 6.3, compute an extreme clique cutset decomposition tree T of G, and call its leaves $H_1, \dots, H_t$.

Step 2. For each $H_i \in \{H_1, \dots, H_t\}$, check whether H_i is chordal or the graph obtained from H_i by removing all universal vertices is a ring of odd length; if one of these checks fails, output No and terminate.

Step 3. For each graph $H_i \in \{H_1, \dots, H_t\}$ that is not a chordal graph, let H_i' be the graph obtained from H_i by removing all universal vertices (so, by Step 2, H_i' is a long ring); now compute a ring partition $(Y_1, \dots, Y_k)$ of H_i' and apply the algorithm of Theorem 6.2 to $H_i' = (Y_1, \dots, Y_k)$ to test whether H_i' is β-perfect. If for some H_i' the algorithm of Theorem 6.2 returns No, then output No and terminate; and otherwise, output Yes.

We now prove that this algorithm correctly decides whether a given claw-free graph G is β-perfect. Suppose G is β-perfect but the algorithm returns No. Suppose the algorithm returns No in Step 2. Then, by Lemma 6.6, G is not (C_4, 3PC, proper wheel)-free, and therefore it follows from Lemma 3.2 that G contains an even hole, in which case G is not β-perfect, a contradiction. So the algorithm does not return No in Step 2, and therefore the algorithm returns No in Step 3, in which case G contains some induced subgraph that was correctly determined by the algorithm of Theorem 6.2 to be β-imperfect, contrary to the β-perfection of G. Thus, if G is β-perfect, then the algorithm returns Yes.

Suppose G is not β-perfect but the algorithm returns Yes. If G contains an even hole, then by Lemma 6.4, one of $H_1, \dots, H_t$ contains an even hole; but by Step 2, each of $H_1, \dots, H_t$ is a chordal graph or consists of a ring of odd length together with a possibly empty set of universal vertices, and so in either case is even-hole-free. Therefore G is even-hole-free. It now follows from Theorem 5.1 that G contains an induced subgraph F isomorphic to H_5, R_5, H_7 or R_7 or to a bad base hyperhole or a bad ring; by Lemmas 4.3, 4.4, 4.6, 4.7, 4.10 and 4.12 respectively, F is not β-perfect. Clearly F has no clique cutset, and hence by Lemma 6.4, F is contained in one of $H_1, \dots, H_t$, say in H_1 without loss of generality. Since F is not chordal, neither is H_1, and therefore (as a result of Step 2) H_1 consists of a long ring together with a possibly empty set of universal vertices. Furthermore, since no vertex of F is universal in F, no vertex of F is universal in H_1, and hence F is contained in H_1', where H_1' is the graph obtained from H_1 by removing all universal vertices. Since F is not β-perfect, neither is H_1', and therefore the algorithm of Theorem 6.2 returns No when given H_1' as input. Thus the algorithm presented above returns No in Step 3, a contradiction.

Finally, we show that this algorithm runs in time $\mathcal{O}(n^5)$. Step 1 takes $\mathcal{O}(nm)$ time. Checking whether a graph is chordal can be done in $\mathcal{O}(n+m)$ time, and checking whether a graph is a ring of odd length can be done in $\mathcal{O}(n^2)$ time (see Lemma 8.14 from [1]), and therefore Step 2 takes $\mathcal{O}(n^2)$ time. In Step 3, for $\mathcal{O}(n)$ many graphs we compute a ring partition and run the algorithm of Theorem 6.2; and therefore Step 4 takes $\mathcal{O}(n^5)$ time. Therefore the algorithm has running time $\mathcal{O}(n^5)$. □

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