Decompositions and algorithms for interpretations of sparse graphs

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The FO model checking problem

Input: finite graph G, sentence φ

Task: determine whether $G \models \varphi$

Example of a FO formula about graphs:

$$\varphi := \exists x_1 \exists x_2 \exists x_3 \forall y \,.\, (y = x_1) \lor (y = x_2) \lor (y = x_3) \lor E(y, x_1) \lor E(y, x_2) \lor E(y, x_3)$$

φ says that graph has dominating set of size at most 3

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Easy: runtime $n^{|\varphi|}$

Wanted: runtime $f(\varphi) \cdot n^c$ for fixed c (FPT)

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Not possible on the class of all finite graphs, but known to be true on many classes of "nice" graph classes, such as bounded degree graphs, planar graphs, …

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Motivation:

- Fundamental problem on its own
- Existence of a FPT algorithm implies the existence of FPT algorithms for many parameterized problems (independent set, dominating set, …)

The FO model checking problem

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Theorem (Courcelle, 1990)
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Theorem (Courcelle, Makowski, Rotics, 2000)
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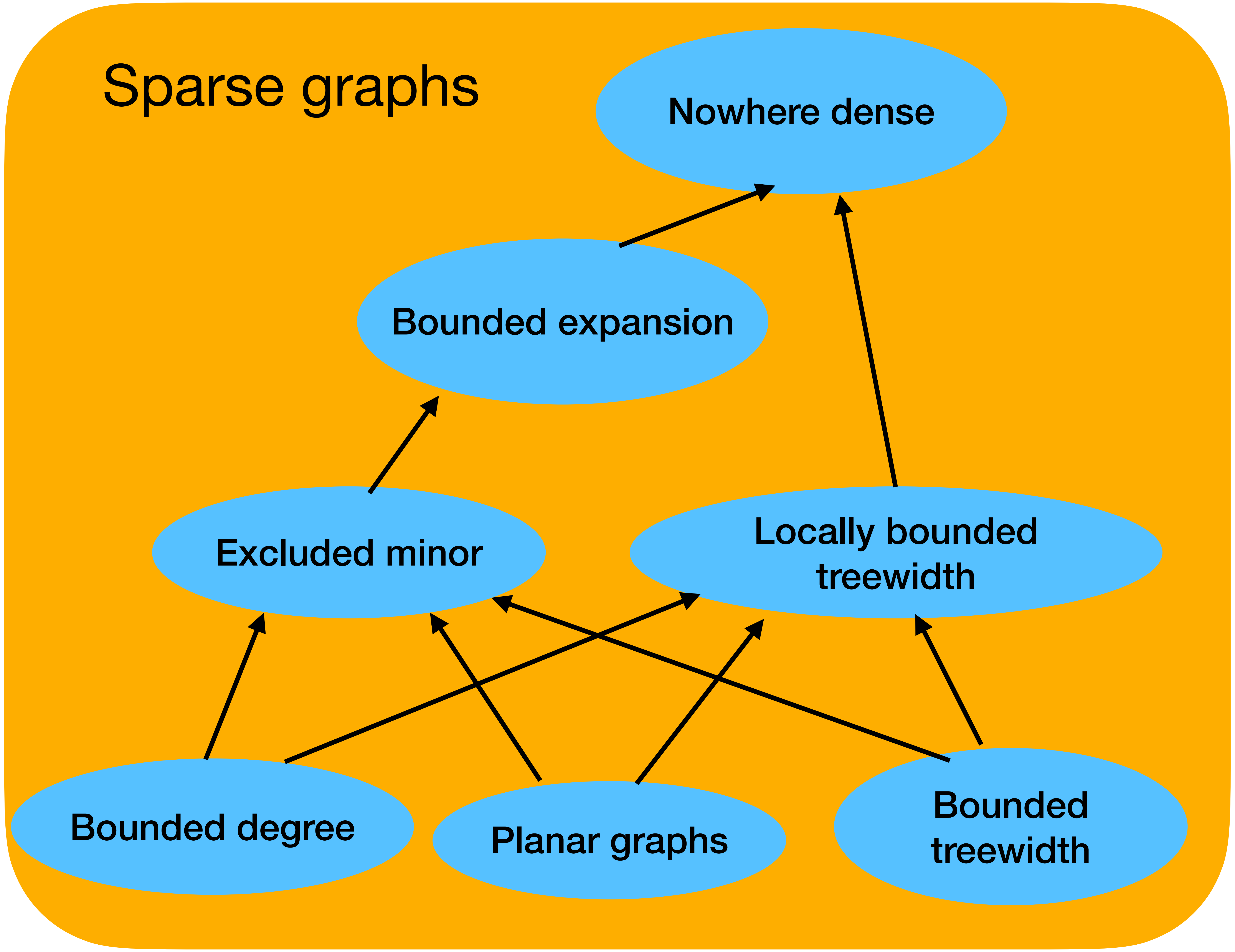
Sparse graphs
Nowhere dense
Bounded expansion
Excluded minor
Locally bounded treewidth
Bounded degree
Planar graphs
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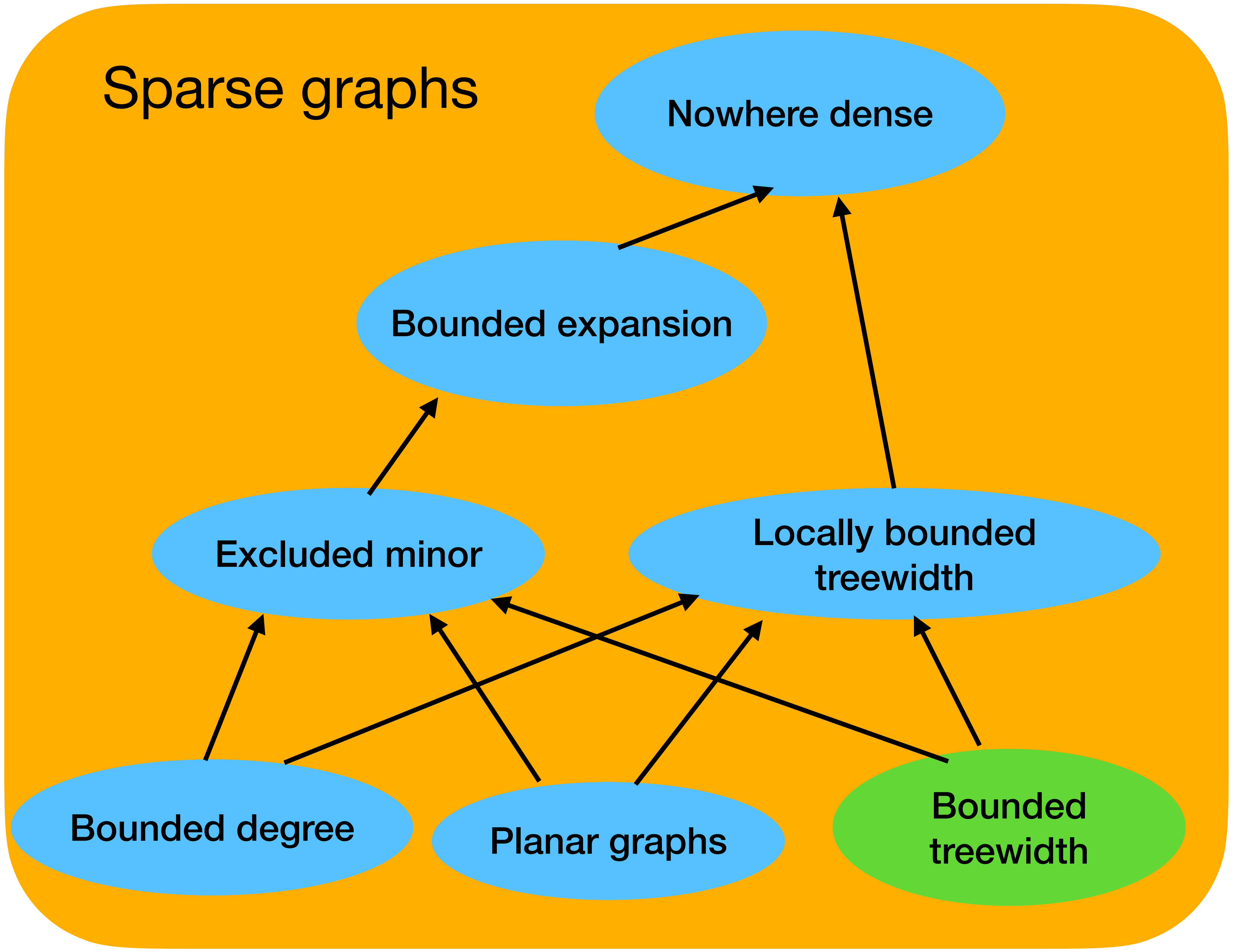
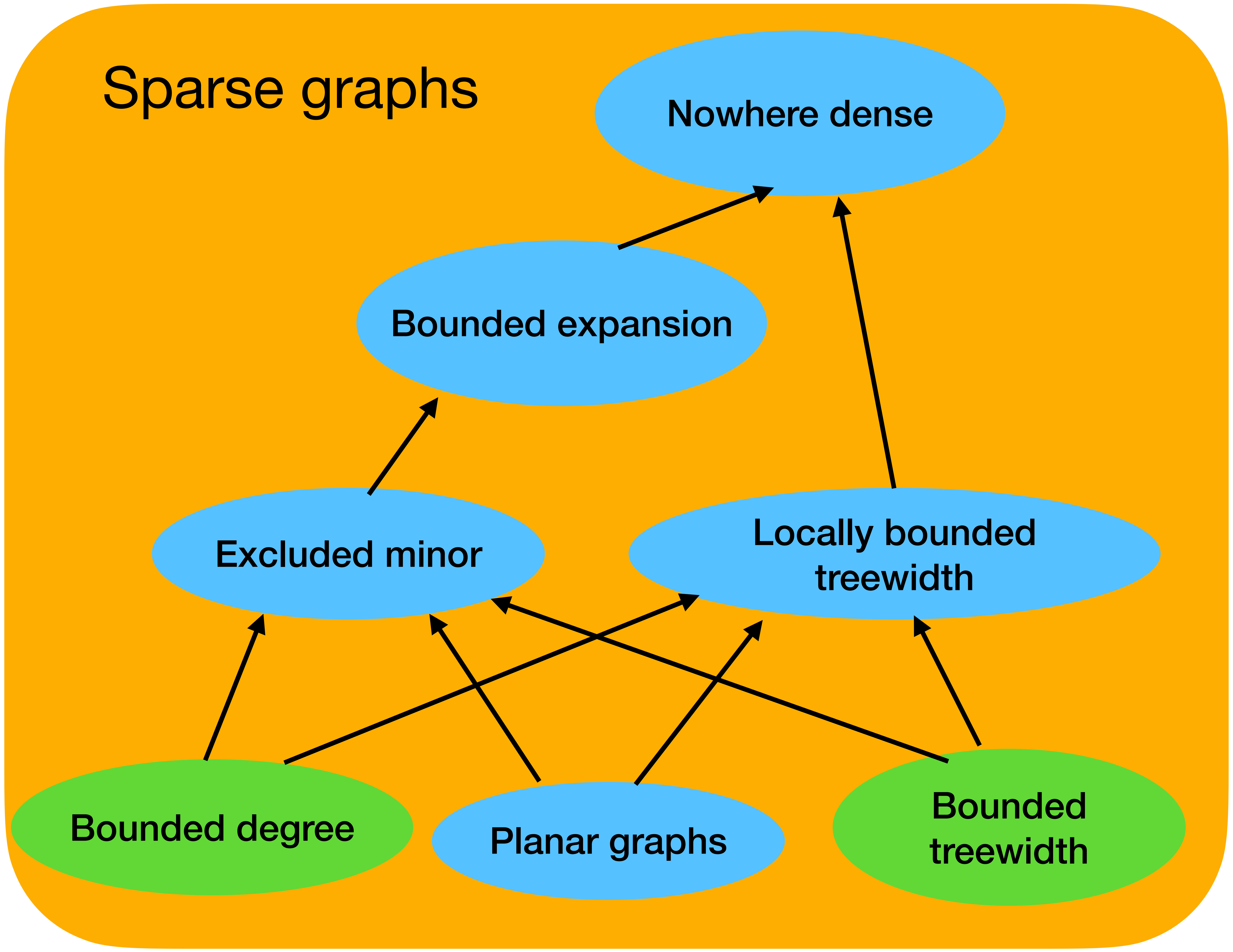


Courcelle, 1990



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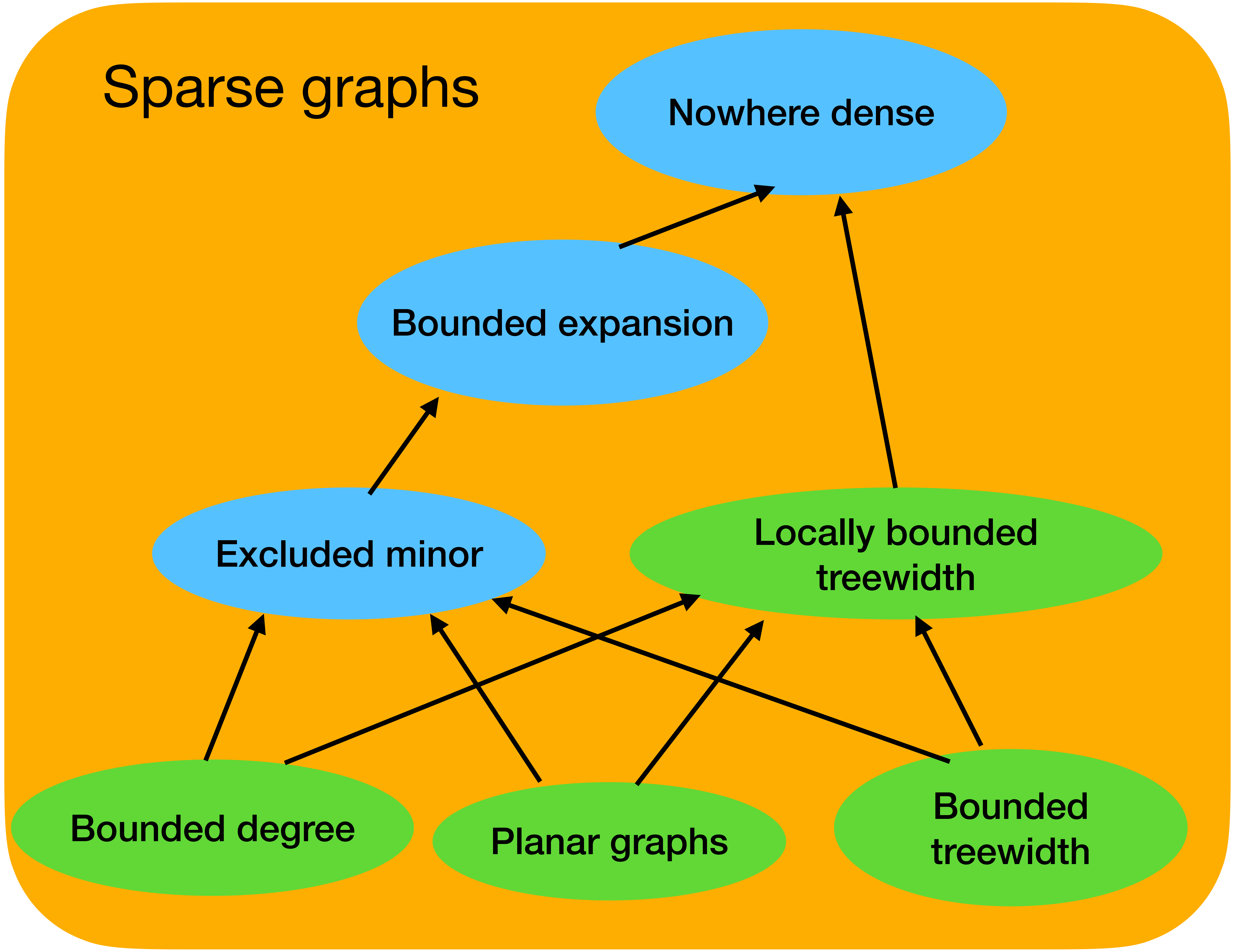

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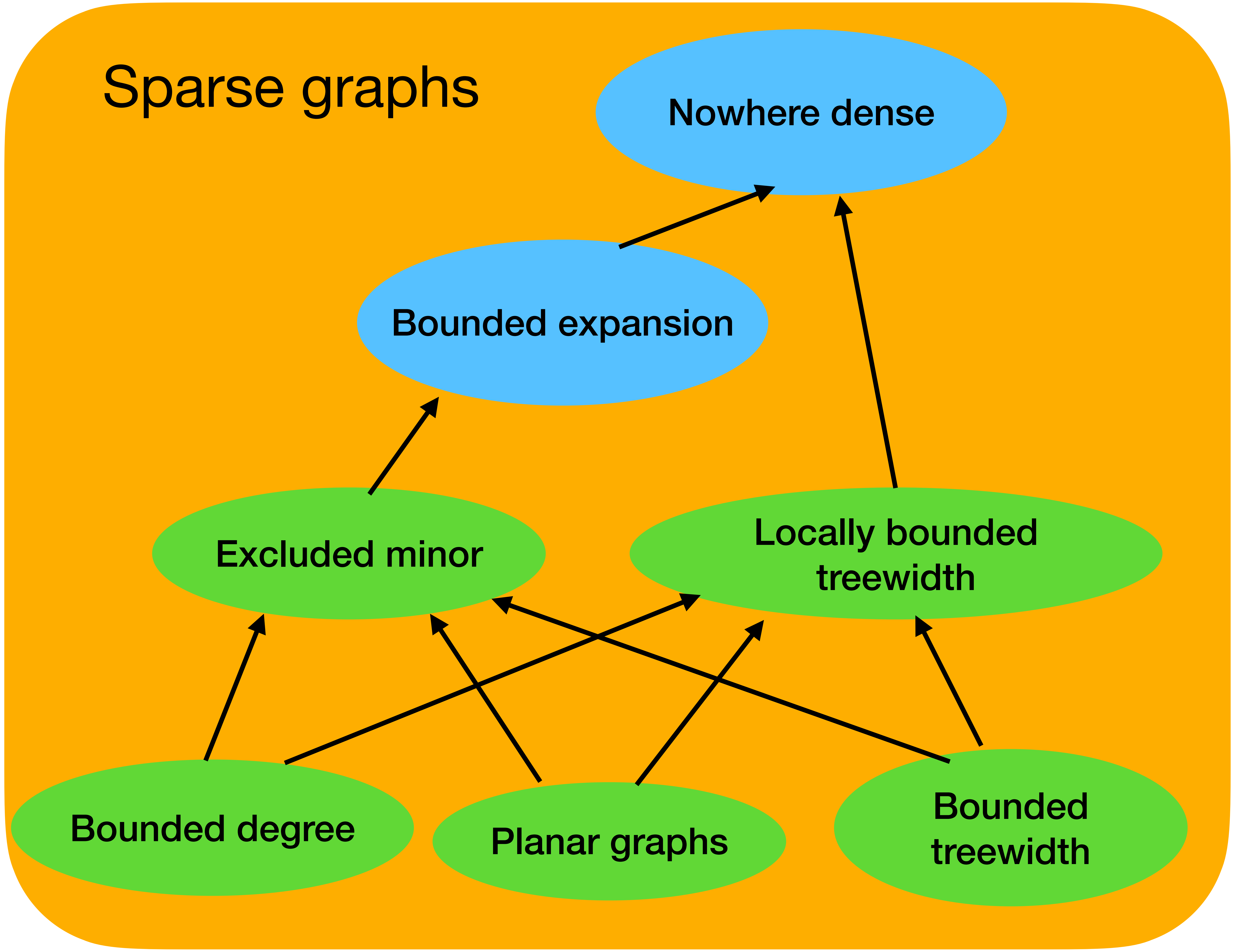

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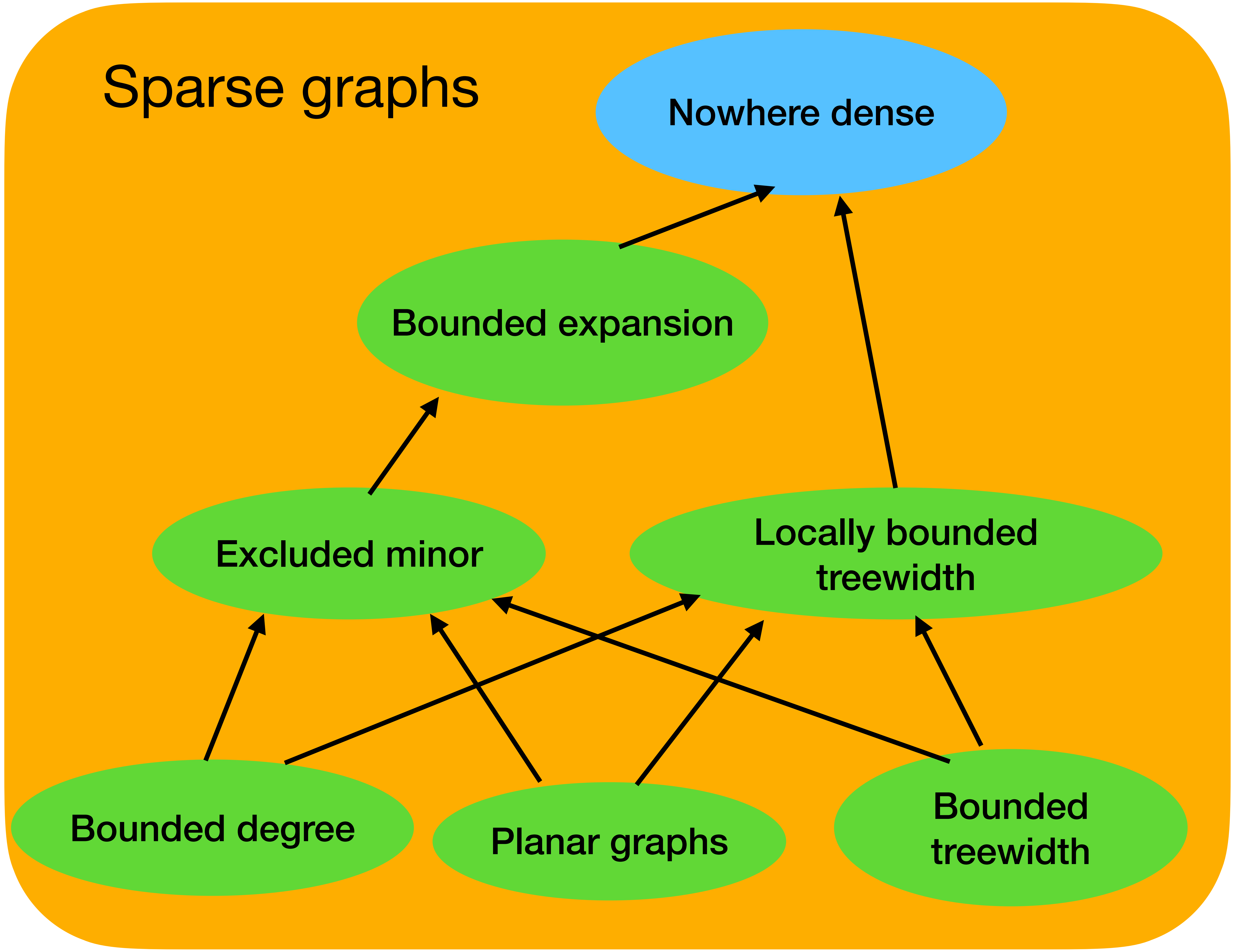

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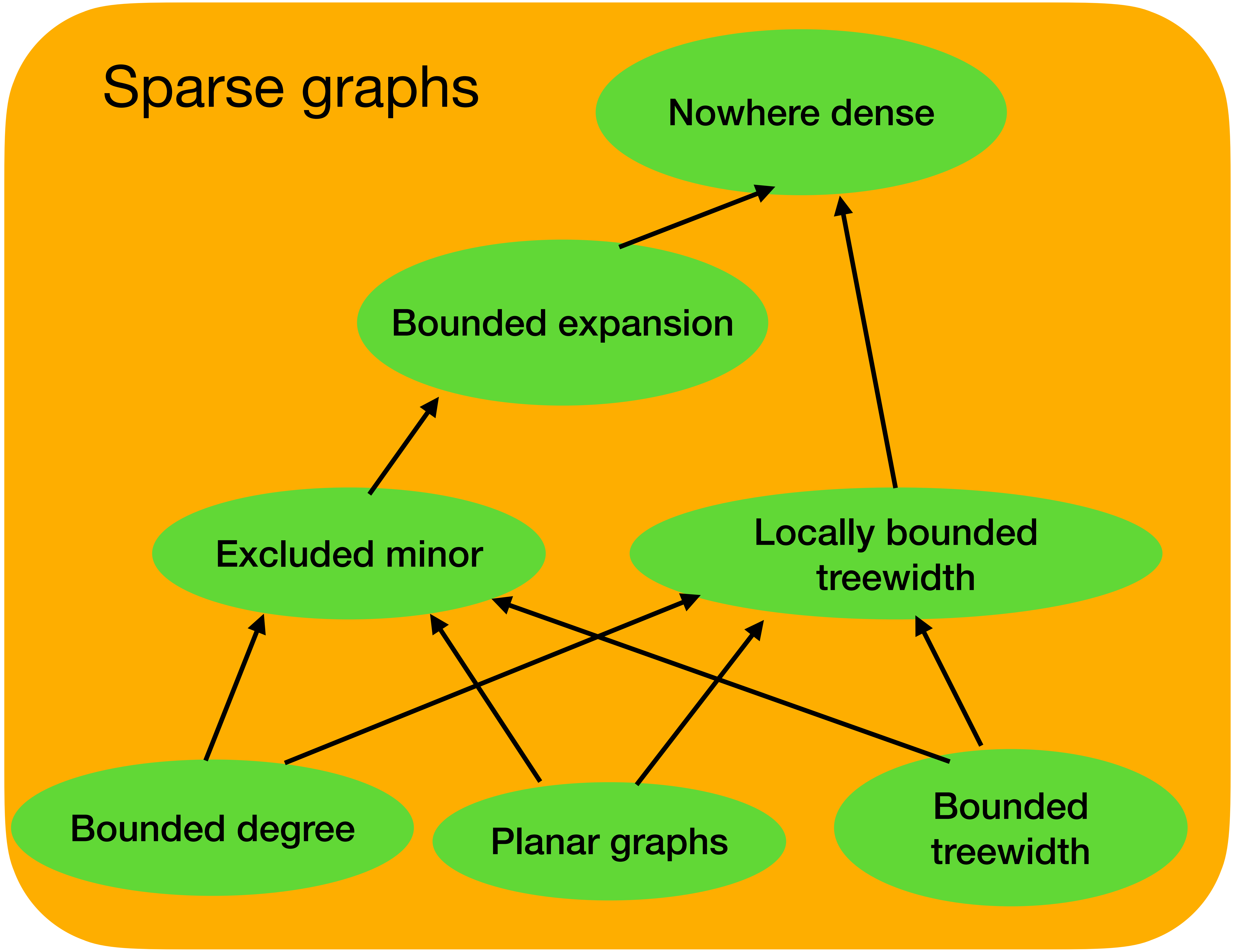

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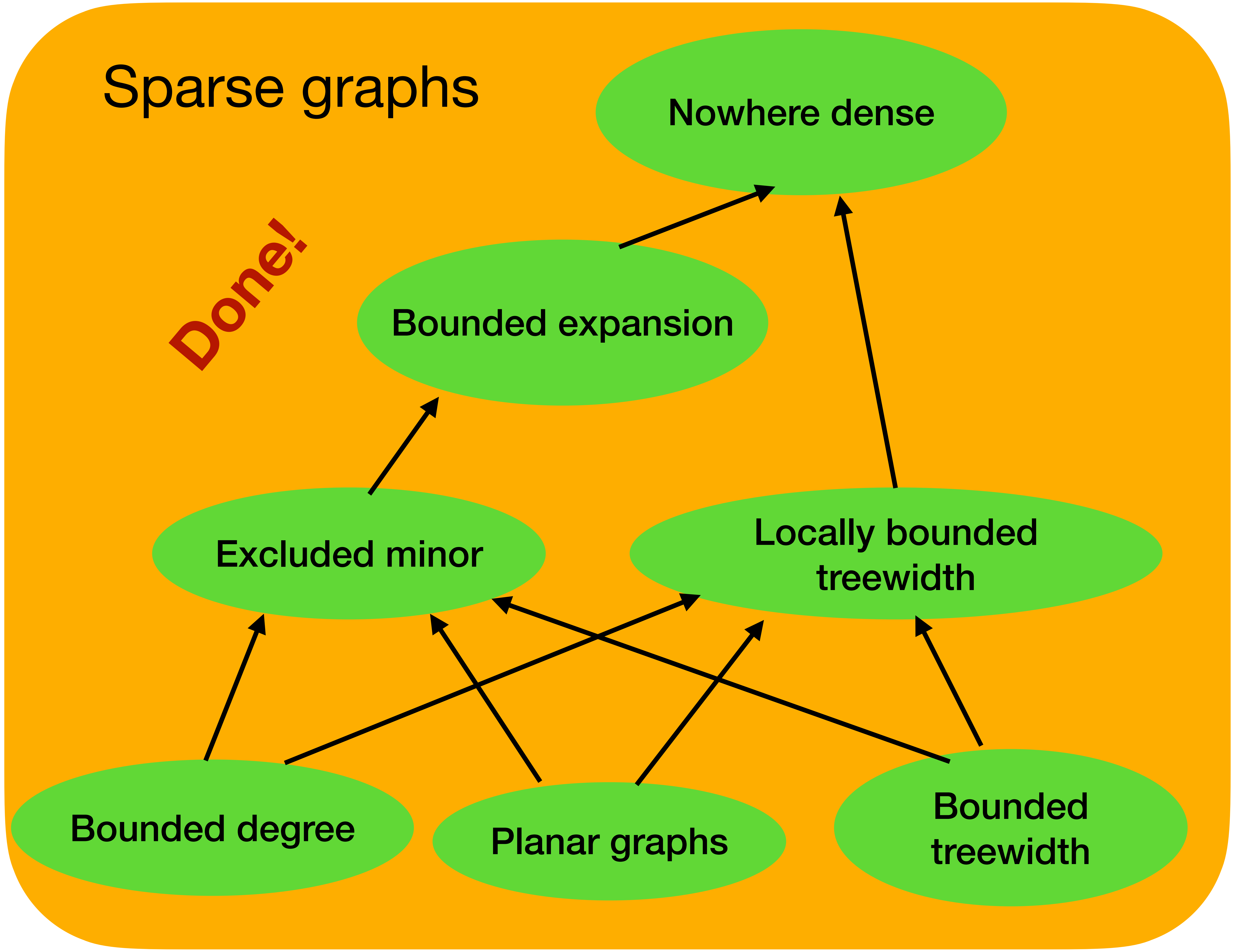

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Problem: No apparent unifying structure, does not go well with logic.

What to do next?

Creating a notion of ‘simple’ graphs which includes dense graphs.

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1. Create a theory ad hoc
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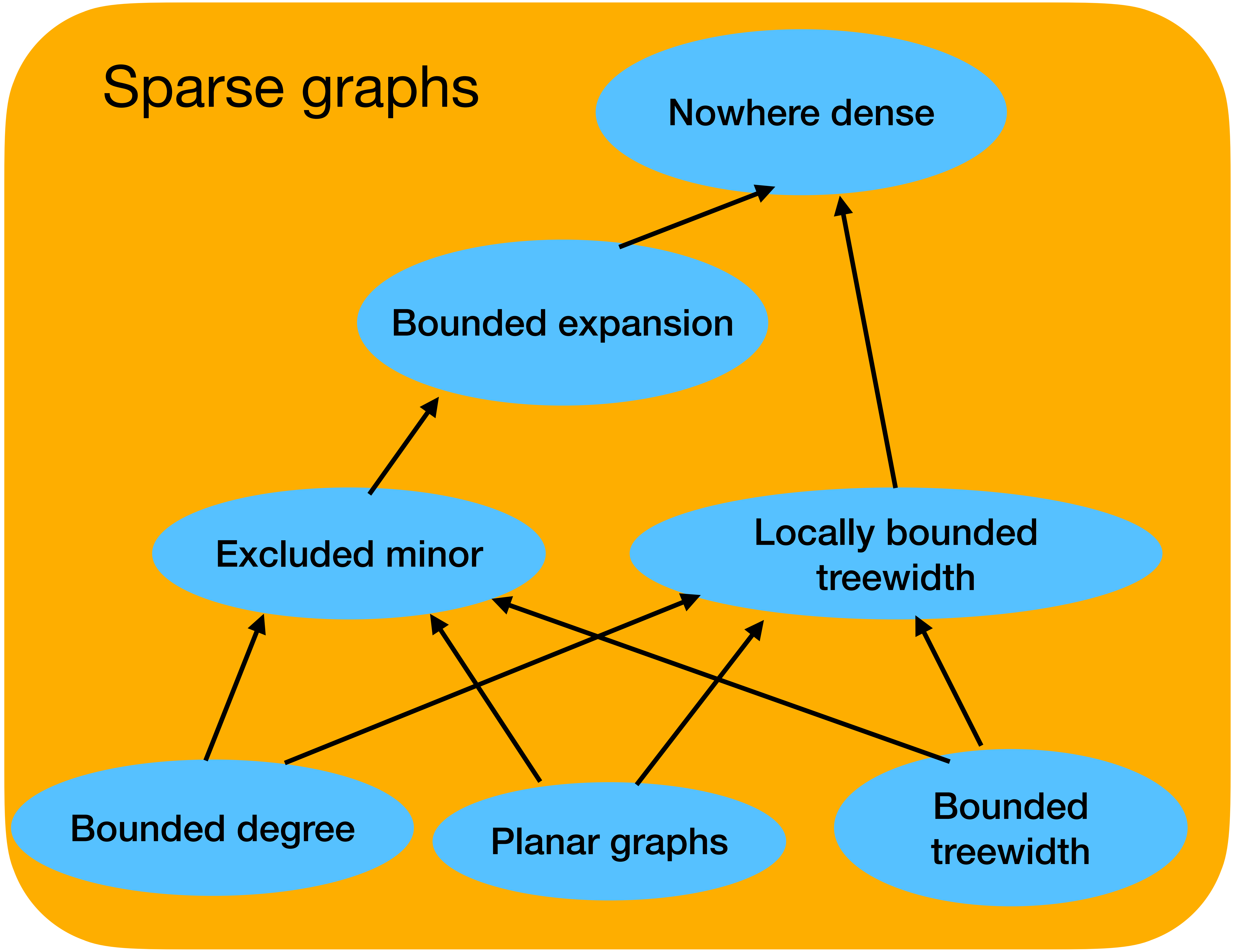
Simple example - complements of planar graphs.



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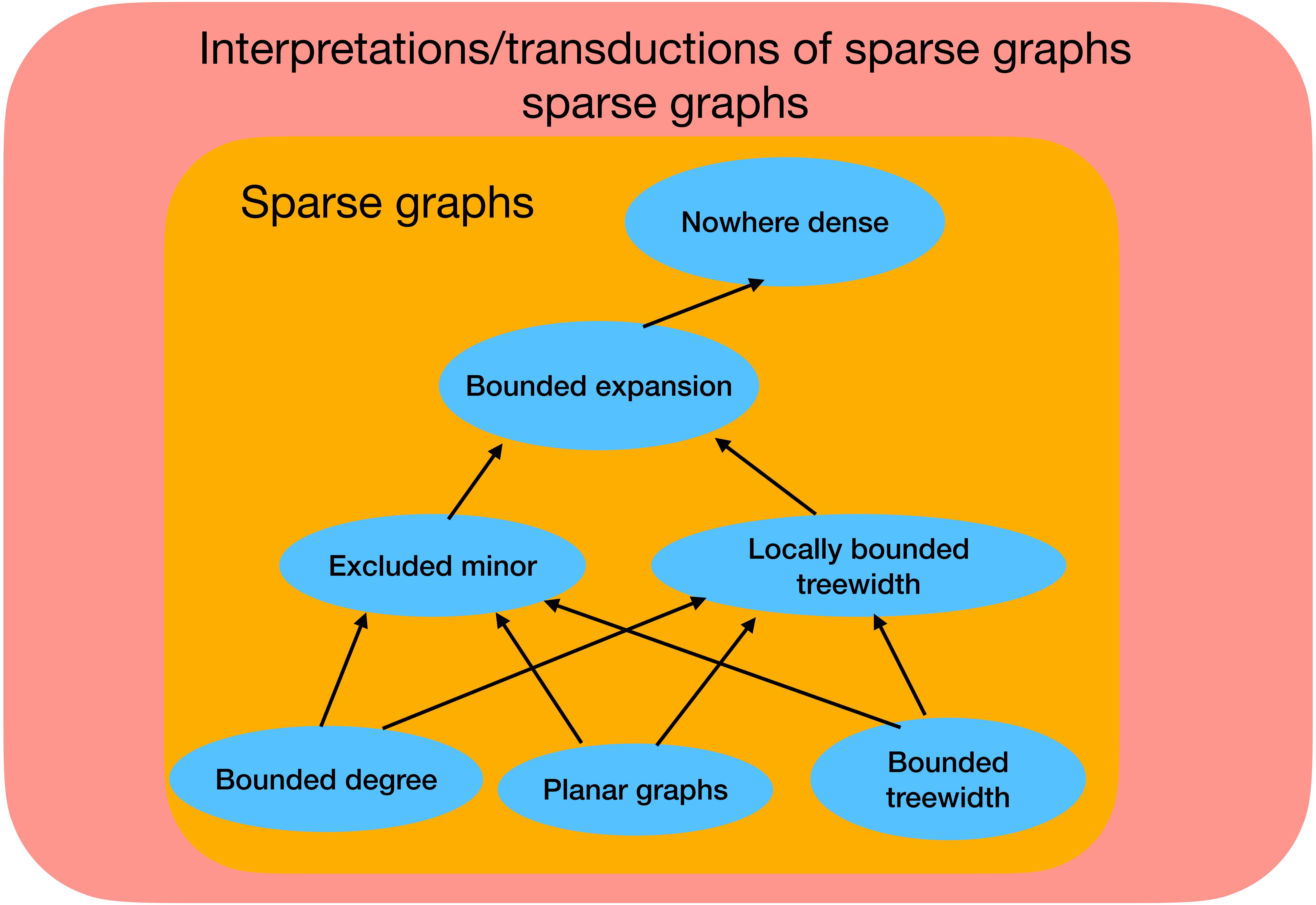
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Interpretations (graph transformations)

- Start with a graph H
- Put an edge between vertices u,v if dist(u,v) > 5 and at least one of u,v has degree at least 4
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One can write the condition “dist(u,v) > 5 and at least one of u,v has degree at least 4” in FO logic:

$$\psi(x, y) := dist(u, v) < 5 \wedge (deg(u) \geq 4 \vee deg(v) \leq 4)$$

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In this case the transformation given by $\psi(x, y)$ is called an interpretation

Interpretations (graph transformations)

Examples — put an edge between u and v if:

- There is no edge between u and v (complementation).
- They are at distance at most 2 (squaring).
- There are exactly two paths of length 17 between u and v and on one of them there is a vertex which is in a triangle.

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Notation: $G = \psi(H)$

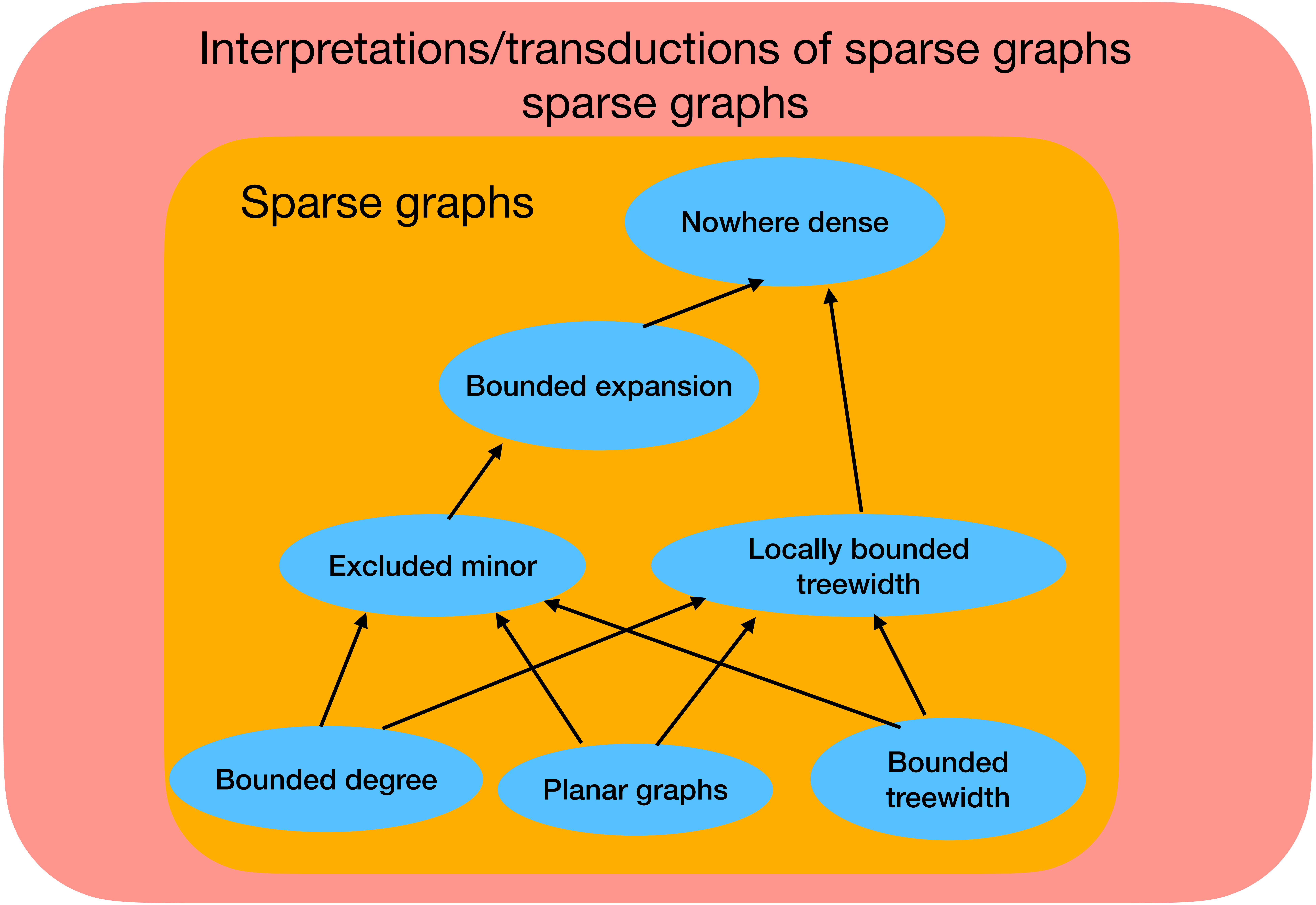
Extends to graph classes: $\mathscr{D} = \psi(\mathscr{C}) = \{\psi(H) \mid H \in \mathscr{C}\}$
(component-wise)

Research program

Let $\mathscr{D}$ be graph class interpretable in a sparse graph class $\mathscr{C}$.

Question: Is FO model checking in FPT on $\mathscr{D}$?

Conjecture: The answer is always YES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Known:

- Interpretations of graphs of bounded treewidth
- Interpretations of graphs of bounded degree (LICS 2016)
- Interpretations of planar graphs, and more generally graphs of locally bounded treewidth and even more generally graph classes of locally bounded clique-width (LICS 2022)

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Why should/could something like this hold?

Why could the conjecture hold?

Setting: Take any sparse graph class $\mathscr{C}$ and any interpretation formula $\psi(x, y)$. Consider $\mathscr{D} = \psi(\mathscr{C})$.

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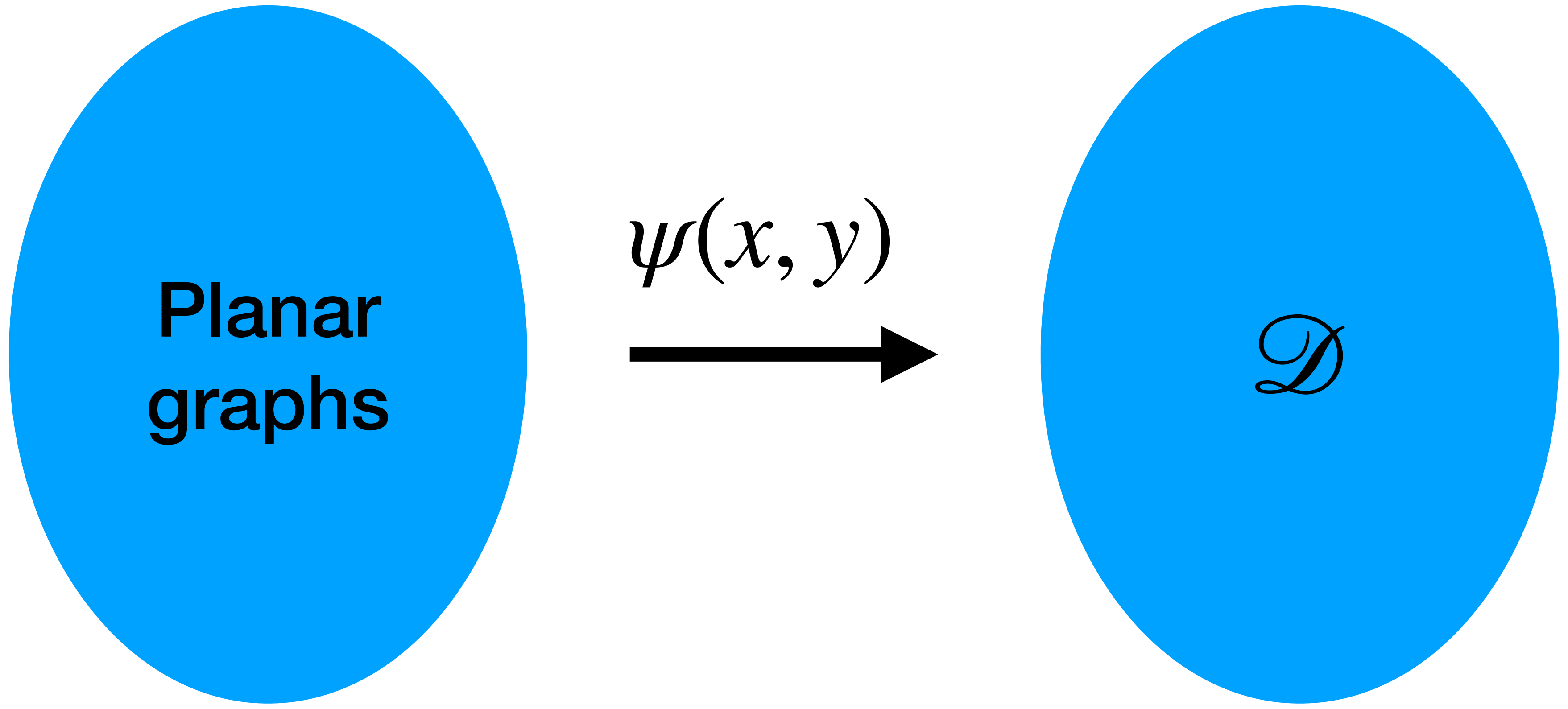

For concreteness:
$\mathscr{C}$ = class of all planar graphs
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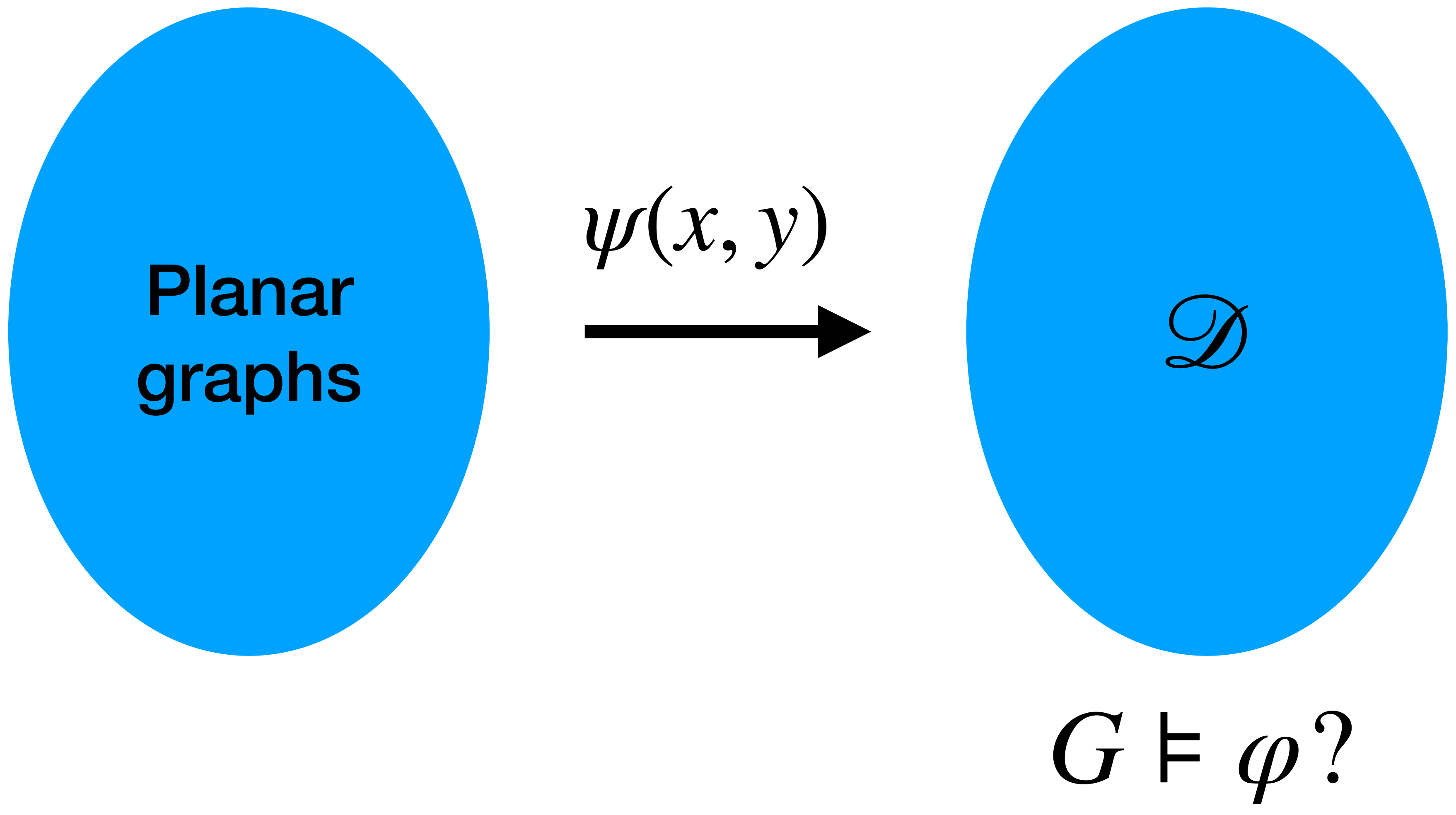

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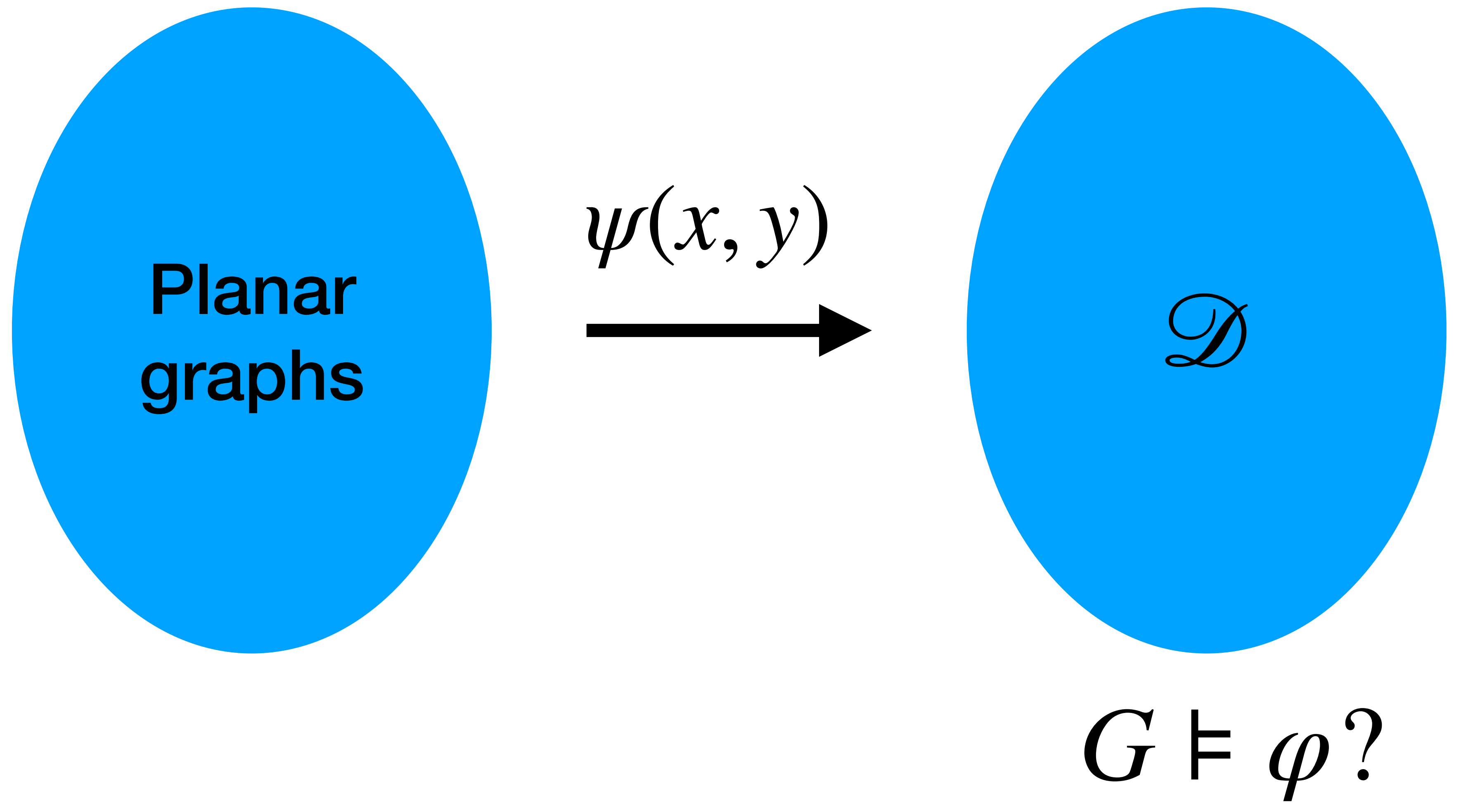

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For concreteness, let φ express "G has a dominating set of size 3."

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G — any graph from $\mathscr{D}$

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Finding H efficiently — the key algorithmic problem

The interpretation reversal problem

Setting: Take any sparse graph class $\mathscr{C}$ and any interpretation formula $\psi(x, y)$. Consider $\mathscr{D} = \psi(\mathscr{C})$.

Task: Find a polynomial algorithm which, given $G \in \mathscr{D}$ computes $H \in \mathscr{C}$ such that $G = \psi(H)$

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We do not expect this to work.

Approximate interpretation reversal problem

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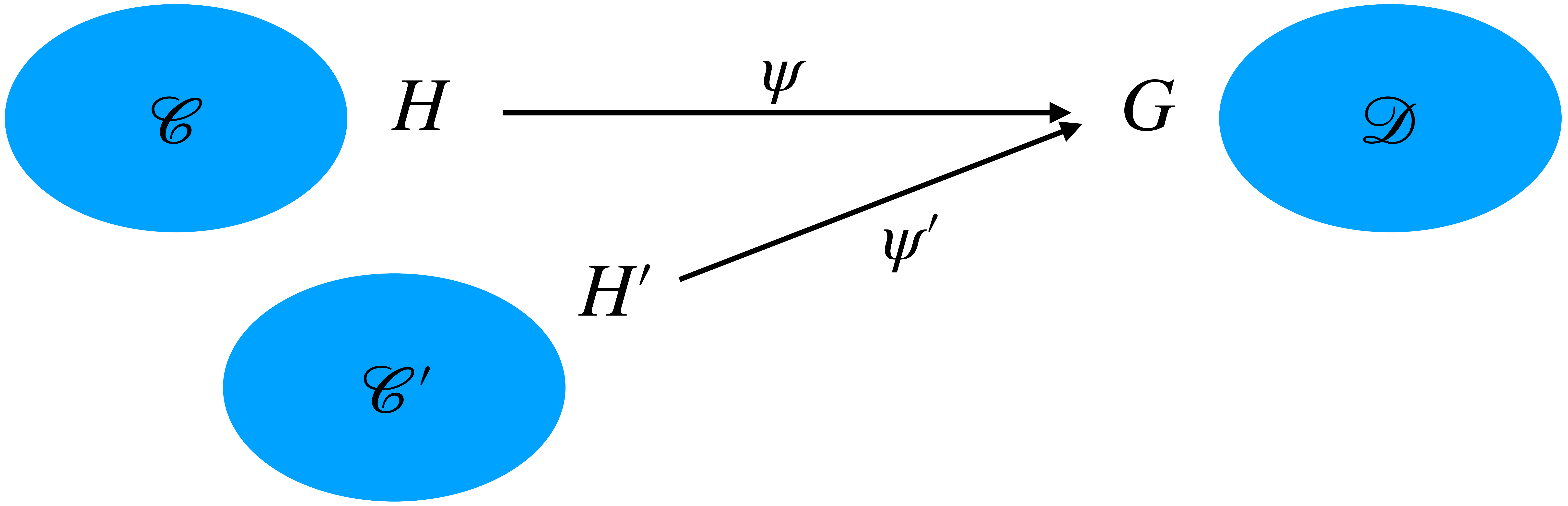
Task: Find a sparse graph class $\mathscr{C}'$, interpretation formula $\psi'(x, y)$ and a polynomial algorithm which given $G \in \mathscr{D}$ computes $H' \in \mathscr{C}$ such that $G = \psi'(H)$



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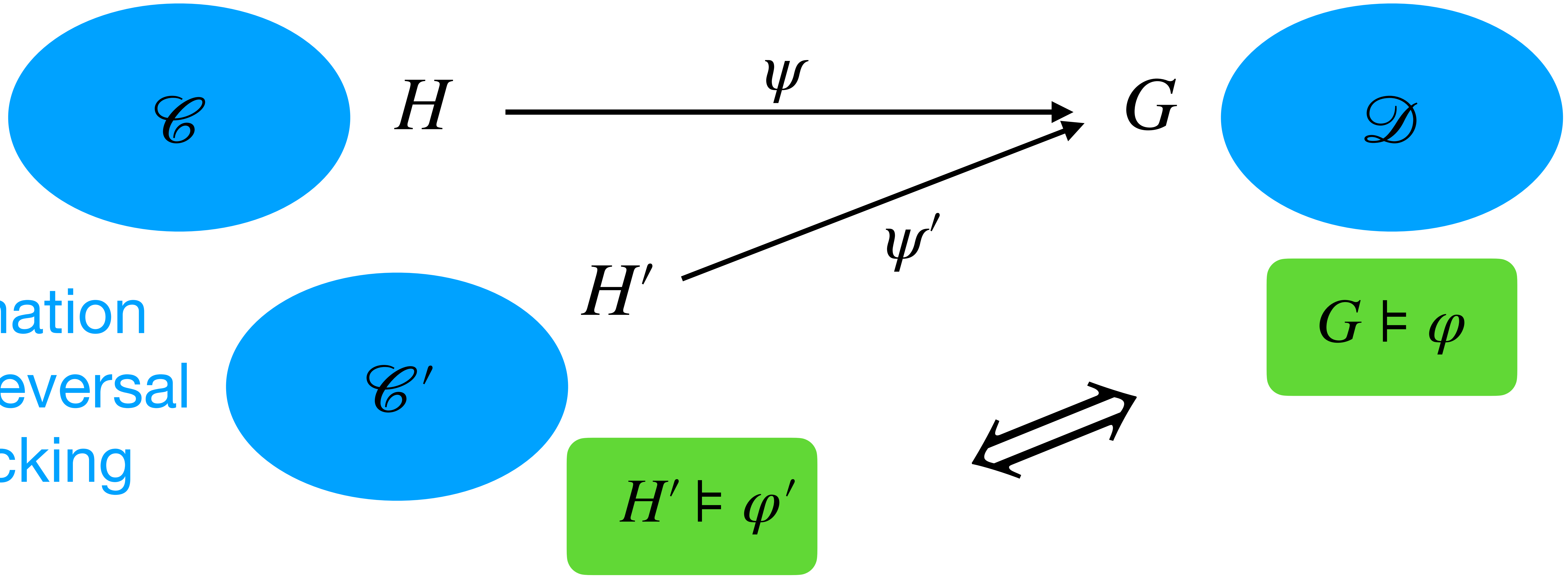

From approximation interpretation reversal the model checking follows.

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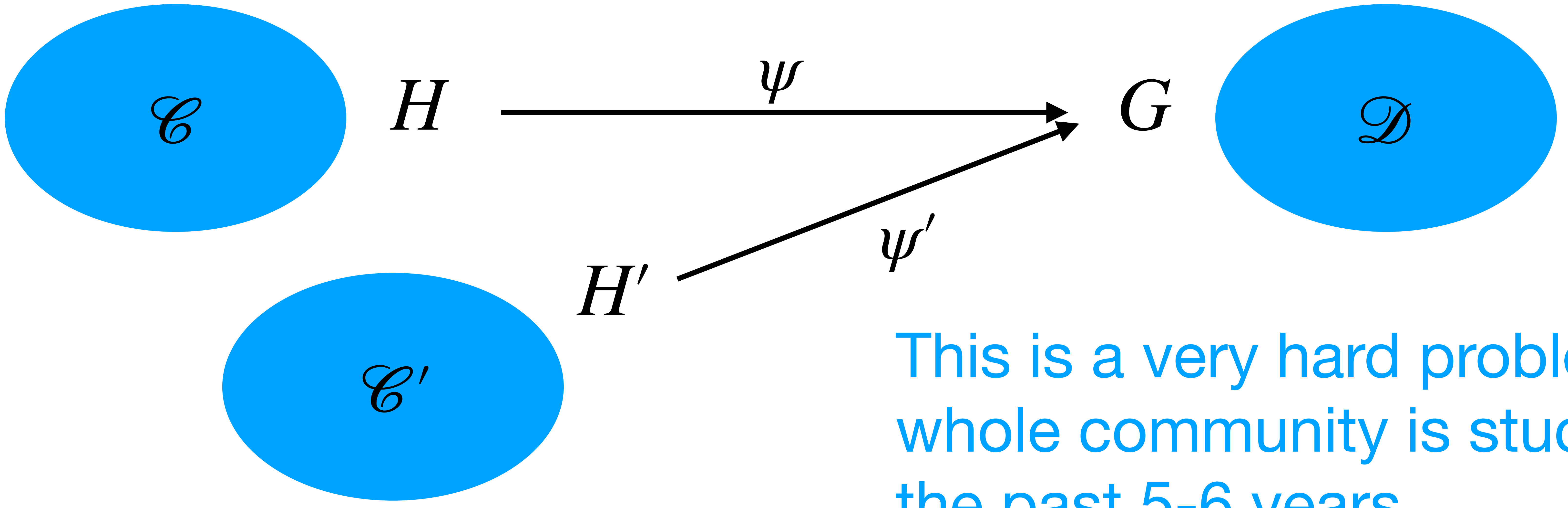

This is a very hard problem, and the whole community is stuck on that for the past 5-6 years.

What we know?

We know how to do this:

- When $\mathscr{C}$ is a class of bounded degree (G., Hliněný, Lokshtanov, Ramanujan, Obdržálek; LICS 2016)
  (Here $\mathscr{C}'$ has also bounded degree, but larger than $\mathscr{C}$)

- When $\mathscr{C}$ is a class of bounded pathwidth (Nešetřil, Ossona de Mendez, Rabinovich, Siebertz; SODA 2020)
  (Here $\mathscr{C}'$ has also bounded pathwidth, but larger than $\mathscr{C}$)

- When $\mathscr{C}$ is a class of bounded treewidth (Nešetřil, Ossona de Mendez, Mi. Pilipczuk, Rabinovich, Siebertz; SODA 2021)
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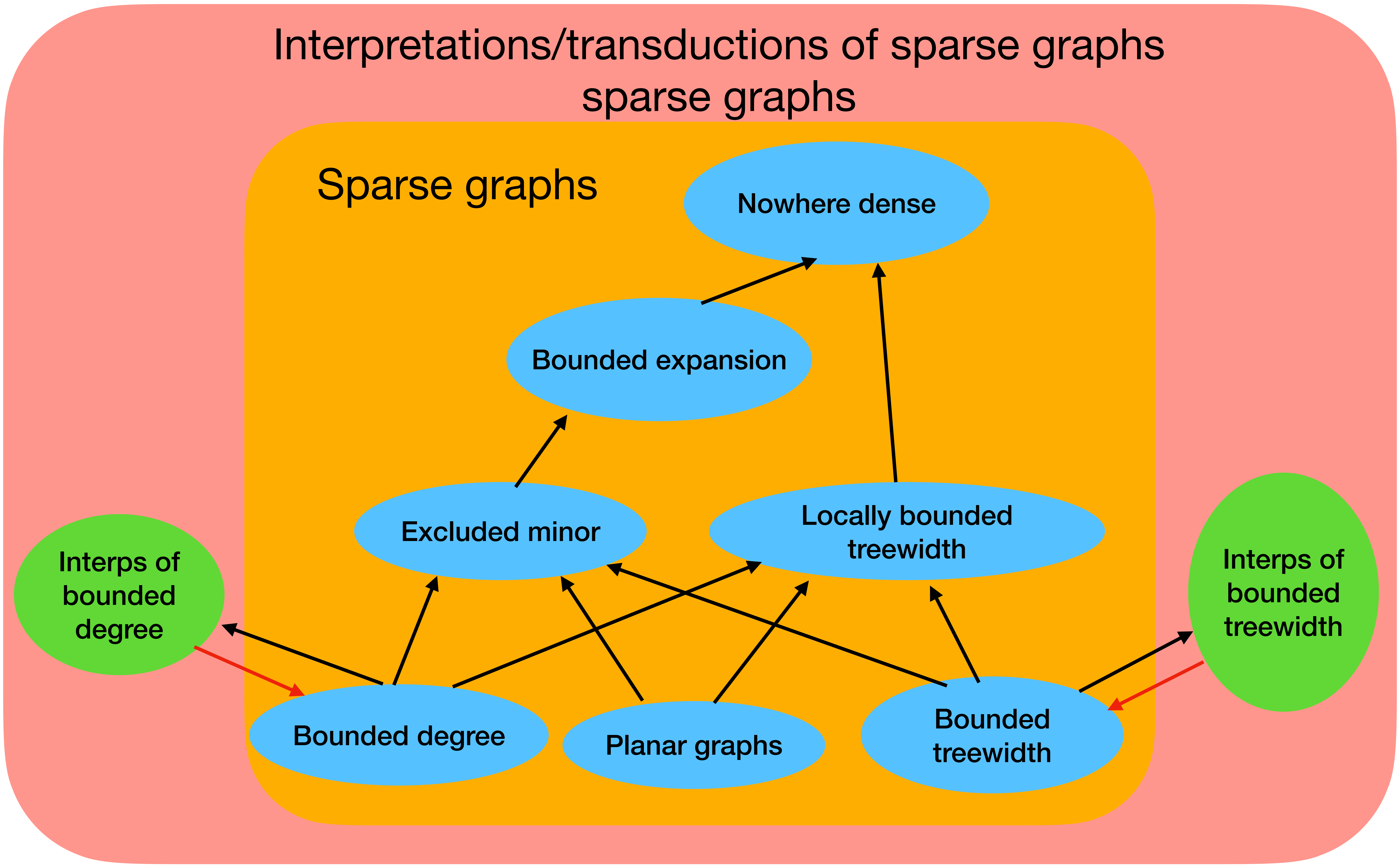
Everything else is open.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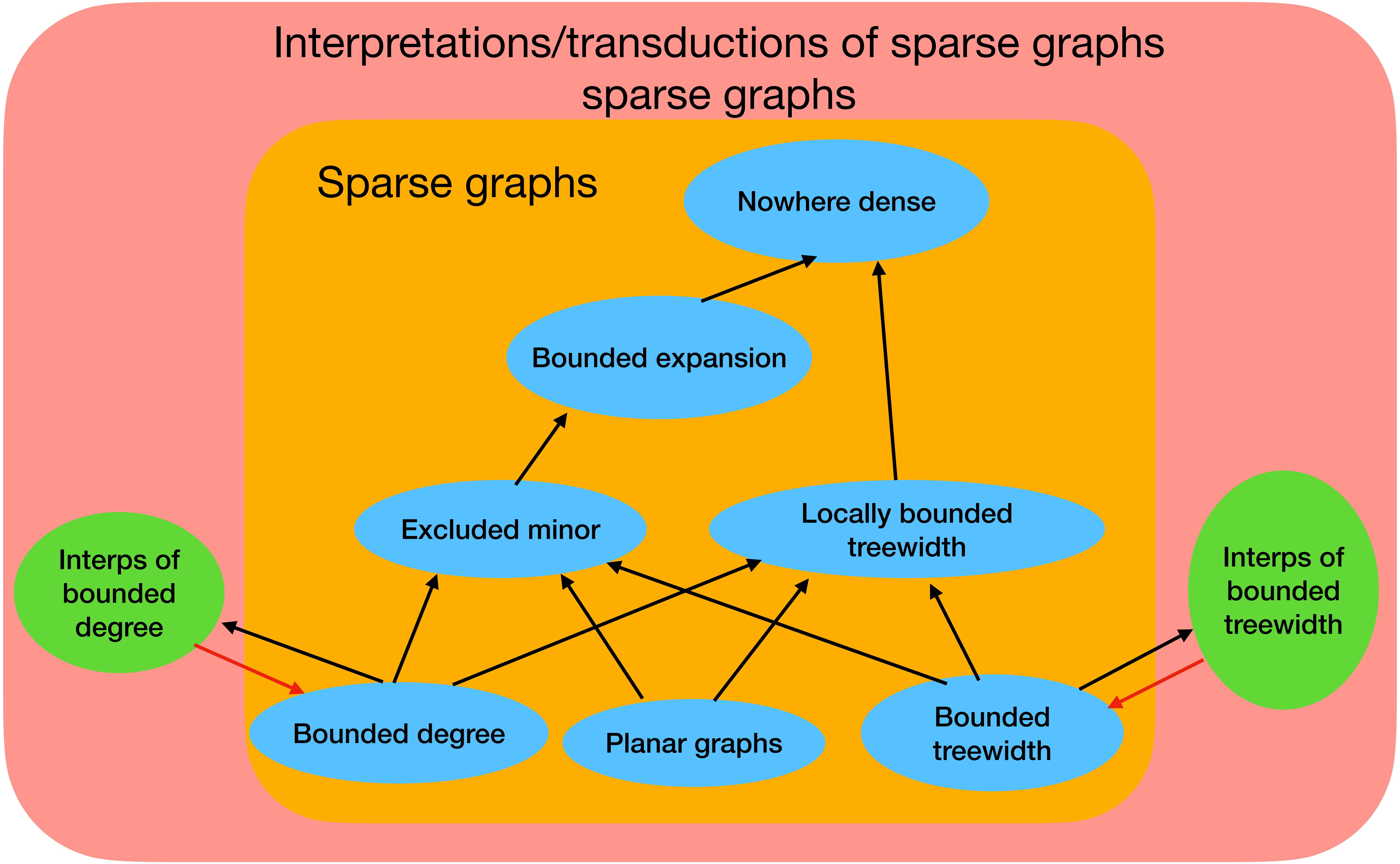
What did we do?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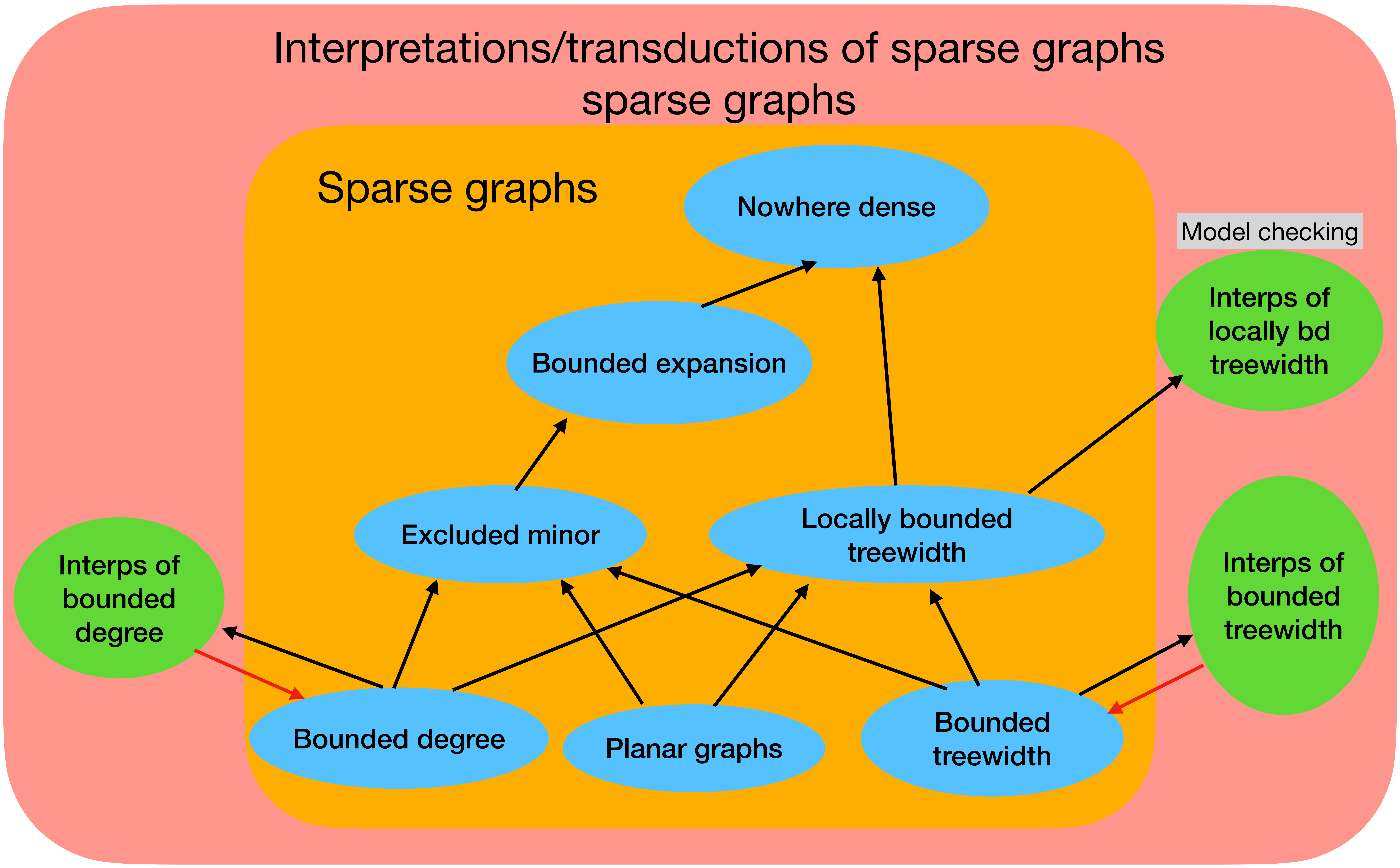
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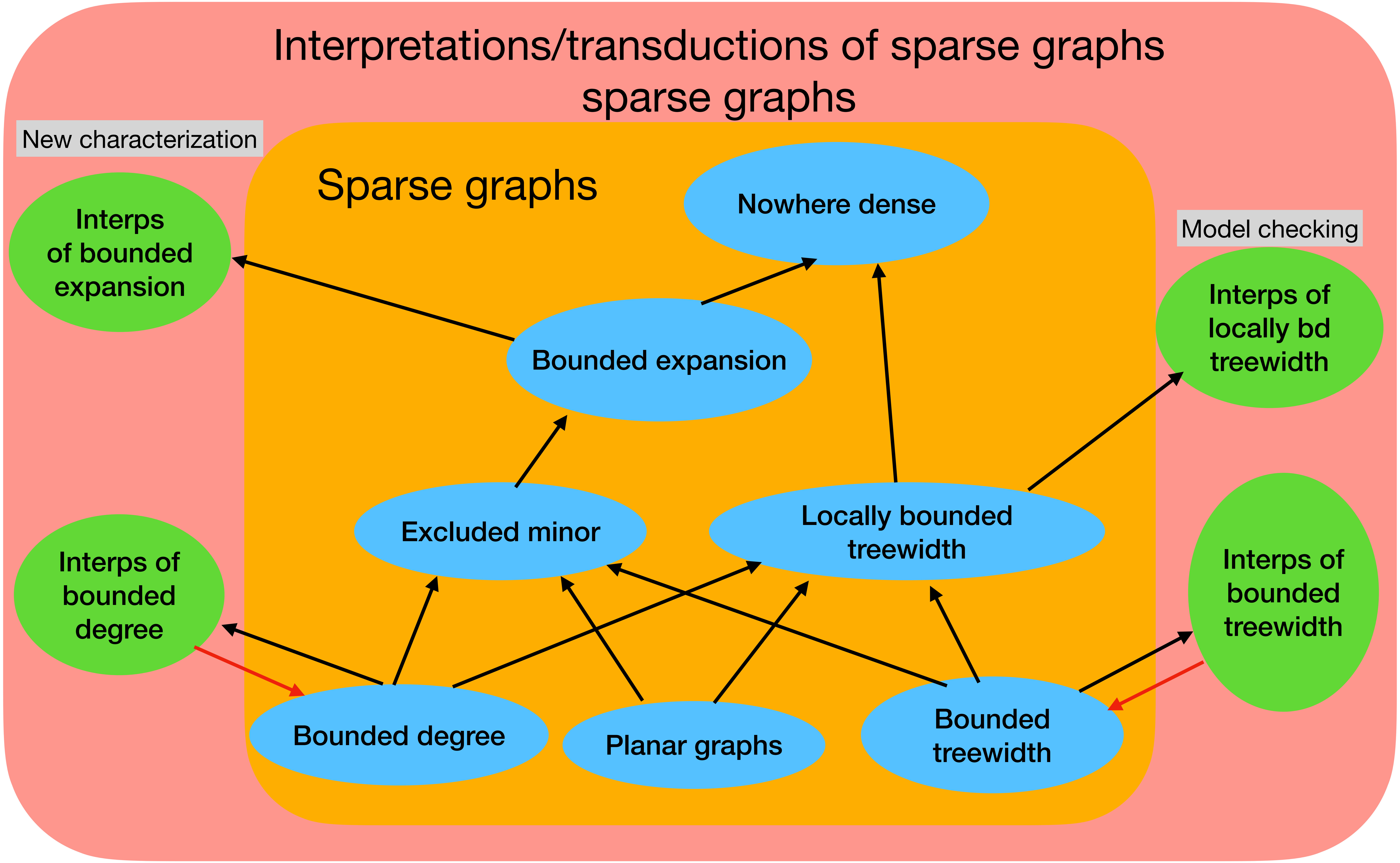
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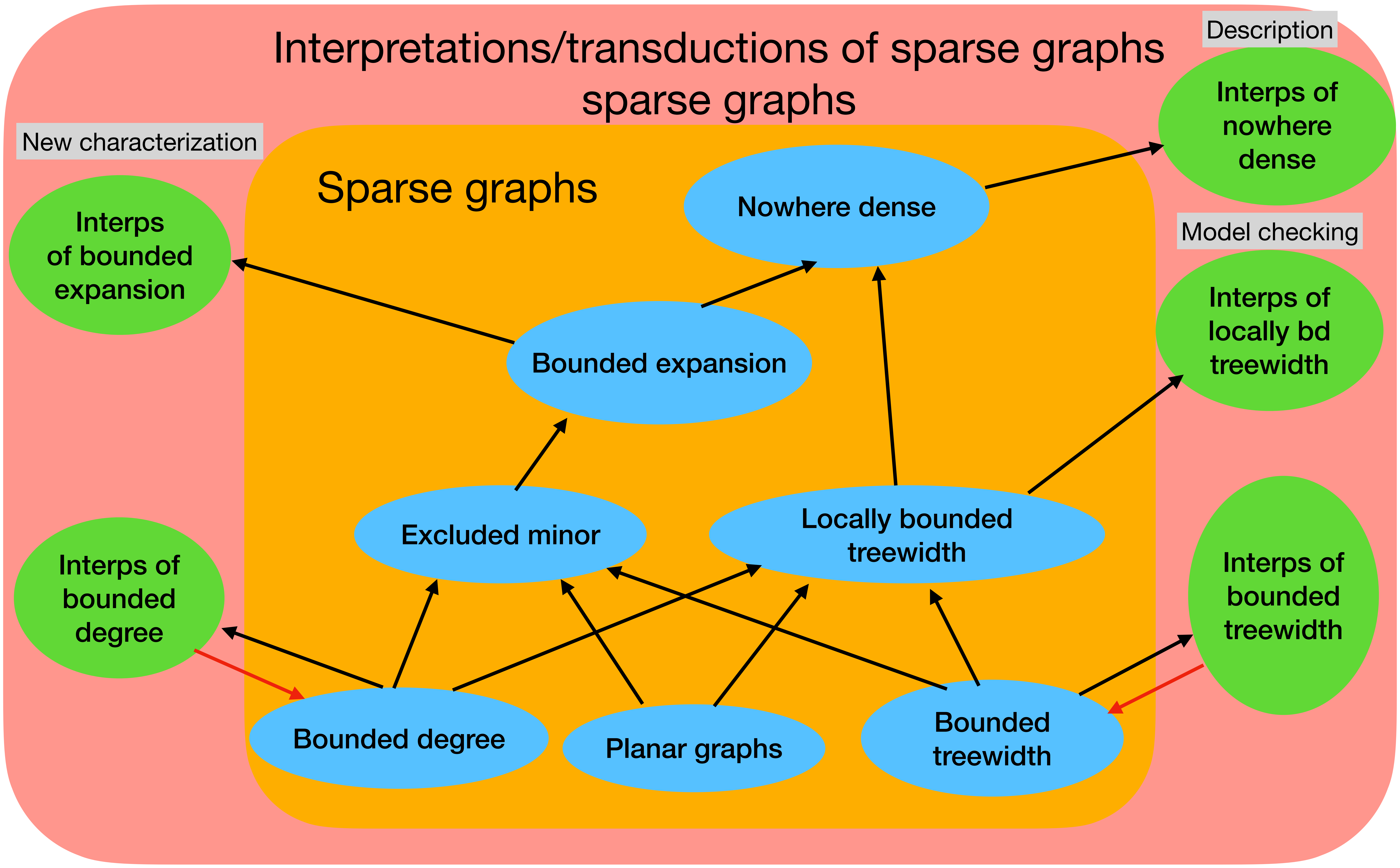
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Our results

Theorem (Bonnet, Dreier, G., Kreutzer, Mählmann, Simon, Toruńczyk; LICS 2022):
Let $\mathscr{D}$ be a graph class interpretable in planar graphs. Then the FO model checking problem is in FPT on $\mathscr{D}$.

Theorem (Dreier, G., Kiefer, Mi. Pilipczuk, Toruńczyk; LICS 2022):
A class $\mathscr{D}$ of graphs is interpretable in a class $\mathscr{C}$ of bounded expansion if and only if there exists a class $\mathscr{B}$ of *bushes* of bounded height and bounded expansion representing $\mathscr{D}$.

Theorem (Dreier, G., Kiefer, Mi. Pilipczuk, Toruńczyk; LICS 2022):
Let $\mathscr{D}$ be a graph class interpretable in a nowhere dense class of graphs $\mathscr{C}$. Then there exists a class $\mathscr{B}$ of quasi-bushes of bounded height which is almost nowhere dense and which represents $\mathscr{D}$.

Model checking on interpretations of planar graphs

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Not proven by approximate interpretation reversals, but somewhat similarly.

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Before we explain the algorithm, we need to explain the model checking algorithms for (some) classes of sparse graphs based on Gaifman's locality theorem.

Interlude:
Gaifman’s locality theorem and model checking on sparse graphs

Gaifman's locality theorem

For every FO sentence φ there exist r, m and formulas $\gamma_1(x), \ldots, \gamma_m(x)$ such that to evaluate φ on any graph G one needs to:

1. For every $v \in V(G)$ look at $G[N_r(v)]$
2. Evaluate all formulas $\gamma_1(x), \ldots, \gamma_m(x)$ on $G[N_r(v)]$ and store the results
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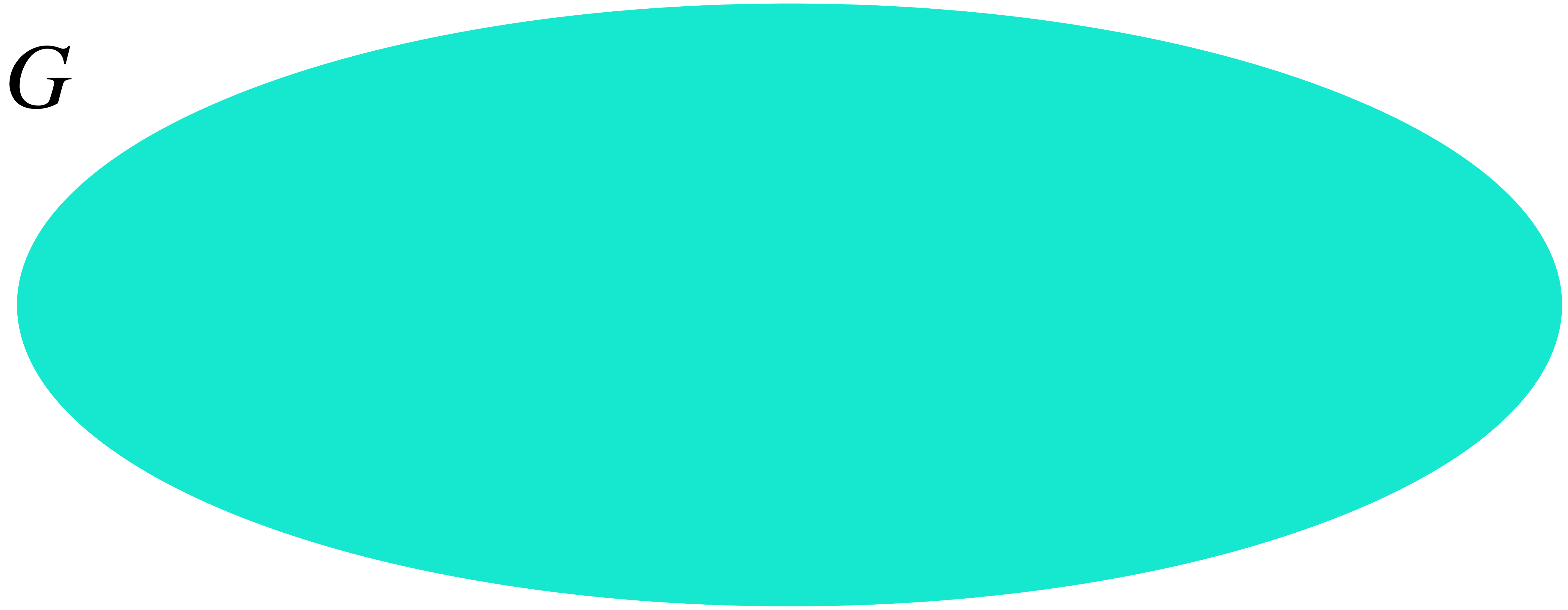

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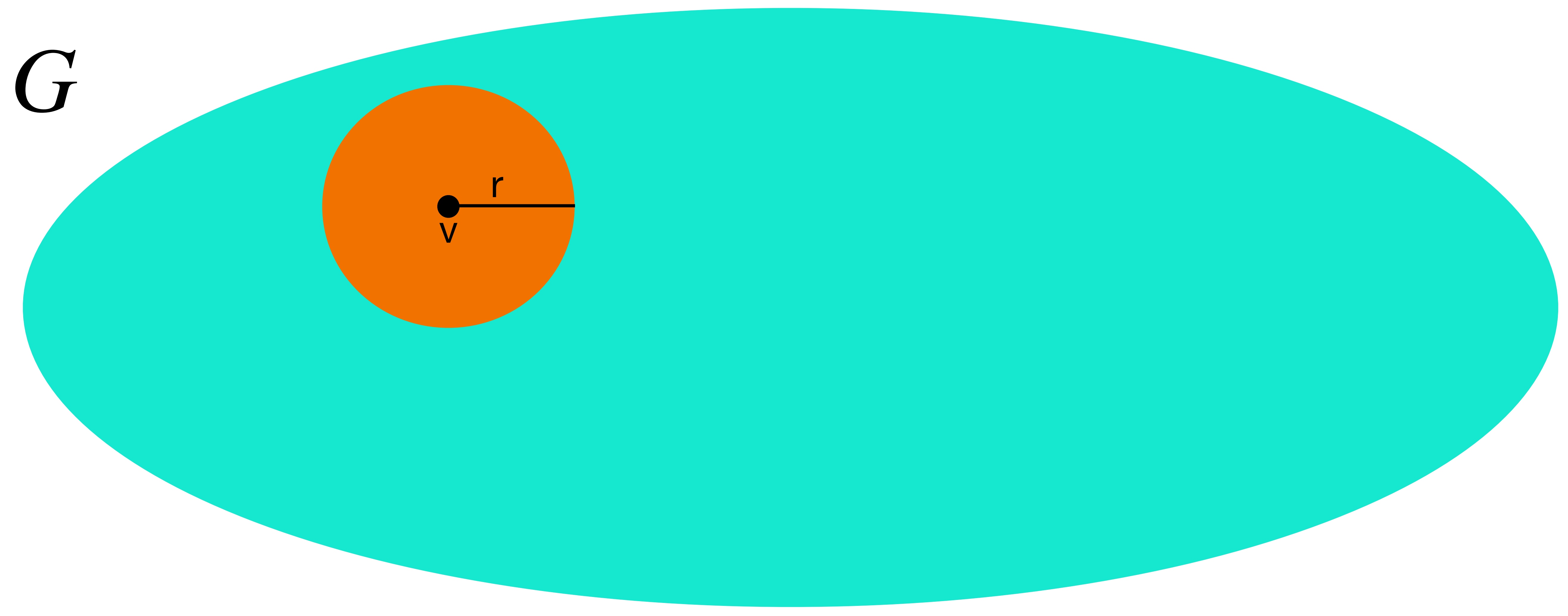

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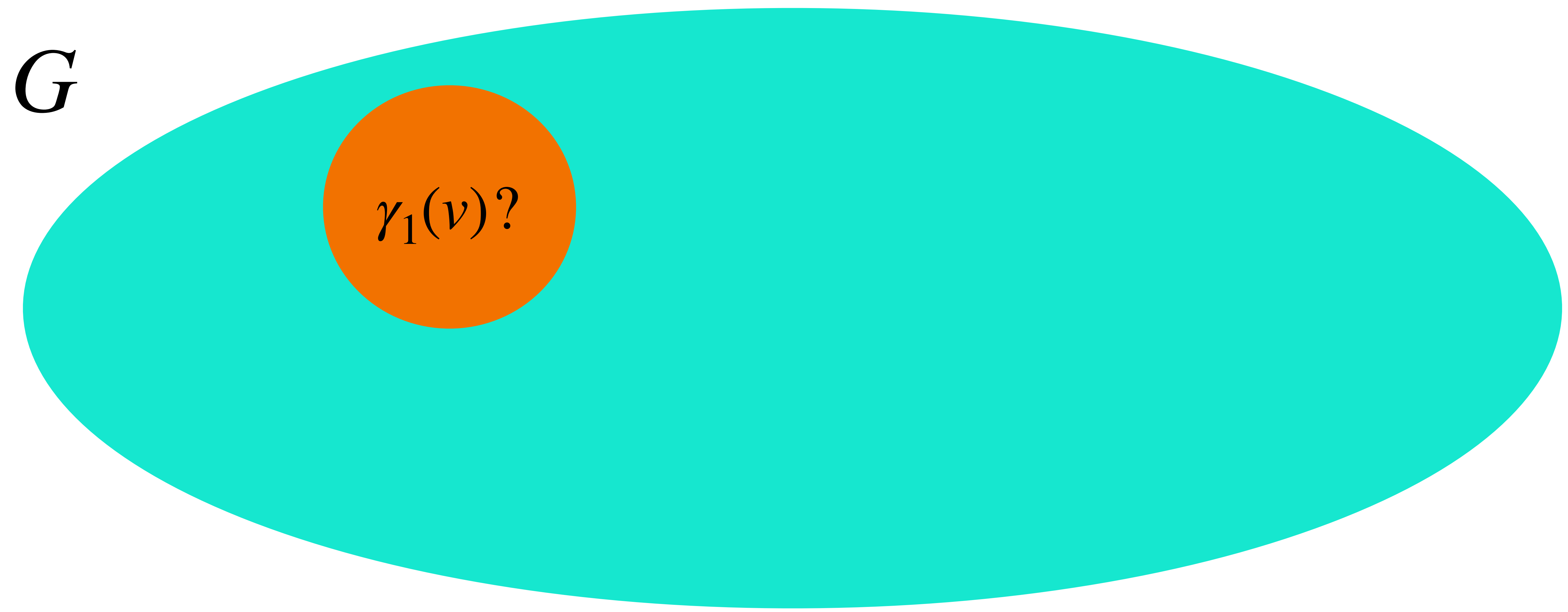

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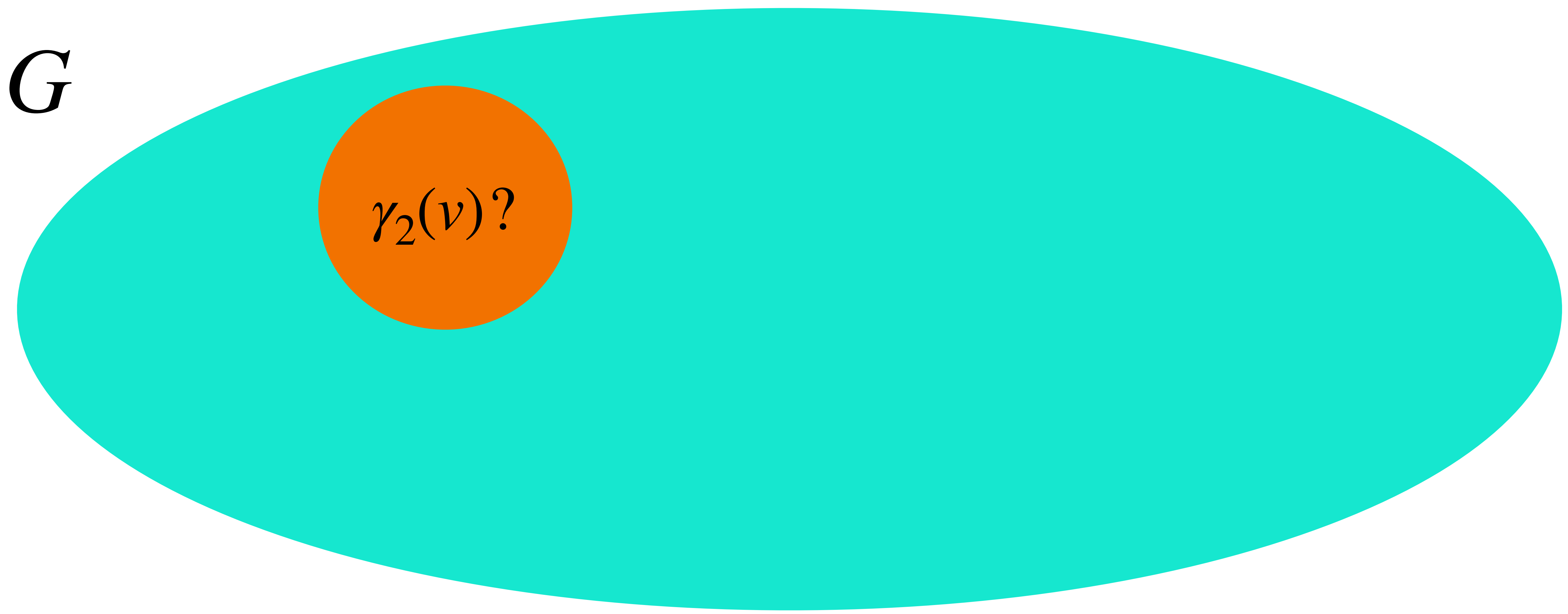

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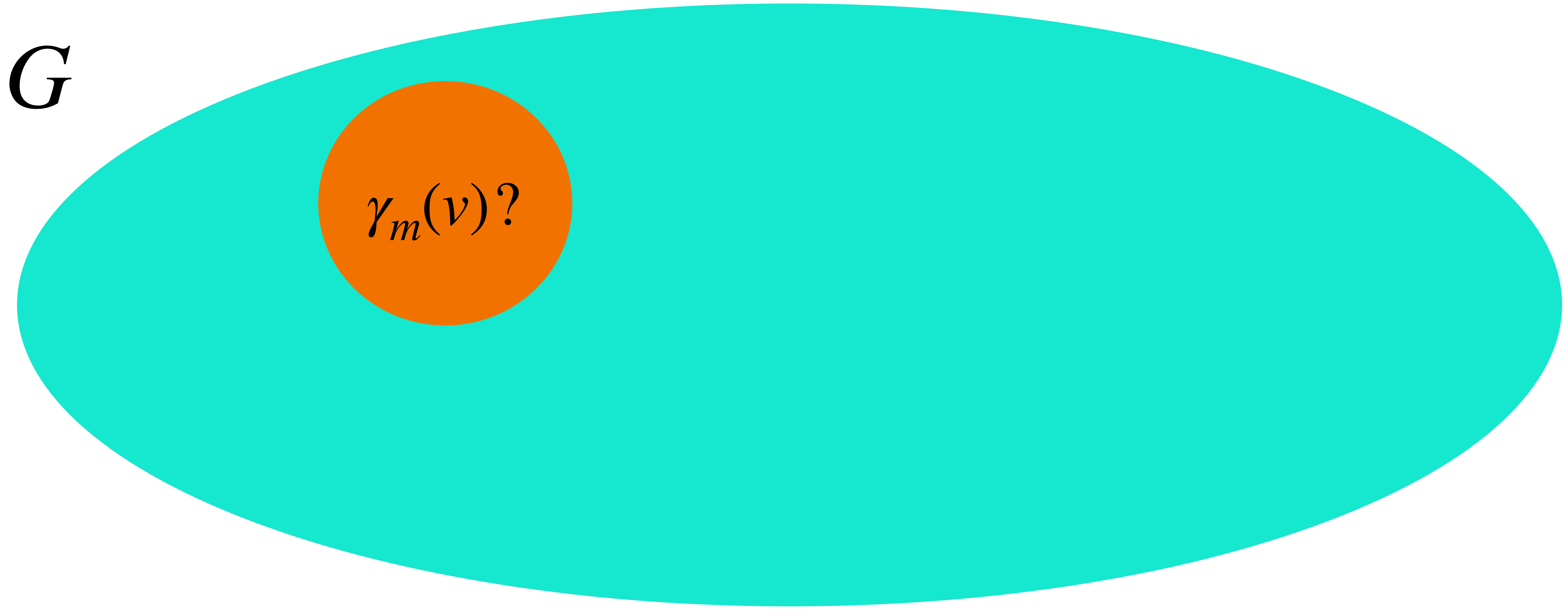

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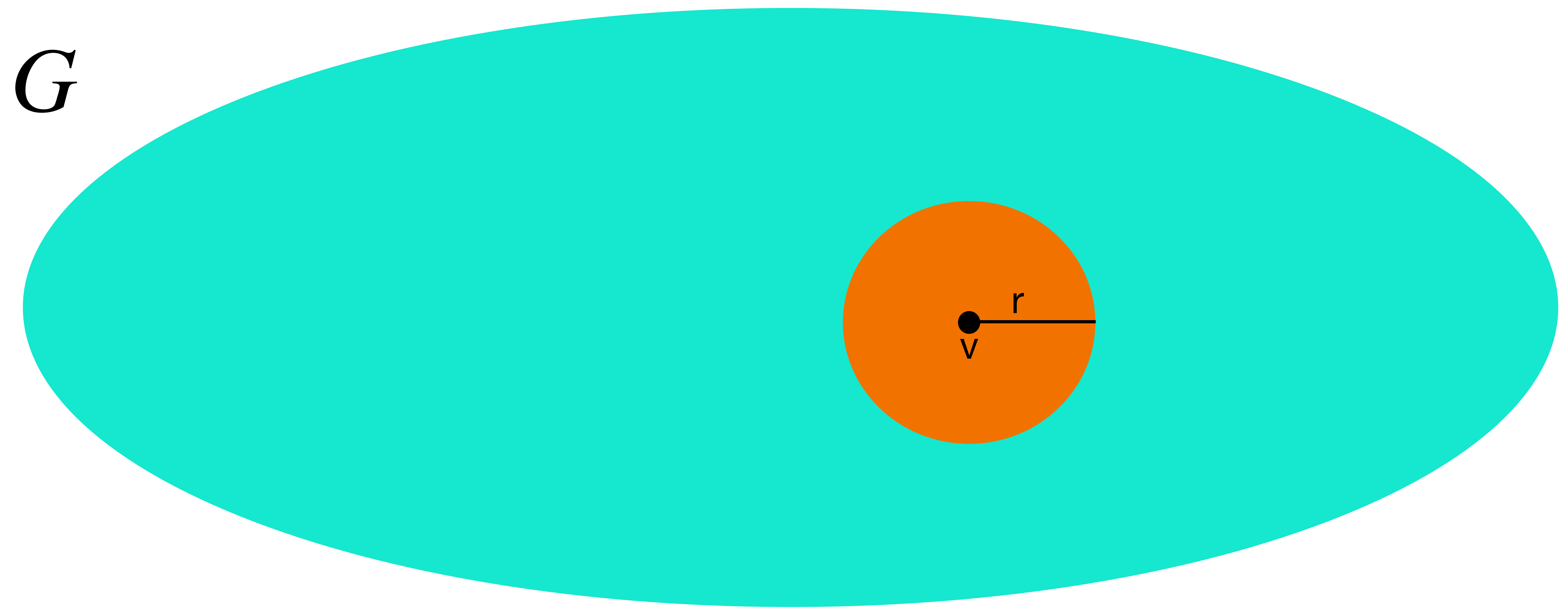

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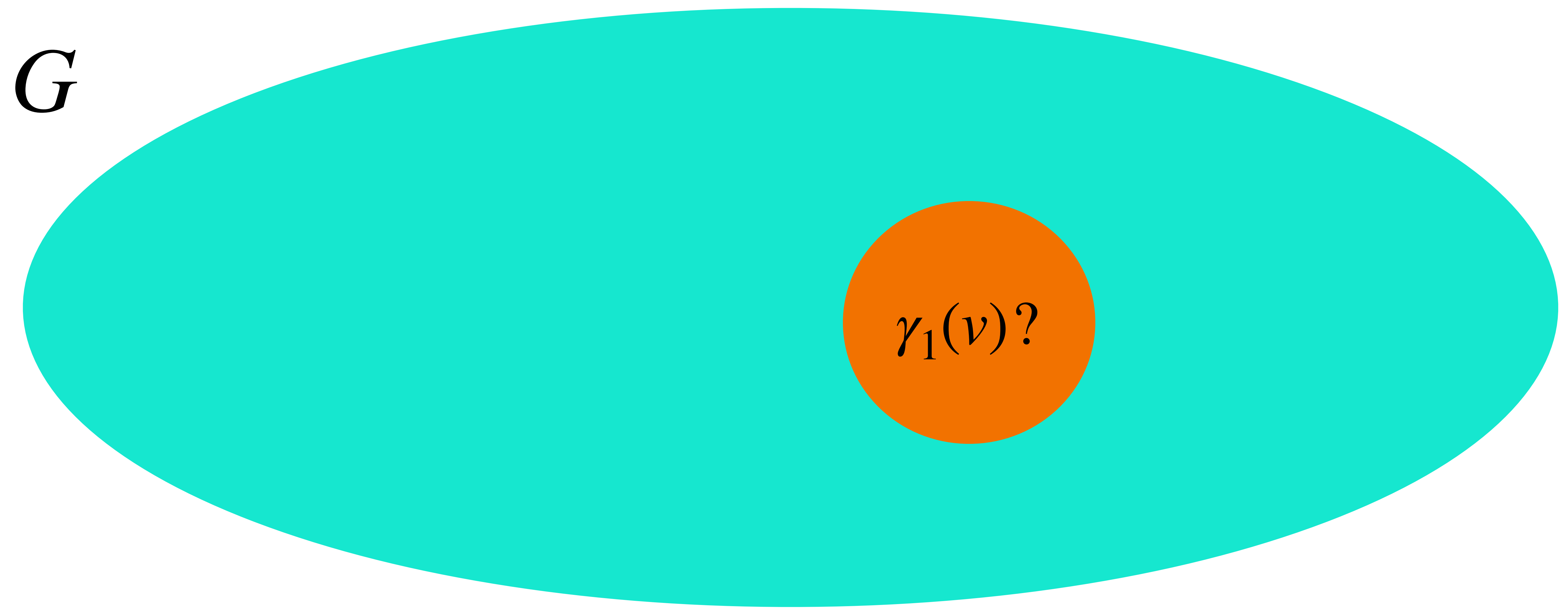

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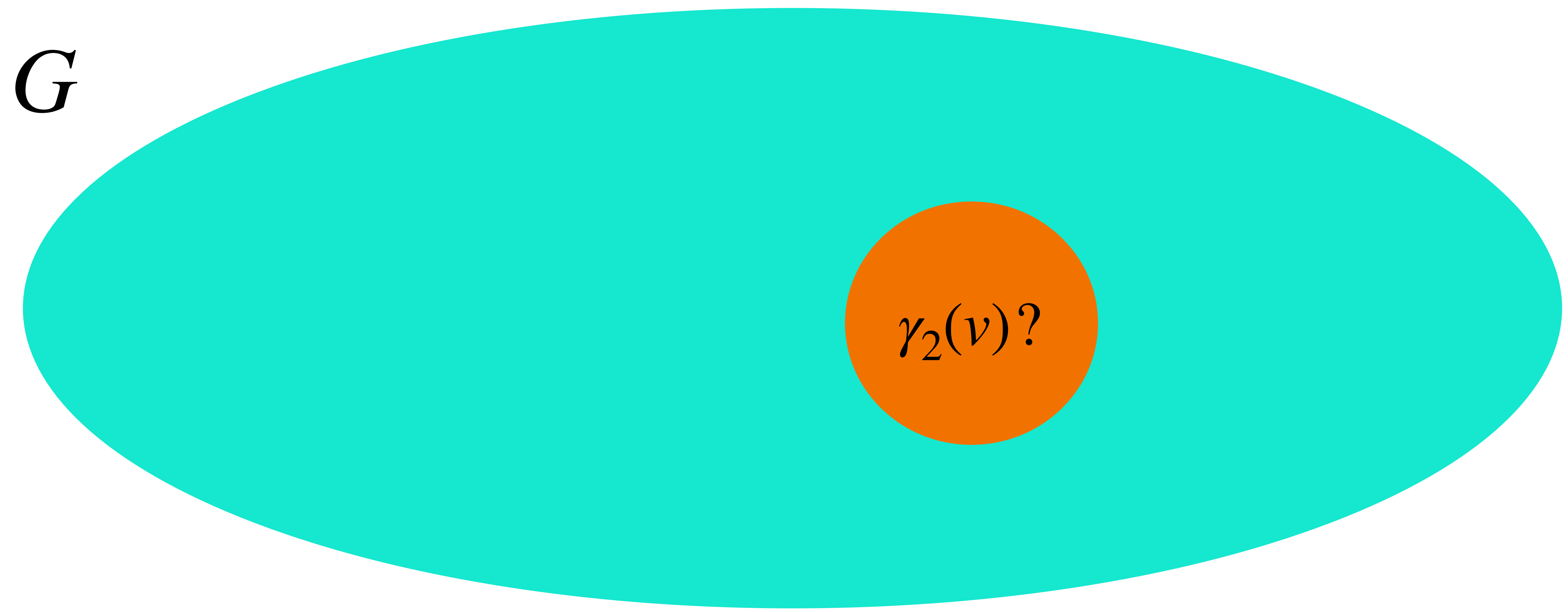

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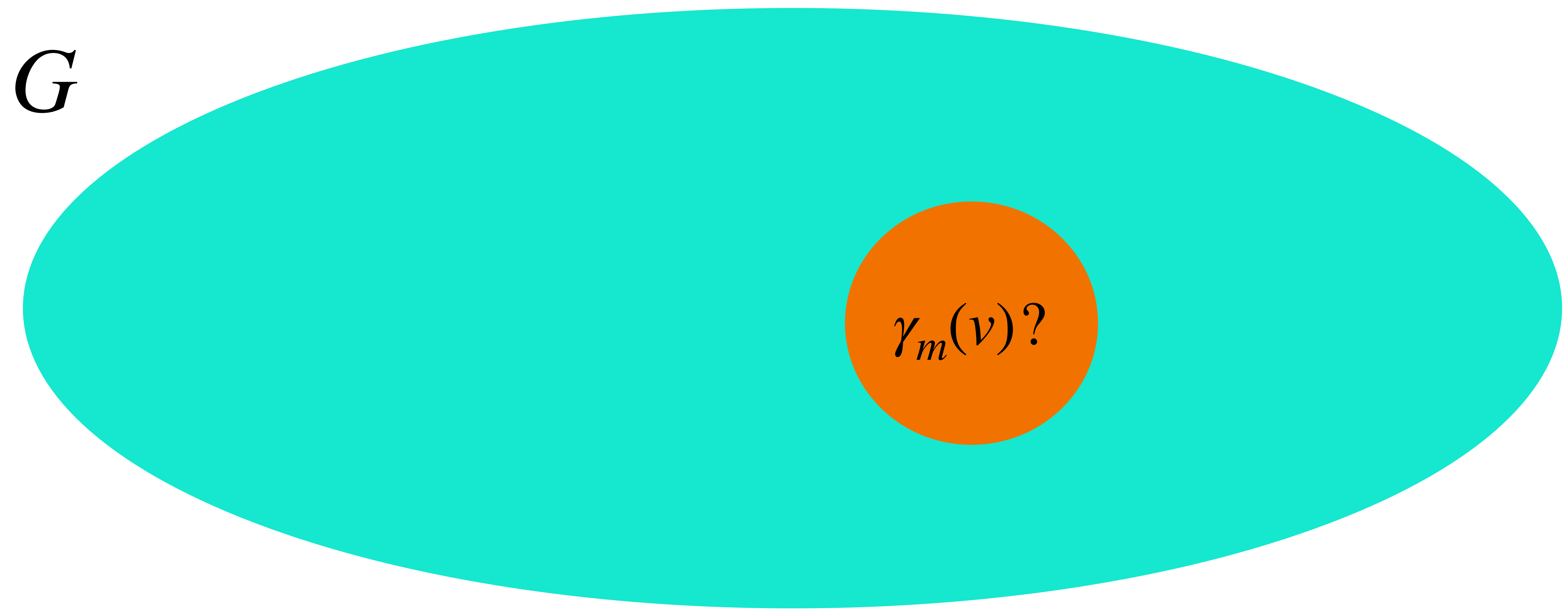

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1. For every $v \in V(G)$ look at $G[N_r(v)]$ (this has size at most $d^r + 1$)
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3. After doing this for all $v \in V(G)$ combine the results together (easy)

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Definition:
Class $\mathscr{C}$ has locally bounded treewidth if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that: For any $G \in \mathscr{C}$ and any $v \in V(G)$, the graph induced by $N_r(v)$ has treewidth at most $f(r)$.

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End of interlude

Our algorithm

Setting:

We work with a graph class $\mathscr{D}$ interpretable in planar graphs using formula $\psi(x, y)$.

We are given $G \in \mathscr{D}$ and φ as input.

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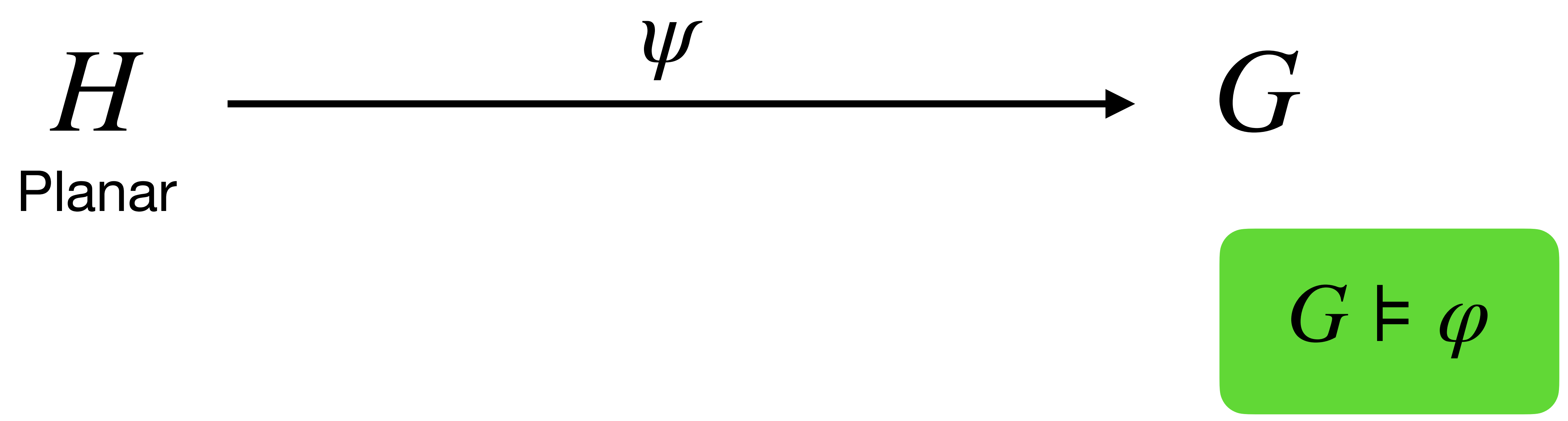

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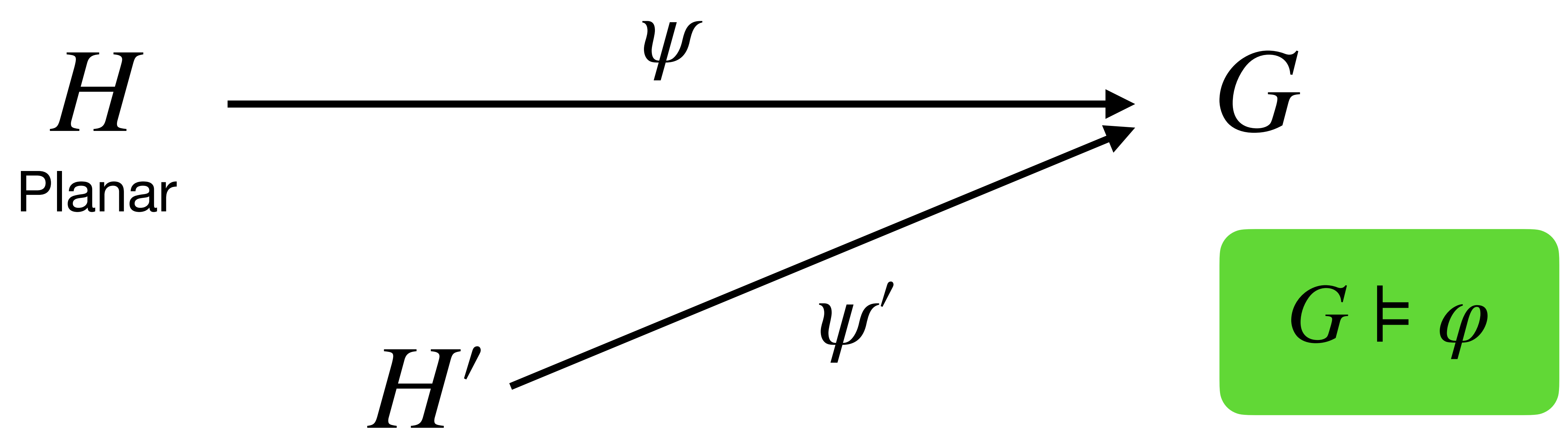

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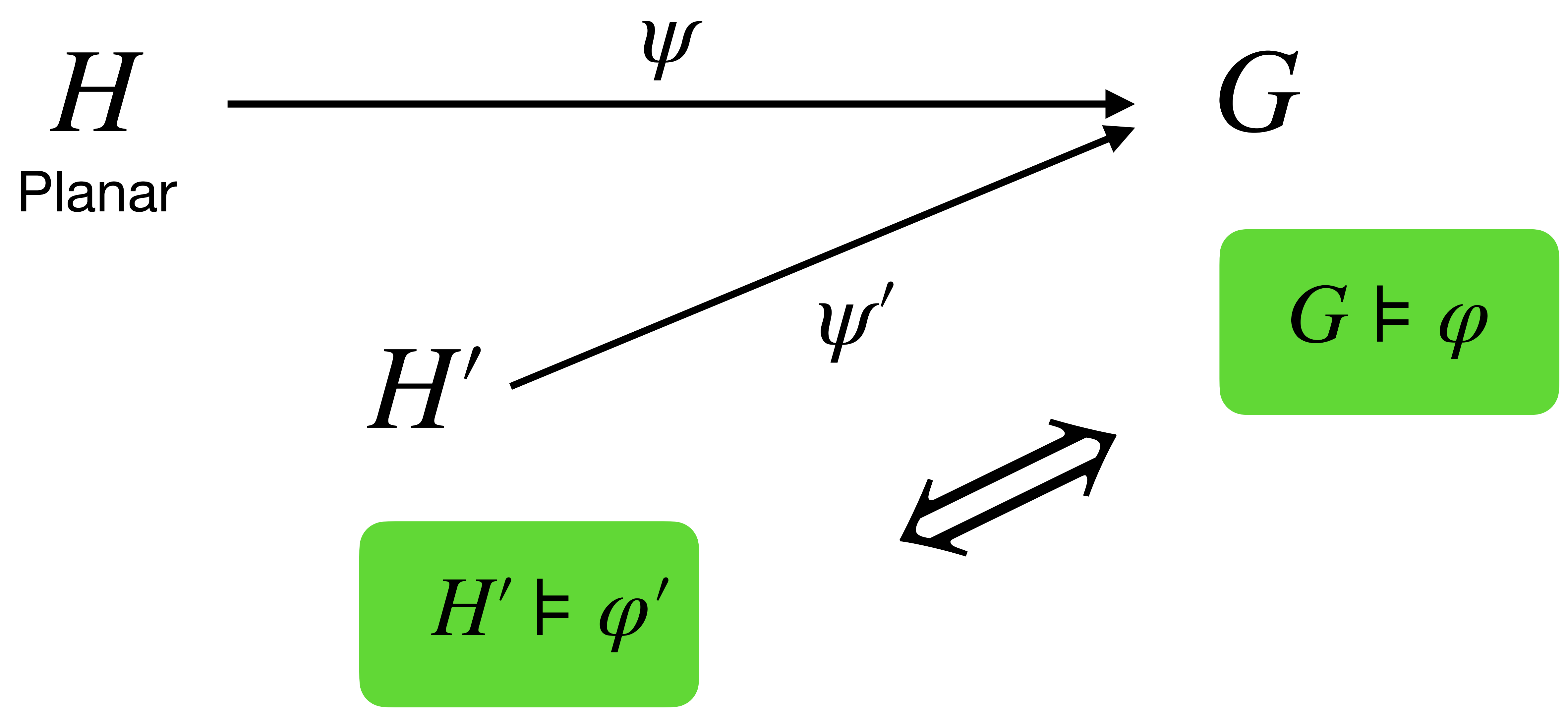

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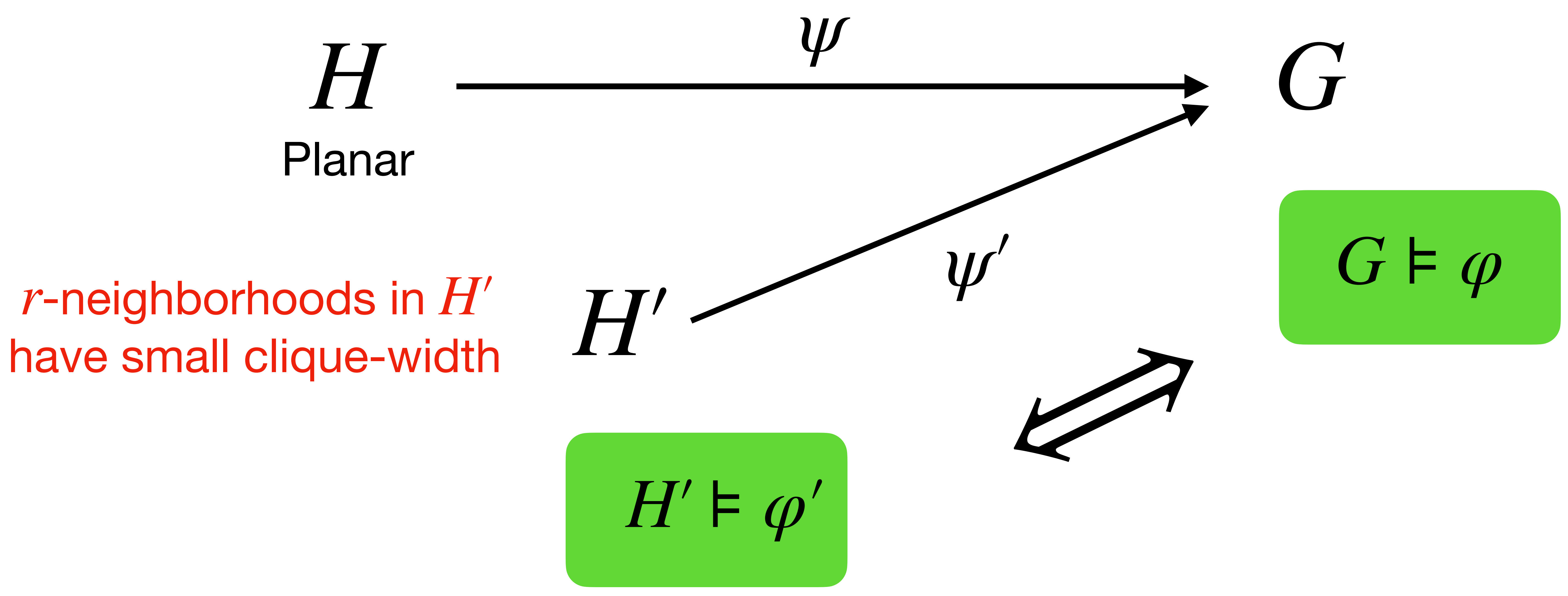

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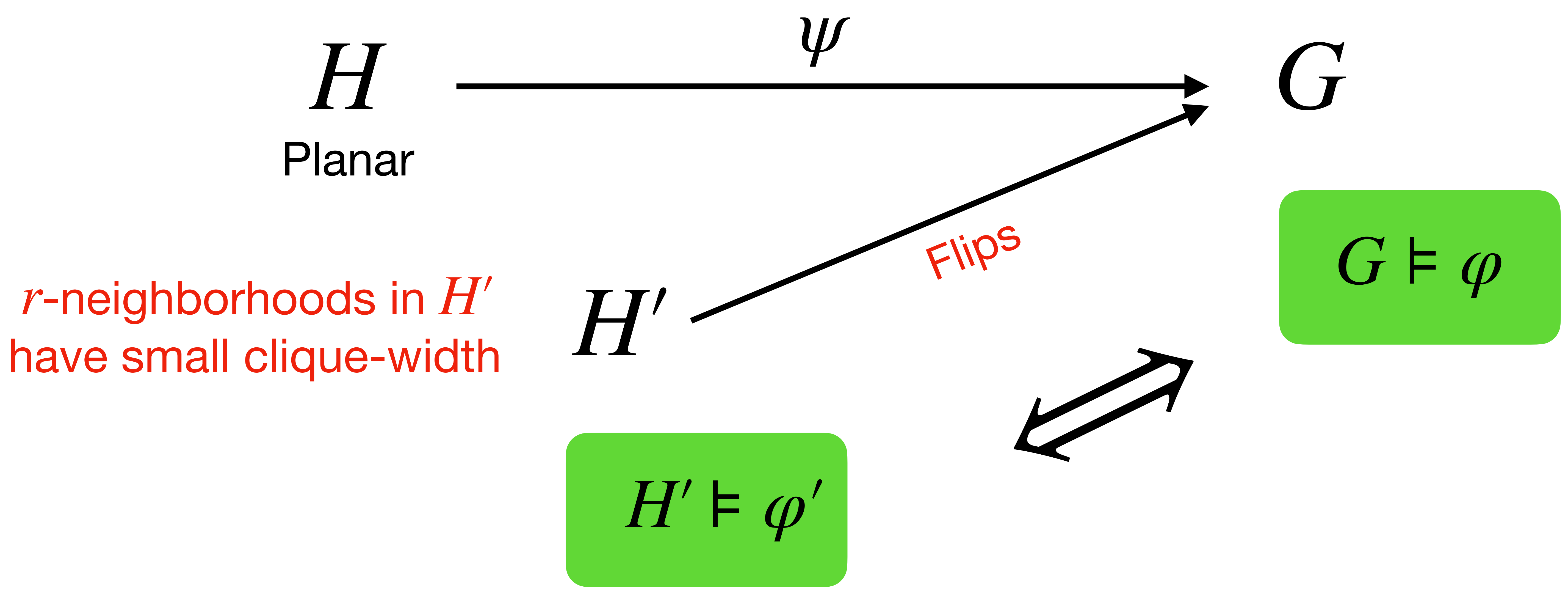

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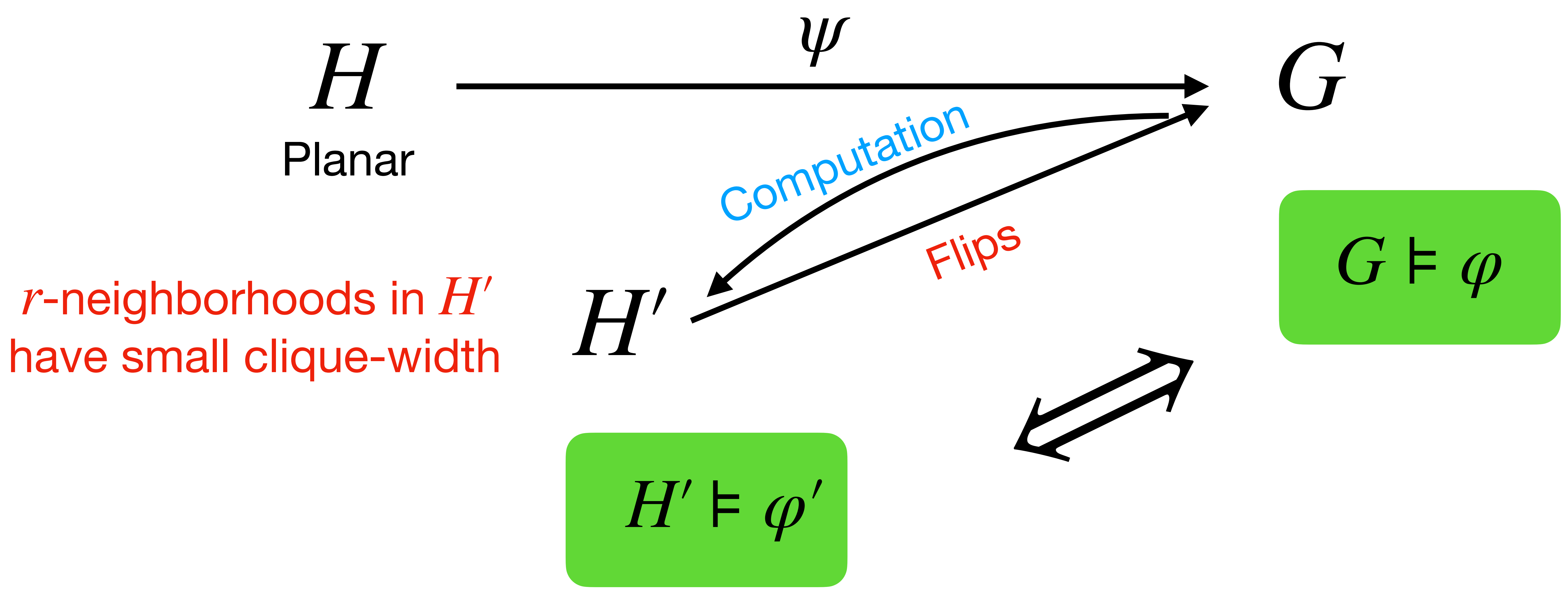

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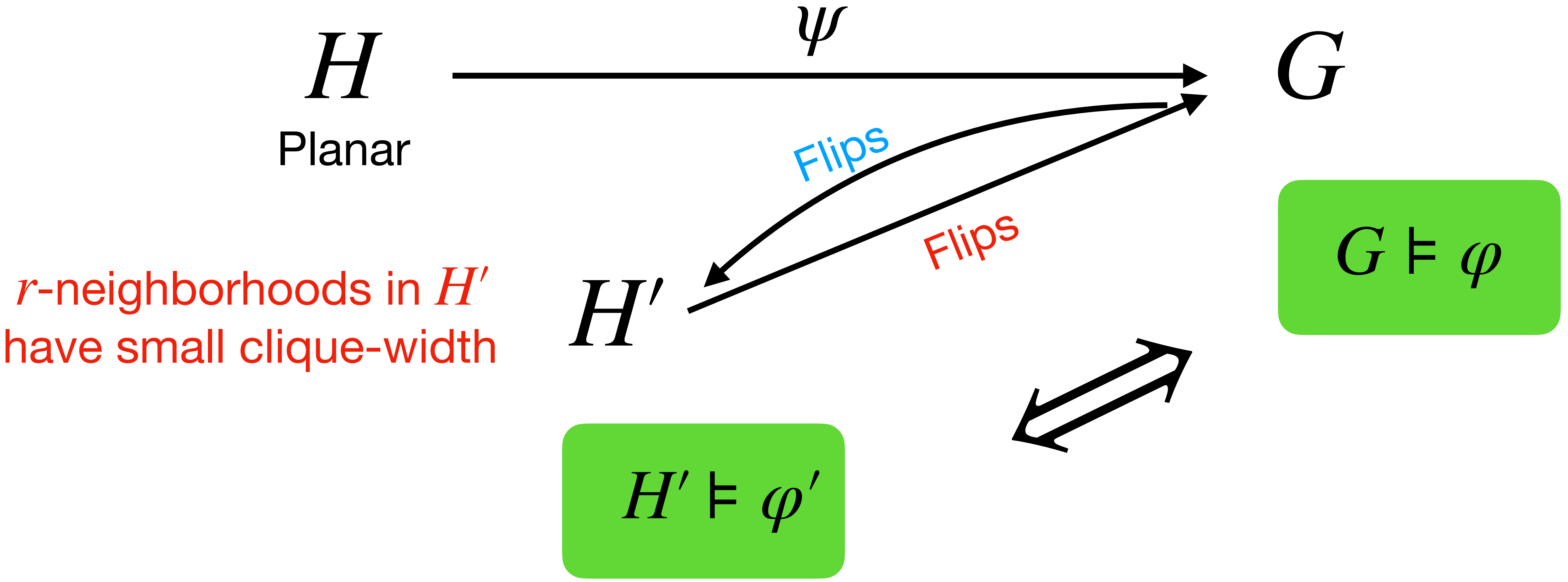

The algorithm for computing H'

Given: G, φ

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Runtime: $|G|^k$ guesses of S, for each of them $2^{(2^k+k)^2}$ choices for flips, and for each of them we check clique-width: $|G|^k \cdot 2^{(2^k+k)^2} \cdot f(cw) \cdot |G|^3$

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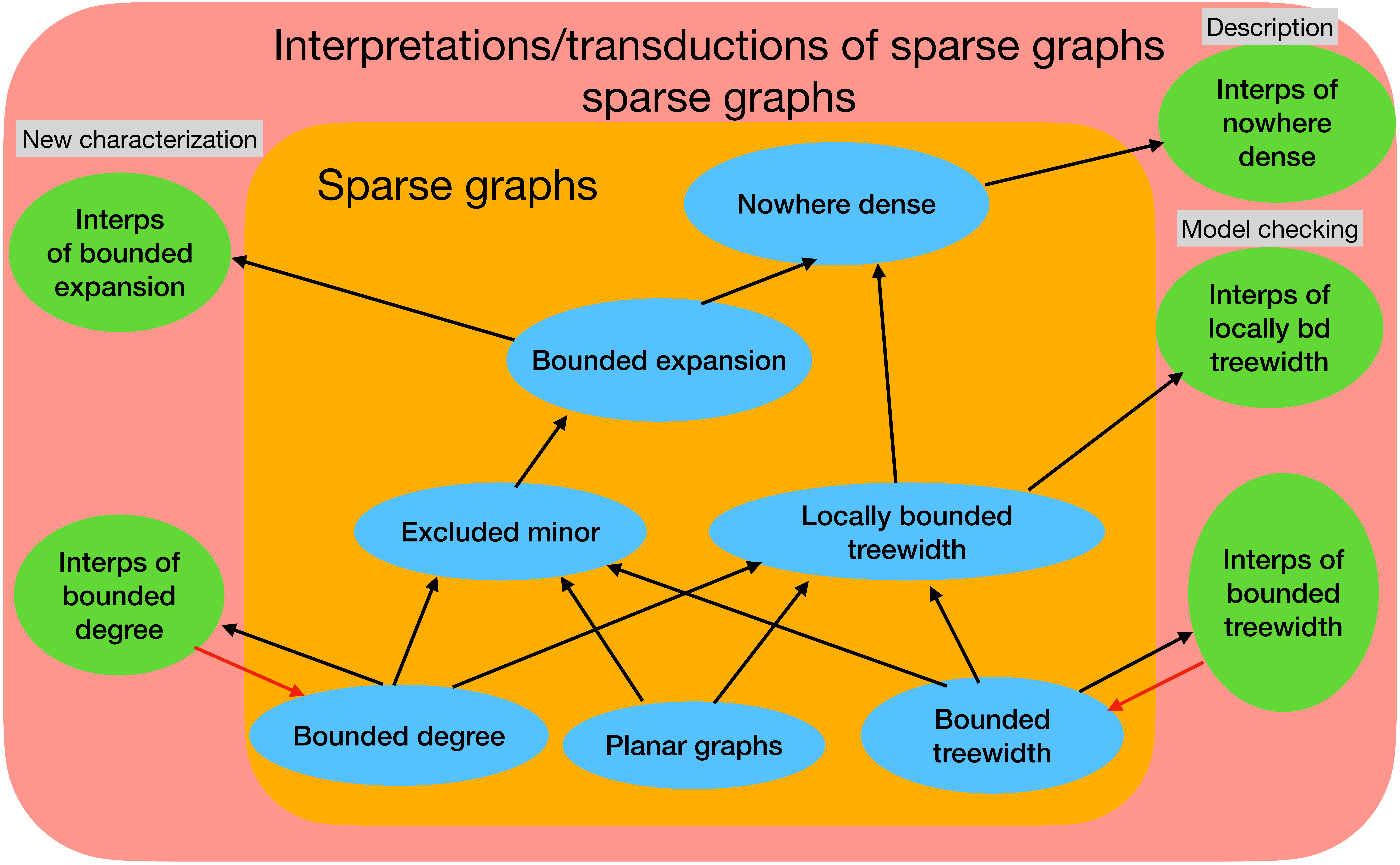
Technical core of the paper — proving that for some choice of S and some choices of flips the resulting graph will have locally small clique-width.


Interpretations/transductions of sparse graphs
sparse graphs
Description
Interps of nowhere dense
New characterization
Sparse graphs
Nowhere dense
Interps of bounded expansion
Model checking
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Interps of bounded treewidth
Bounded degree
Planar graphs
Bounded treewidth

Characterising interpretations of graph classes of bounded expansion and describing interpretations of nowhere dense graph classes

Our results

Theorem (Dreier, G., Kiefer, Mi. Pilipczuk, Toruńczyk; LICS 2022):
A class $\mathscr{D}$ of graphs is interpretable in a class $\mathscr{C}$ of bounded expansion if and only if there exists a class $\mathscr{B}$ of *bushes* of bounded height and bounded expansion representing $\mathscr{D}$.

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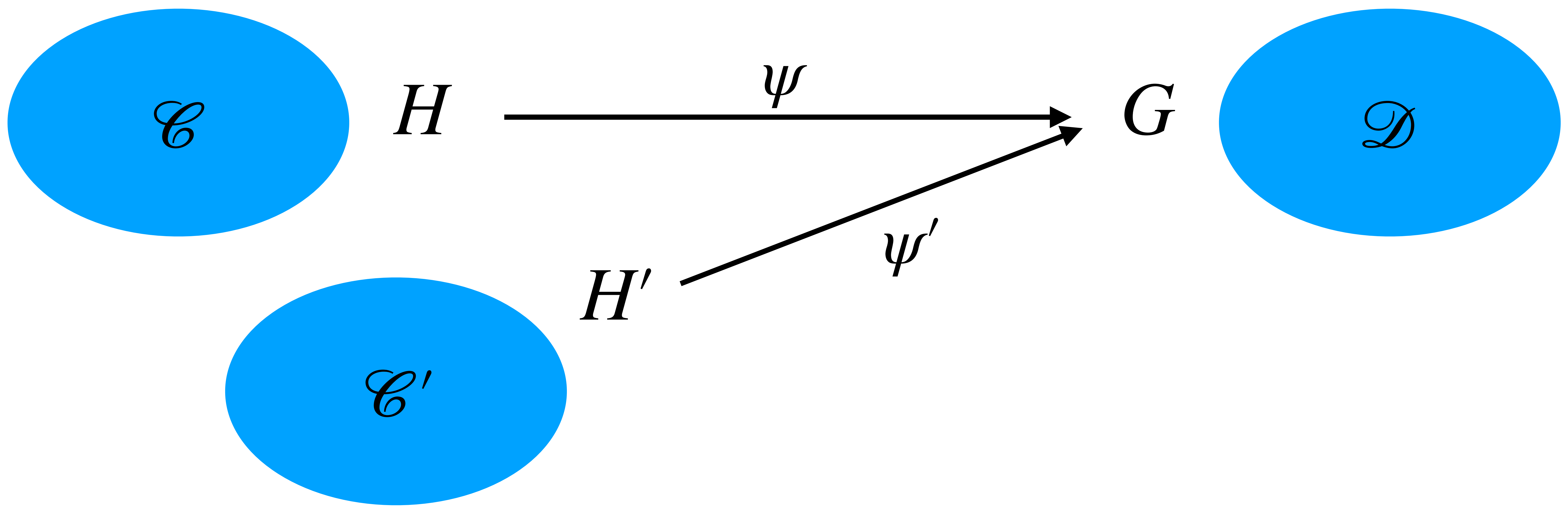
A class $\mathscr{D}$ of graphs is interpretable in a class $\mathscr{C}$ of bounded expansion if and only if there exists a class $\mathscr{B}$ of *bushes* of bounded height and bounded expansion representing $\mathscr{D}$.

Theorem (Dreier, G., Kiefer, Mi. Pilipczuk, Toruńczyk; LICS 2022):
Let $\mathscr{D}$ be a graph class interpretable in a nowhere dense class of graphs $\mathscr{C}$. Then there exists a class $\mathscr{B}$ of quasi-bushes of bounded height which is almost nowhere dense and which represents $\mathscr{D}$.

What is it good for?



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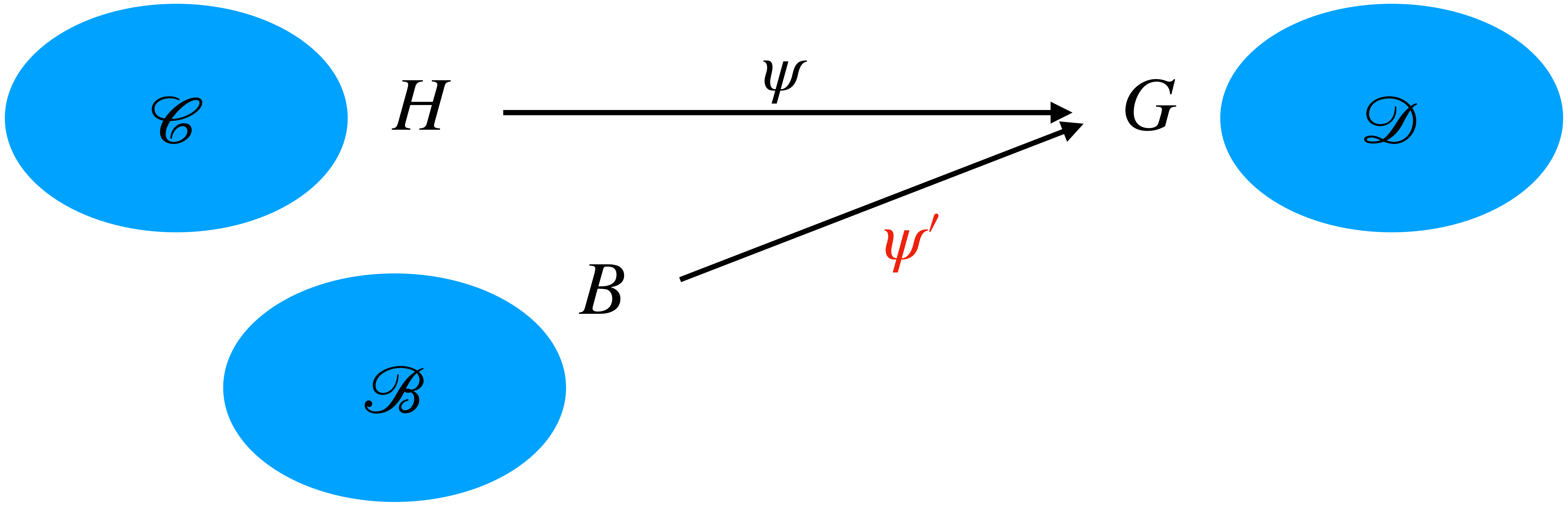
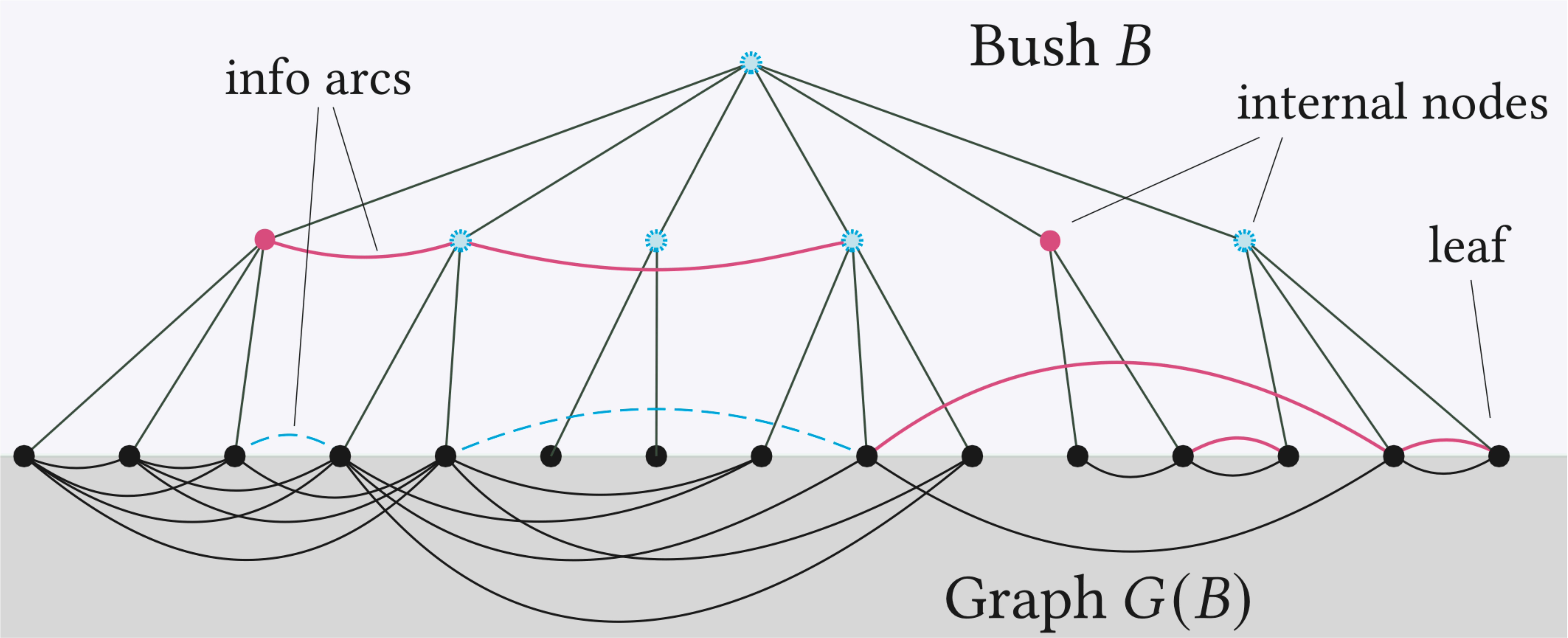
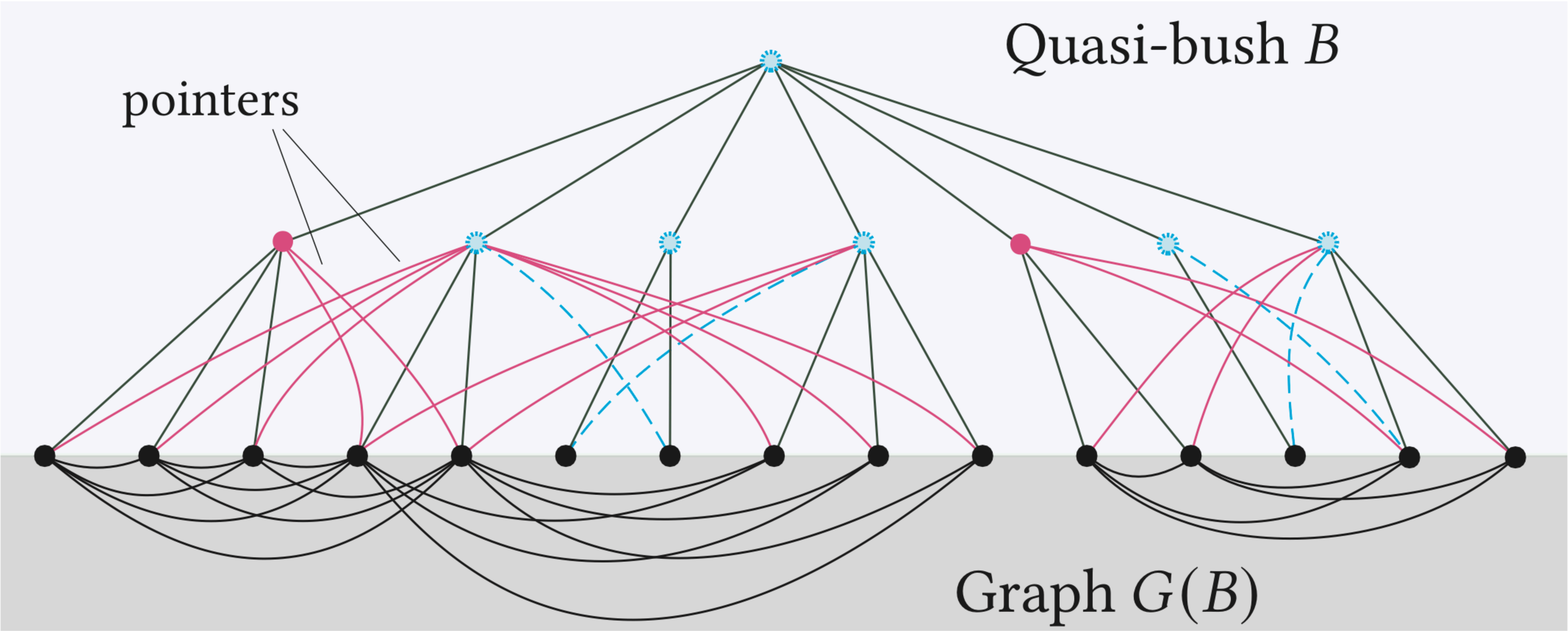


Bushes



Quasi-bushes

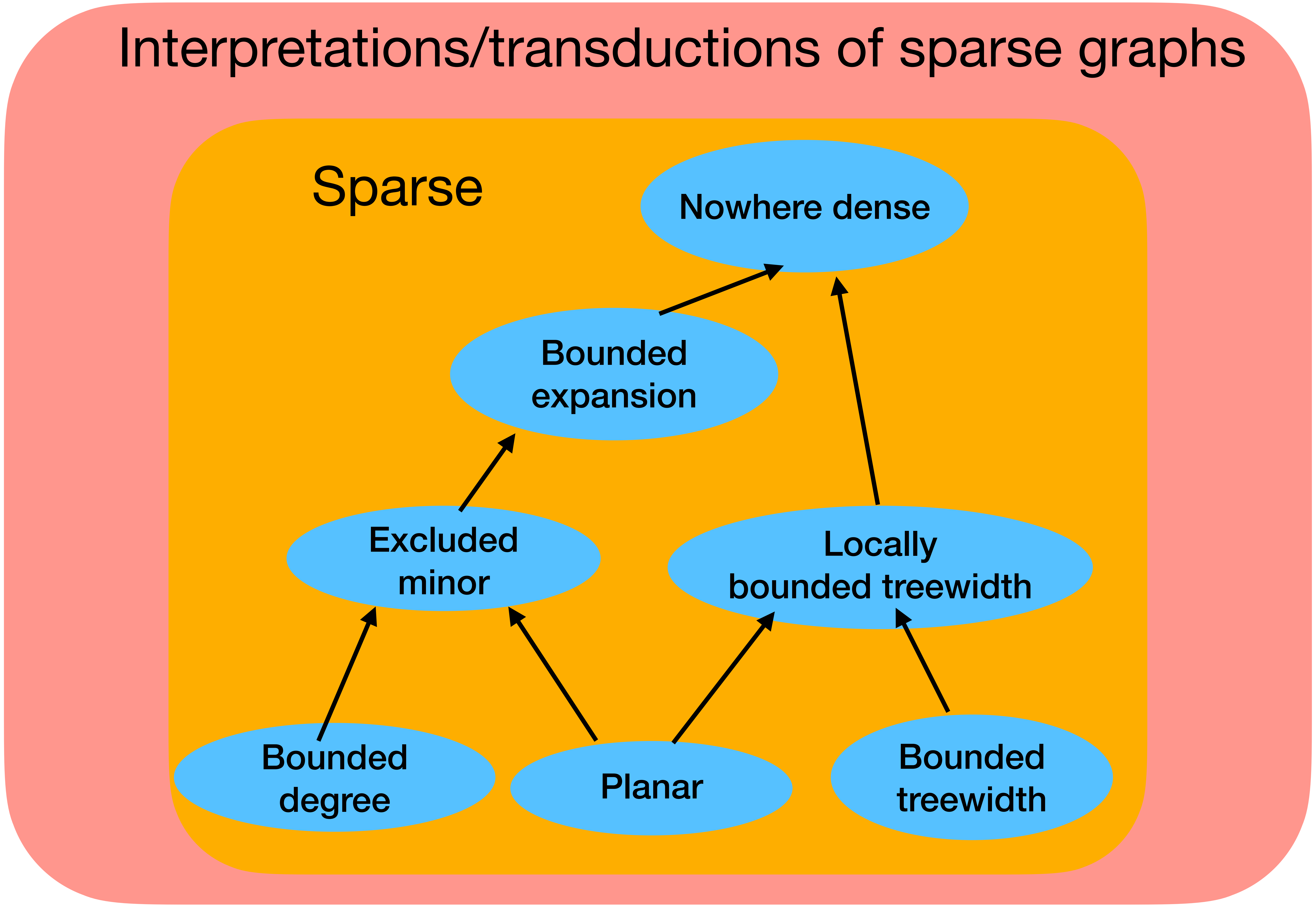


Open problems

Model checking algorithms for interpretations of other classes of sparse graphs.

Approximate interpretation reversal for interpretations of classes of sparse graphs.

Conjecture:
Let $\mathscr{C}$ be a class of graphs such that one cannot interpret every graph in $\mathscr{C}$ (for every interpretation formula ψ there exists graph G such that $G \notin \mathscr{C}$). Then $\mathscr{C}$ has an efficient model checking algorithm.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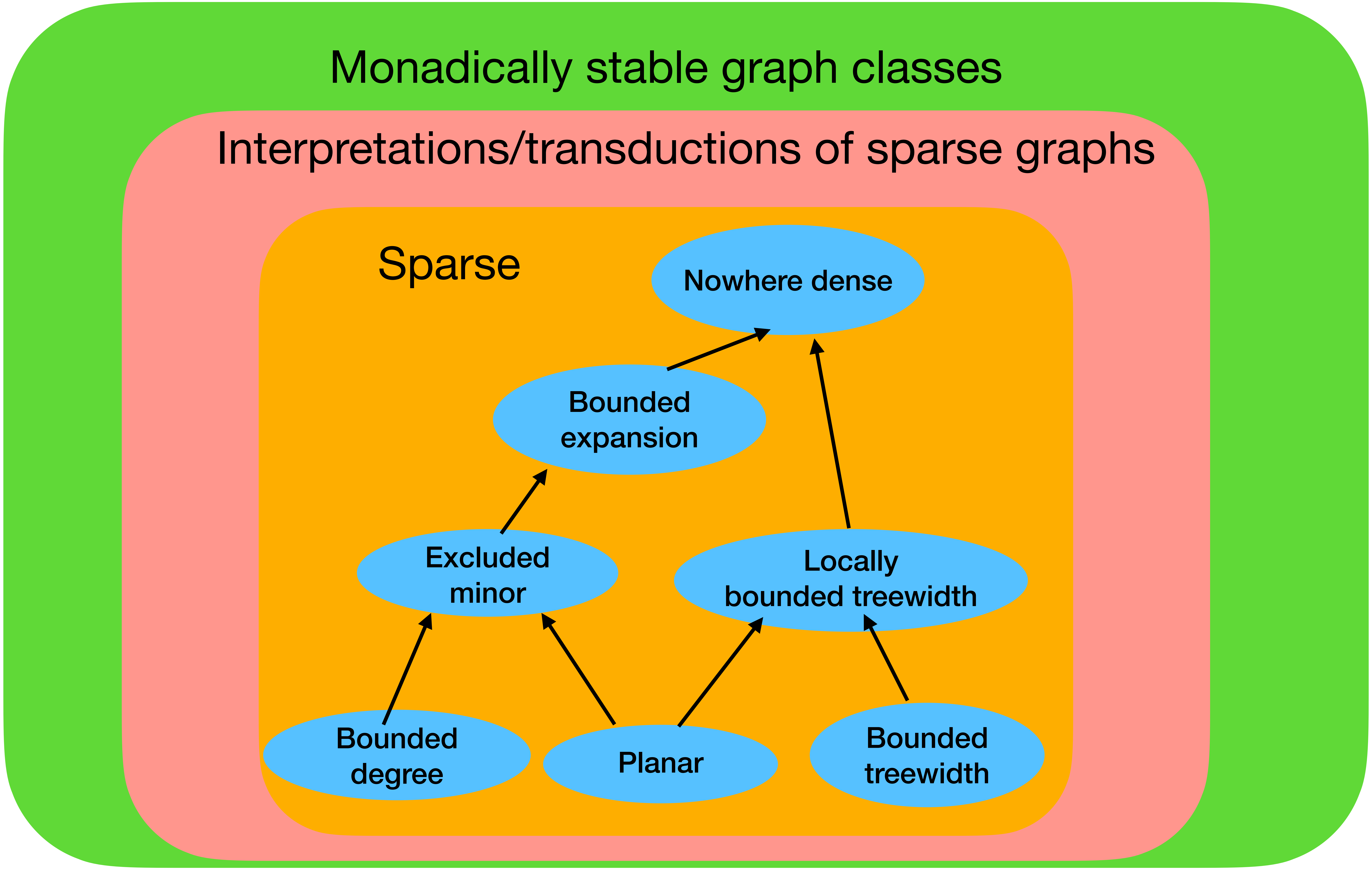
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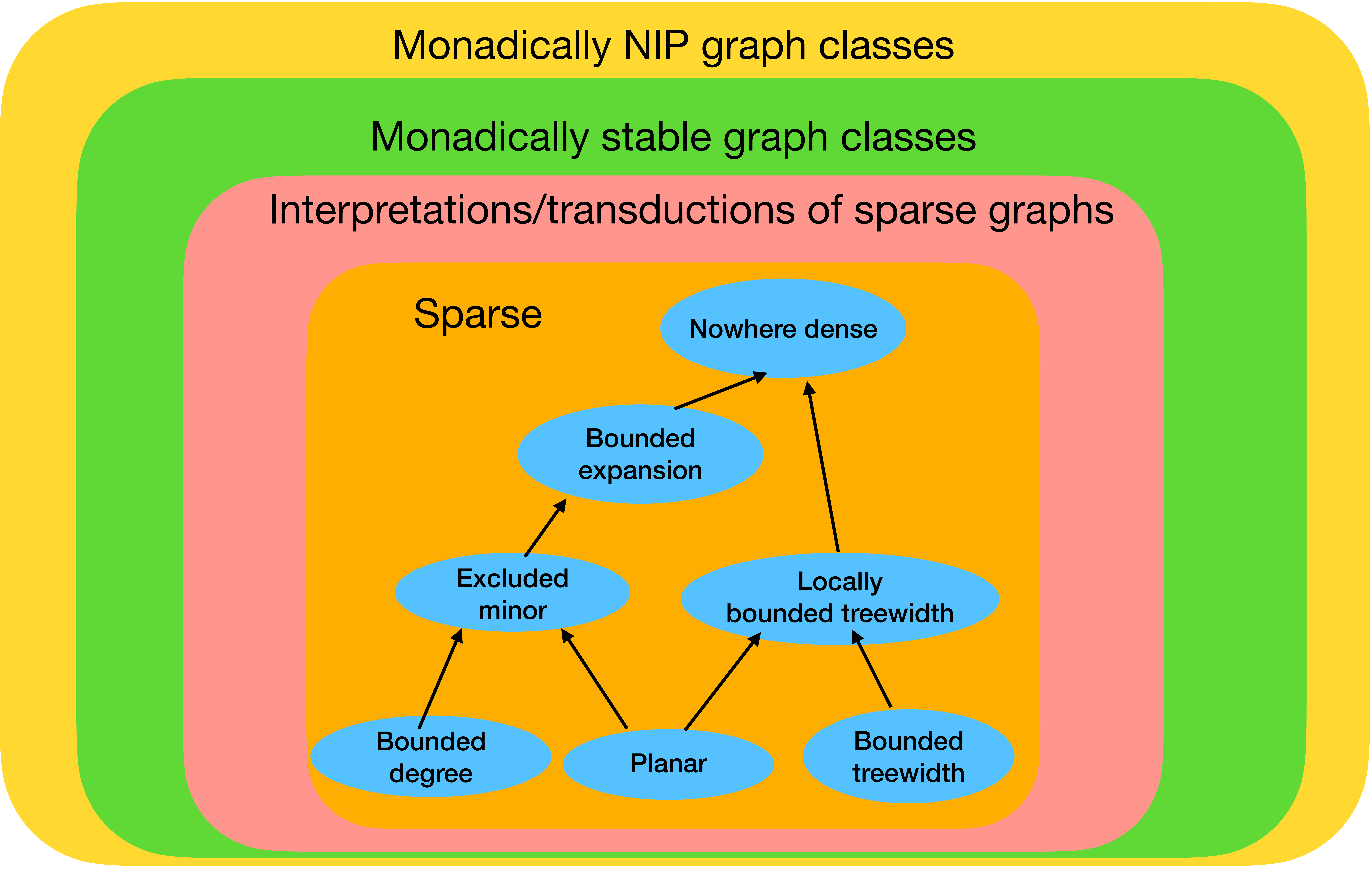
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