UNIVERSITY OF BERGEN

Kemeny Rank Aggregation: Width Measure and Diversity Notion

Emmanuel Arrighi[1], Henning Fernau[2], Daniel Lokshtanov[3], Mateus de Oliveira Oliveira[1], Petra Wolf[2].

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[1] University of Bergen, Norway
[2] University of Trier, Germany
[3] University of California Santa Barbara, CA, USA

Introduction

Kemeny Rank Aggregation (KRA)

Problem (KRA)

Given a list of votes Π over a set of candidates C and $k \in \mathbb{N}$. Is there a ranking τ on C such that the sum of the KT-distances of τ from all the votes is at most k.

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$$
\begin{array}{lccccccccccccc}
\Pi: & c_1 & < & c_7 & < & c_4 & < & c_2 & < & c_5 & < & c_6 & < & c_3 \\
 & c_4 & < & c_7 & < & c_2 & < & c_6 & < & c_1 & < & c_5 & < & c_3 \\
 & c_4 & < & c_6 & < & c_7 & < & c_2 & < & c_5 & < & c_3 & < & c_1 \\
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$$\text{KT-dist}(\pi, \tau) = |\{\{i,j\} \mid c_i <_\pi c_j, c_i >_\tau c_j\}|$$

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Complexity of KRA

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KRA is NP-complete. KRA can be solved neither in time $\mathcal{O}^*(2^{o(|C|)})$ nor in time $\mathcal{O}^*(2^{o(\sqrt{k})})$ unless ETH fails.

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Complexity of with respect to number of votes $|\Pi|$

- $|\Pi| = 2$: polynomial
- $|\Pi| \geq 4$: NP-complete
- $|\Pi| = 3$: open

Completion of an Ordering (CO)

Intuitive definition

$V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

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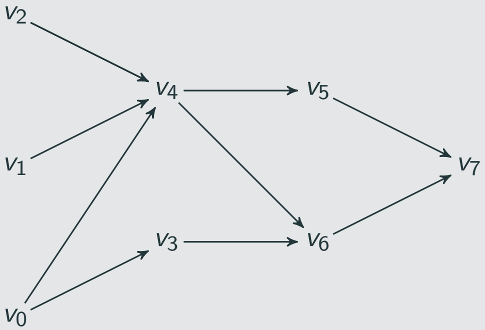

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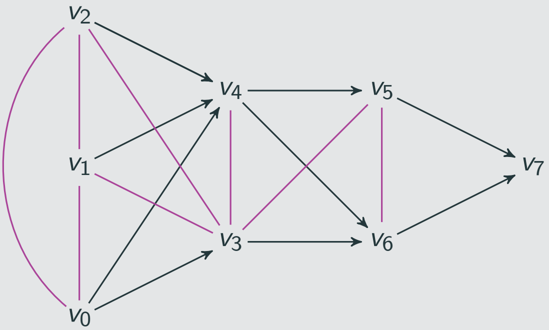

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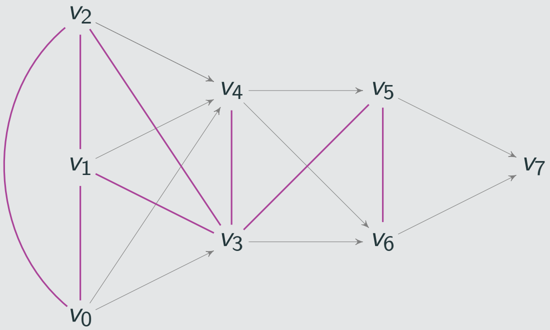

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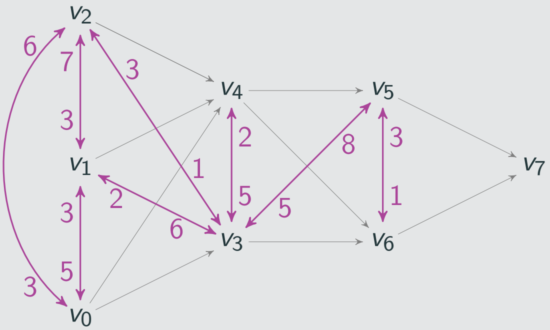

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Given a partial order $\rho \subseteq V \times V$*, a cost function* $c : V \times V \to \mathbb{N}$*, and* $k \in \mathbb{N}$*. Is there a linear extension* $\tau \supseteq \rho$ *with* $c(\tau \setminus \rho) = \sum_{(x,y) \in \tau \setminus \rho} c(x, y) \leq k$*?*

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The problem CO is NP-complete even for cost in $\{1, 2\}$.

PCO can be solved neither in time $\mathcal{O}^*(2^{o(|V|)})$ nor in time $\mathcal{O}^*(2^{o(\sqrt{k})})$ unless ETH fails.

Reduction from KRA to CO

KRA: input

A set of candidates $C = \{c_1, \ldots, c_n\}$ and a list of votes $\Pi = \{\pi_1, \ldots, \pi_m\}$

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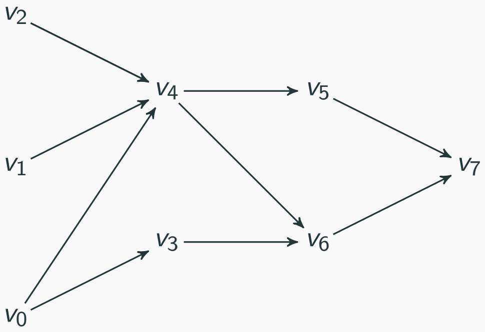
The width measure

Cocomparability graph

Definition (Cocomparability graph)

Given a partial order $\rho \subseteq V \times V$, the graph $G_\rho = (V, E)$ with $E = \{\{u, v\} \in V \times V \mid u \neq v, (u, v) \notin \rho, (v, u) \notin \rho\}$ is the cocomparability graph of ρ.



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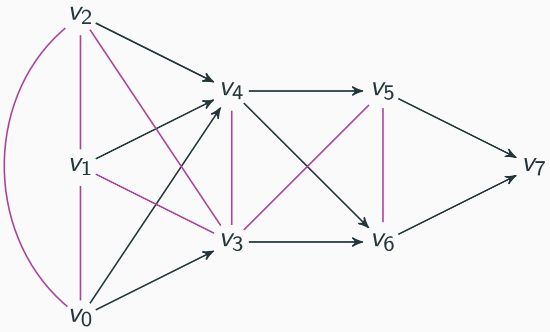

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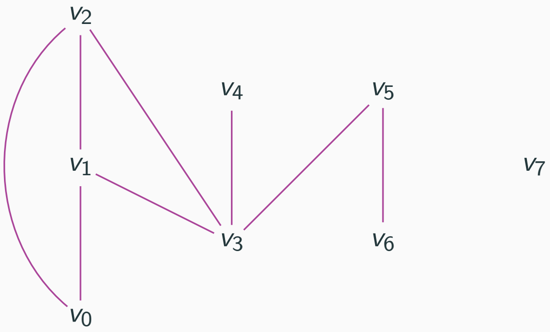
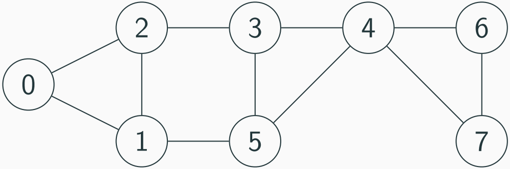


ρ-Consistent Path-decomposition



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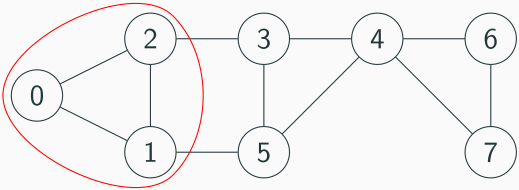

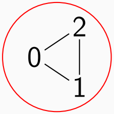

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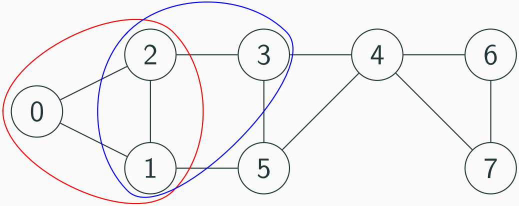

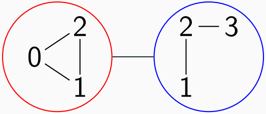

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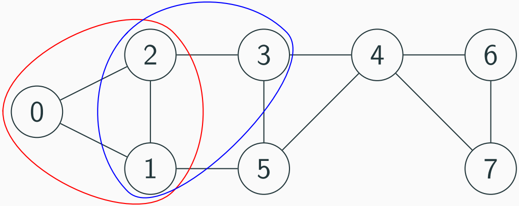

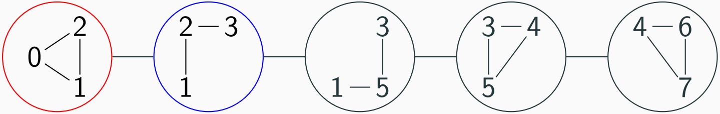

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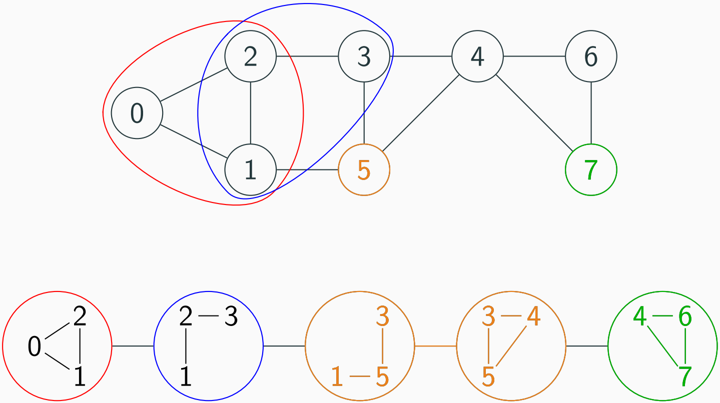

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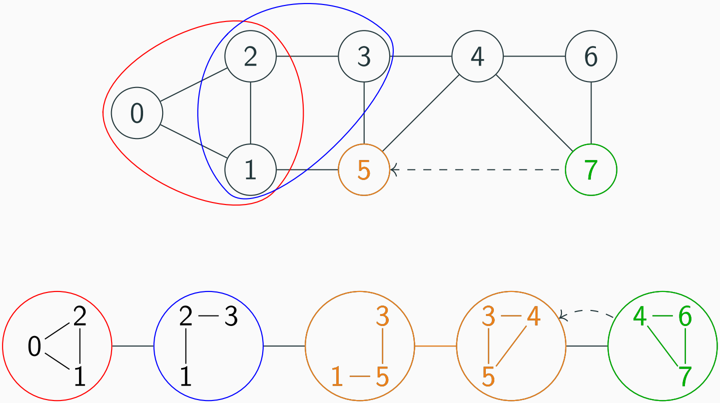

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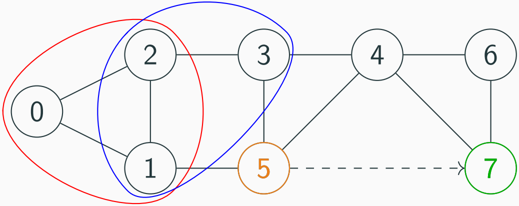

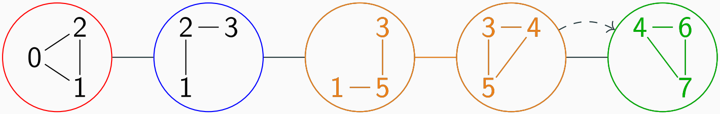

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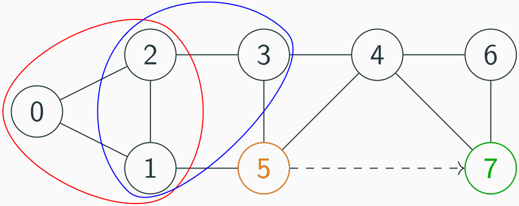

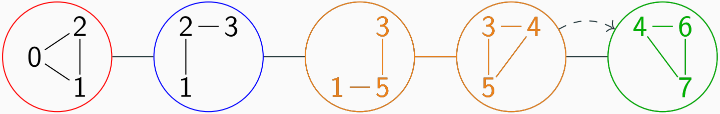

Definition (Consistency)

P consistent with ρ if there is no pair of vertices $(u, v) \in \rho$ with $\max\{i \in [l] \mid v \in B_i\} < \min\{i \in [l] \mid u \in B_i\}$.

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Lemma

Given a partial order ρ, $cpw(G_\rho, \rho) = pw(G_\rho)$.

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Lemma

Given a partial order ρ, we can construct ρ-consistent path decomposition of width $\mathcal{O}(cpw(G_\rho, \rho))$ in time $2^{\mathcal{O}(cpw(G_\rho,\rho))} \cdot |V|$.

Maximum range of candidates

$$c_4 < c_7 < c_2 < c_6 < c_1 < c_5 < c_3$$

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$$r(c_1) = 4$$

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$$r(c_2) = 4$$

Maximum range of candidates

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$$r(c_3) = 2$$

Maximum range of candidates

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KRA can be solve in $\mathcal{O}^\star(2^{5 \cdot r_{\max}})$

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DP Algorithm for CO

Result for Completion of an Ordering (CO)

Theorem

CO *is solvable in time* $\mathcal{O}(|V| \cdot w \cdot 2^{w} \cdot \log(k) + |V|^2 \cdot \log(k))$ *where* w *is the pathwidth of the cocomparability graph.*

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- $\forall S \subseteq B_i$, we keep c_S the cost of an optimal solution to $L_i \cup S$
  - B_i introduces v:

$$c_S = \min_{v' \in \max_\rho(S)} \{c_{S\setminus\{v'\}} + c(L_i \cup S \setminus \{v'\}, \{v'\})\}$$

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CO *is solvable in time* $\mathcal{O}(|V| \cdot w \cdot 2^{w} \cdot \log(k) + |V|^2 \cdot \log(k))$ *where* w *is the pathwidth of the cocomparability graph.*

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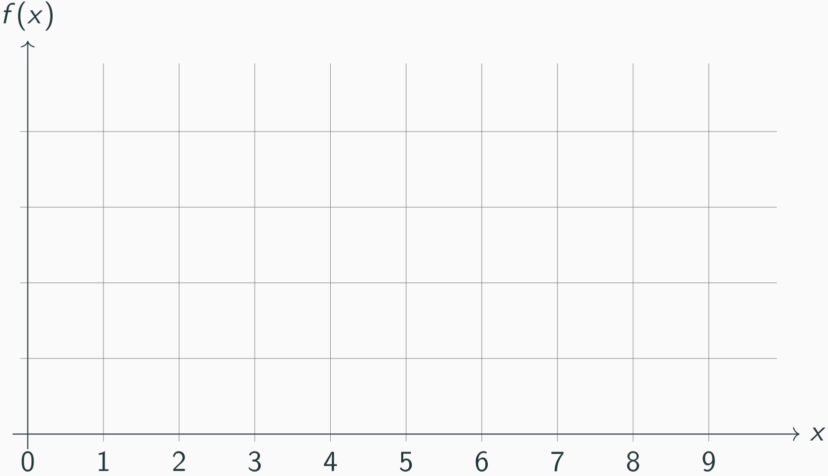
Diverse Kemeny Rank Aggregation and Completion of Ordering

Intuition for Diversity of Solutions Framework

Given $f : \mathbb{R} \to \mathbb{R}$



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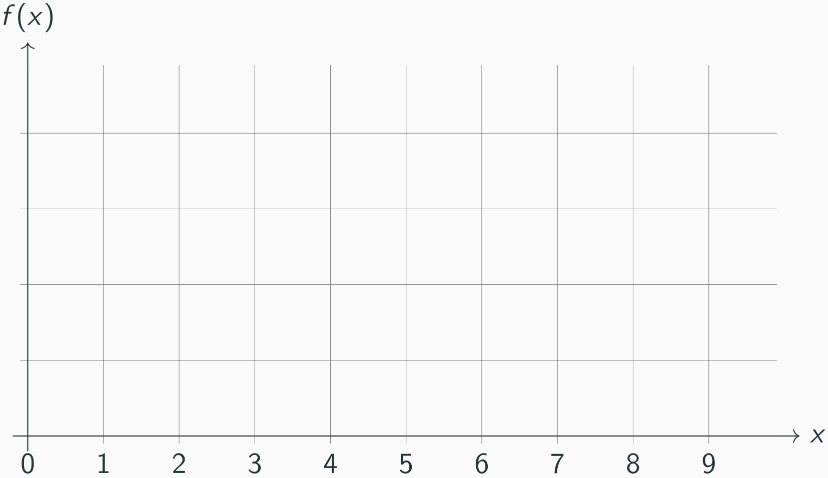

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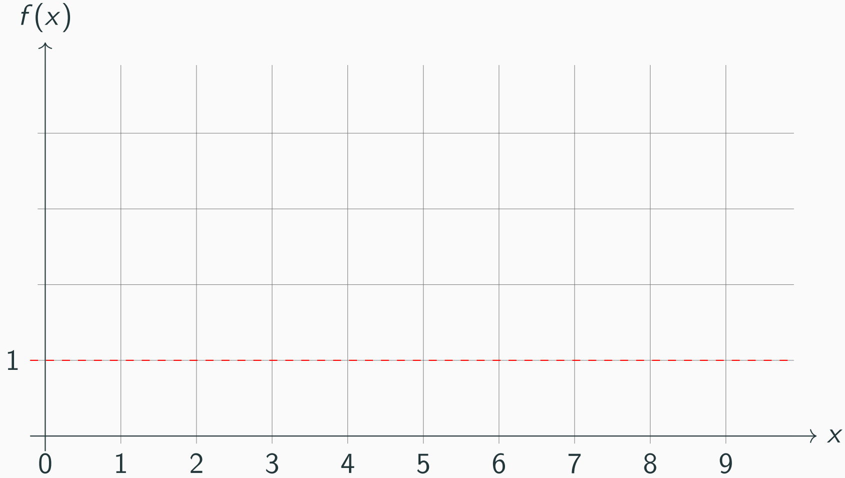

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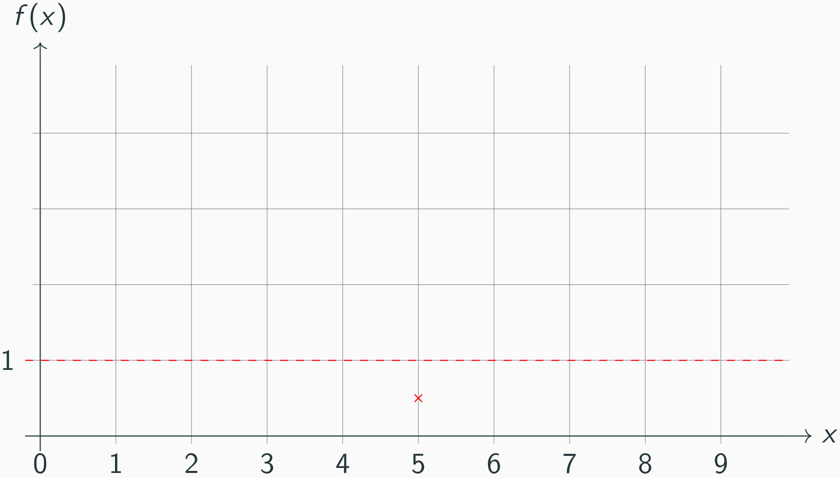

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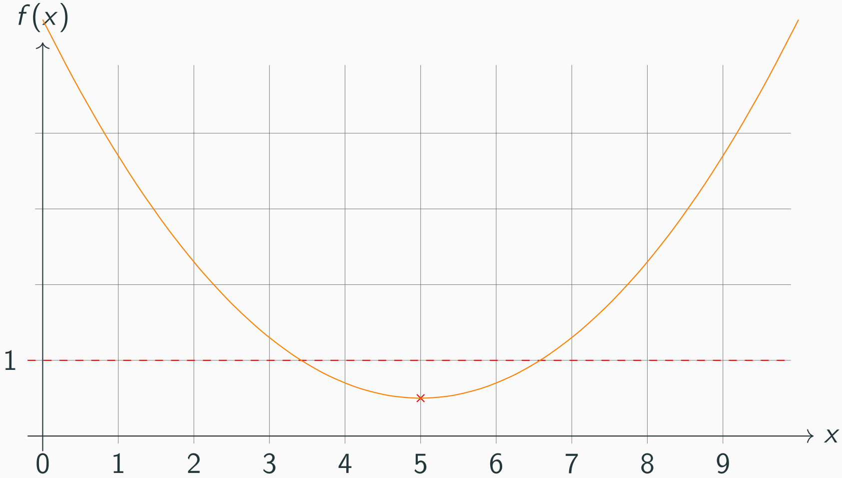

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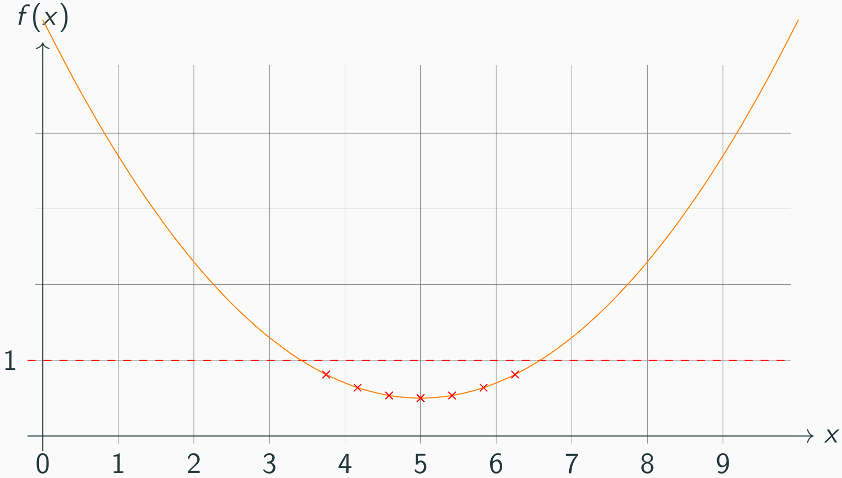

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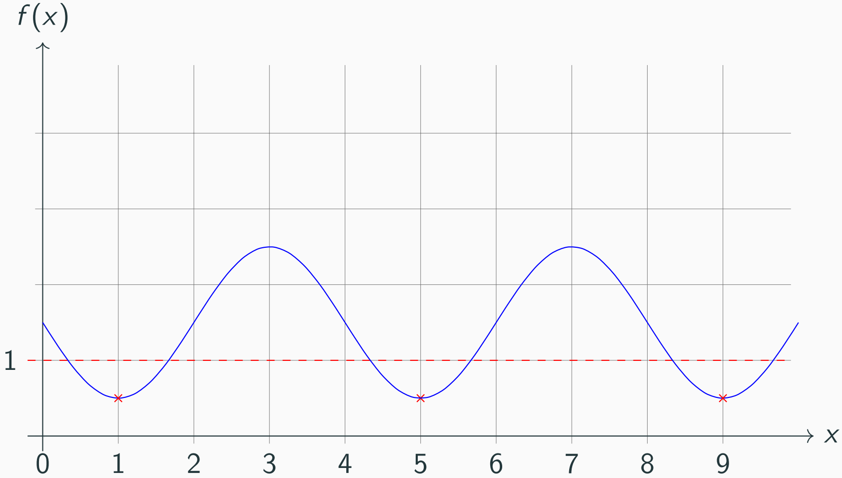

Diverse Set of Solutions: Parameters

Let S be a set of solution.

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Diverse Kemeny Rank Aggregation (Diverse-KRA)

Problem (Diverse-KRA)

Given a list of partial votes Π *over a* ***set of candidates*** C *and* $k, r, d \in \mathbb{N}$*. Is there a set* $R = \{\tau_1, \ldots, \tau_r\}$ *of rankings of* C *with* $\sum_{\pi \in \Pi} \text{KT-dist}(\pi, \tau_i) \leq k$ *for each* $i \leq r$ *and* $\text{KT-Div}(R) \geq d$*.*

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Diverse Completion of an Ordering (CO)

Problem (Diverse-CO)

*Given a **partial order** $\rho \subseteq V \times V$, a **cost function** $c : V \times V \to \mathbb{N}$, and $k, r, d \in \mathbb{N}$.*

Is there a set $R = \{\tau_1, \ldots, \tau_r\}$ of linear extensions of ρ with $c(\tau_i \setminus \rho) \leq k$ for each $i \leq r$ and $\text{KT-Div}(R) \geq d$?

Diverse Completion of an Ordering (CO)

Theorem

Determining whether ρ admits r linear extensions at distance $\leq \delta$ from optimum, diversity $\geq d$, and scatteredness $\geq s$ is possible in time $\mathcal{O}((w! \cdot \delta)^{\mathcal{O}(r)} \cdot (s+1)^{r^2} \cdot d \cdot |V| \cdot \log(|V|^2 \cdot k))$.

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  - keep track of the diversity and scatteredness.

Conclusion

- CO can model other problems:
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- Is $w!$ needed for the diverse-CO?
- What can be models by CO?
- Can we do something similar for other ordering problems?

Thank You!

Width notions for ordering-related problems
Arrighi, Fernau, de Oliveira Oliveira, and Wolf (2020)

Diversity in kemeny rank aggregation: A parameterized approach
Arrighi, Fernau, Lokshtanov, de Oliveira Oliveira, and Wolf (2021)

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Problem (PCO)

Like CO but for all incomparable pairs $(x, y) \in V \times V$ *in* ρ *the cost* $c(x, y)$ *is positive.*

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Let (ρ, c, k) be a YES-instance of PCO. Then, $pw(G_\rho) = \mathcal{O}(\sqrt{k})$.

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Definition (Two-layer drawing)

Let $G = (V_1, V_2, E \subseteq V_1 \times V_2)$ be a bipartite graph.

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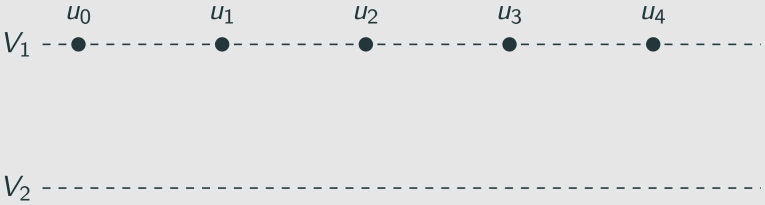

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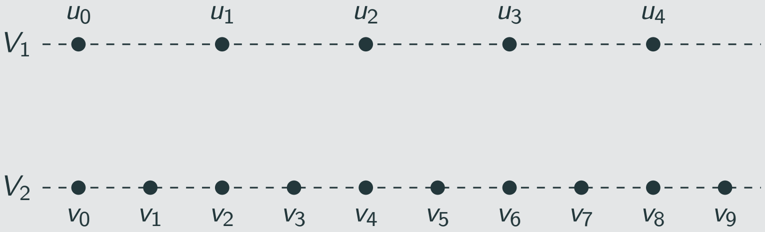

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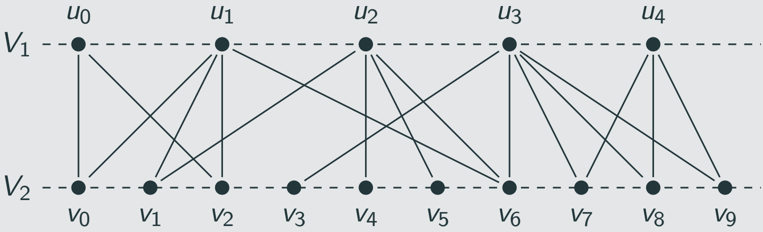

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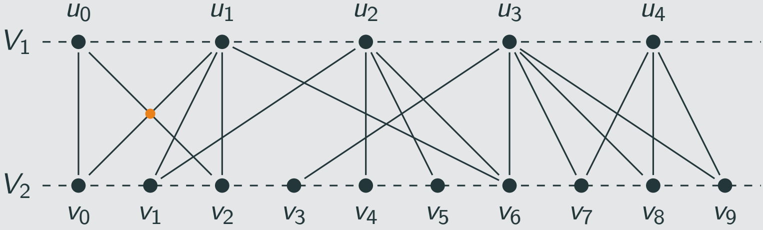

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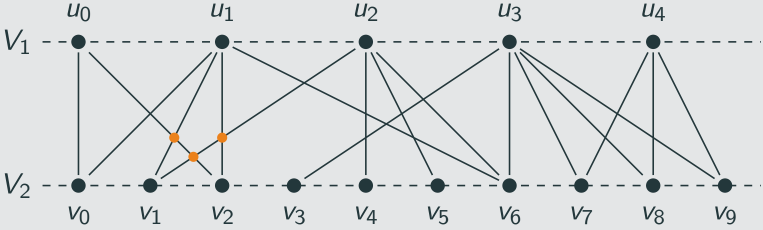

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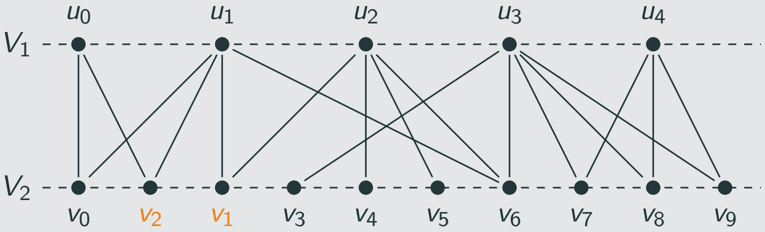

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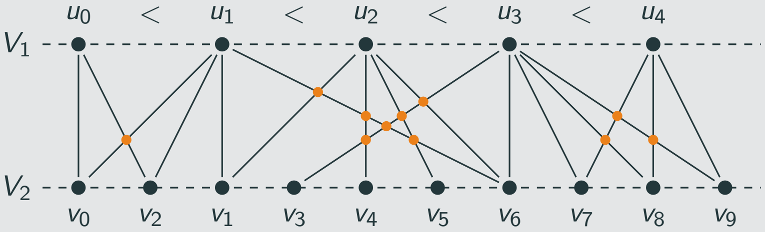

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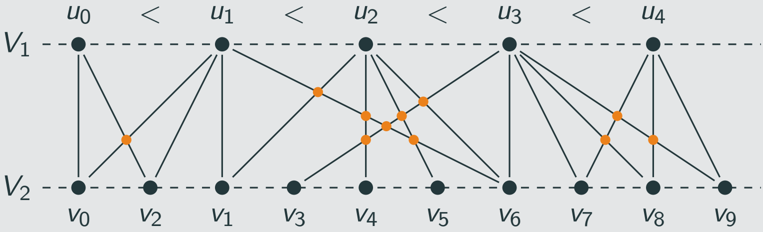


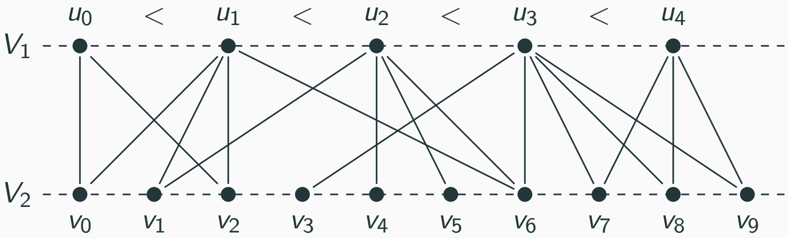
Problem (OSCM)

Given a bipartite graph $G = (V_1, V_2, E)$, a linear order τ_1 on V_1 and $k \in \mathbb{N}$. Is there a linear order τ_2 on V_2 such that the two-layer drawing specified by (τ_1, τ_2) has at most k edge crossings?

Reduction from OSCM to CO

OSCM: input

A bipartite graph $G = (V_1, V_2, E)$ and a linear order τ_1 on V_1.



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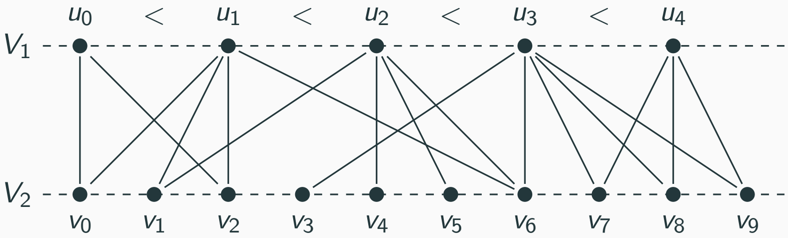

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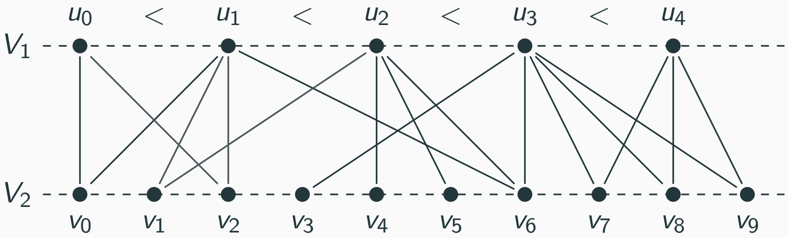

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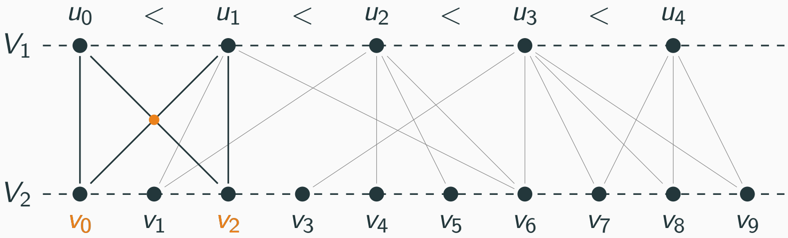

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$$c(v_0, v_2) = 1$$

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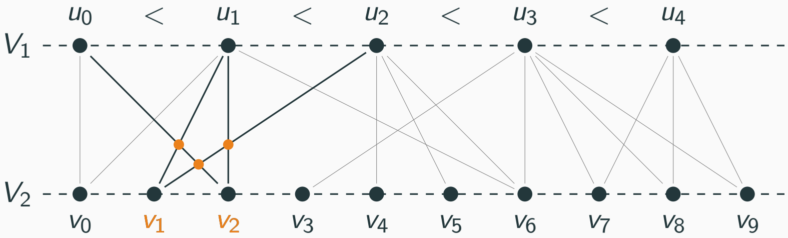

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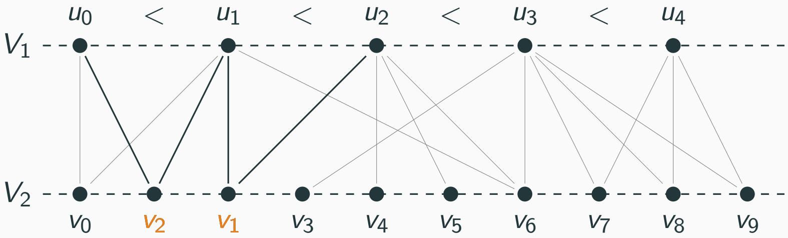

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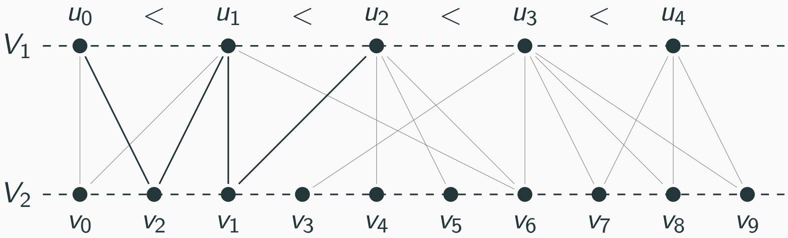
Reduction from OSCM to CO

OSCM: input

A bipartite graph $G = (V_1, V_2, E)$ and a linear order τ_1 on V_1.



- Base set: $V = V_2$
- Cost: $c(v_i, v_j) = |\{(k, l) \mid k > l, (u_k, v_i) \in E, (u_l, v_j) \in E\}|$
- Partial order: $\{(v_i, v_j) \mid c(v_i, v_j) = 0\}$

Grouping by Swapping (GbS)

Definition (Swap)

c	a	d	d	a	a	b	c	c	d	d

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Definition (Blocks string)

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Problem (GbS)

Given a finite alphabet Σ, a string $w \in \Sigma^*$, and $k \in \mathbb{N}$. Can we transform w in a blocks string w' with at most k swaps?

Reduction from GbS to OSCM

GbS: input

An alphabet $\Sigma = \{a, b, c, d\}$ and $w = caddaabccdd$.

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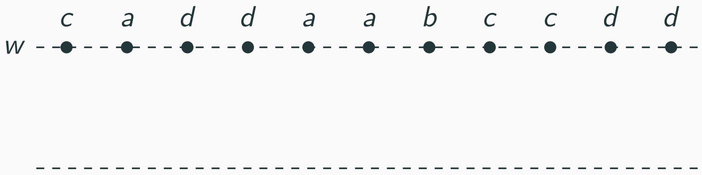

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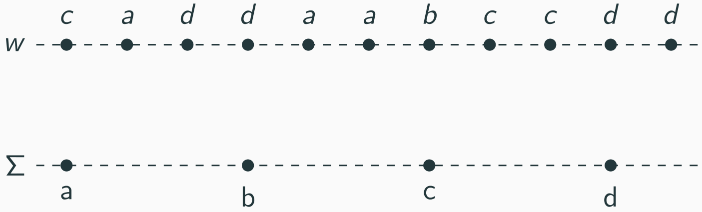

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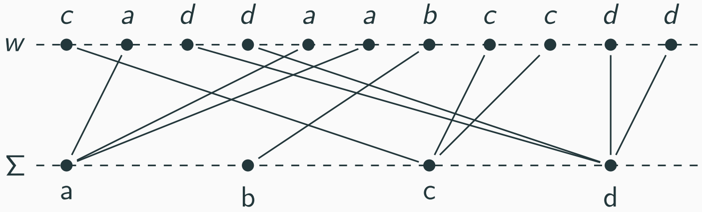

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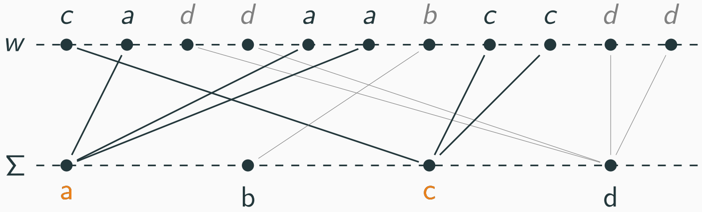

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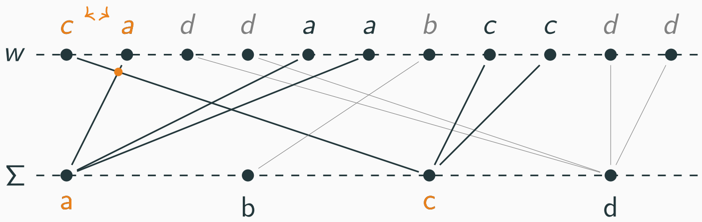

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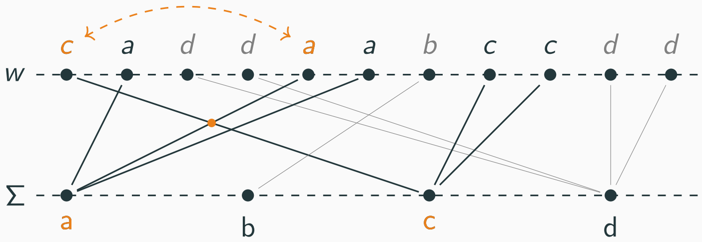

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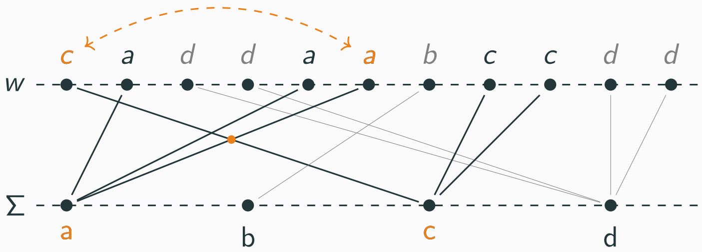

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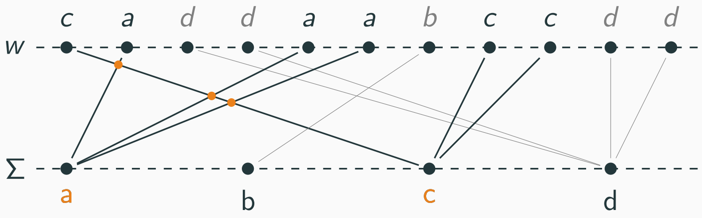

Complexity of GbS

Complexity

GbS is NP-complete. GbS can be solved neither in time $\mathcal{O}^*(2^{o(|\Sigma|)})$ nor in time $\mathcal{O}^*(2^{o(\sqrt{k})})$ unless ETH fails.

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Complexity of r-GbS

- $r = 2$: polynomial
- $r \geq 4$: NP-complete
- $r = 3$: open