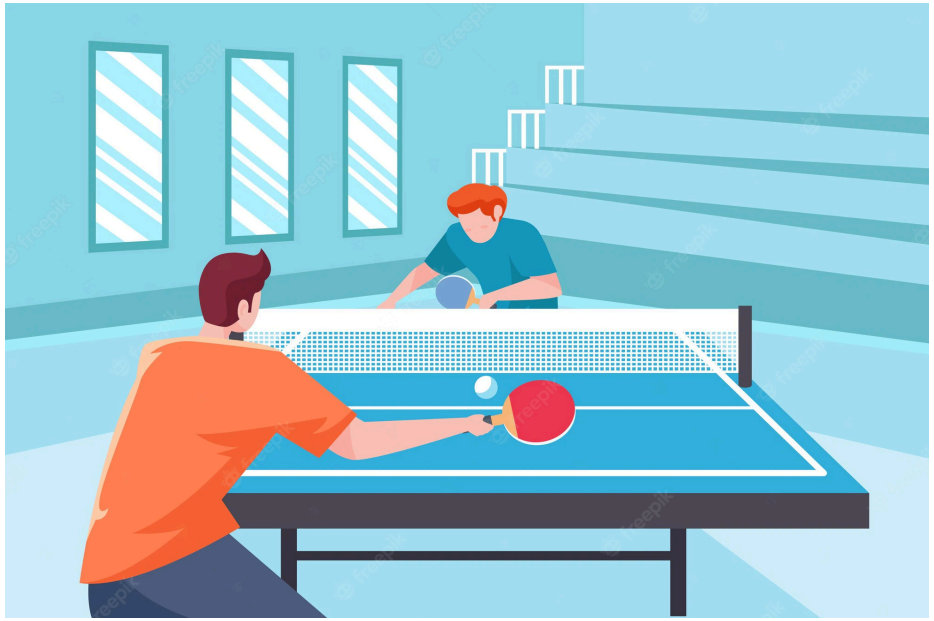
PCCR 2022

The Tournament Fixing Problem



Meirav Zehavi

Outline

Background

FPT algorithm for the TF problem

Kernel for the TF problem

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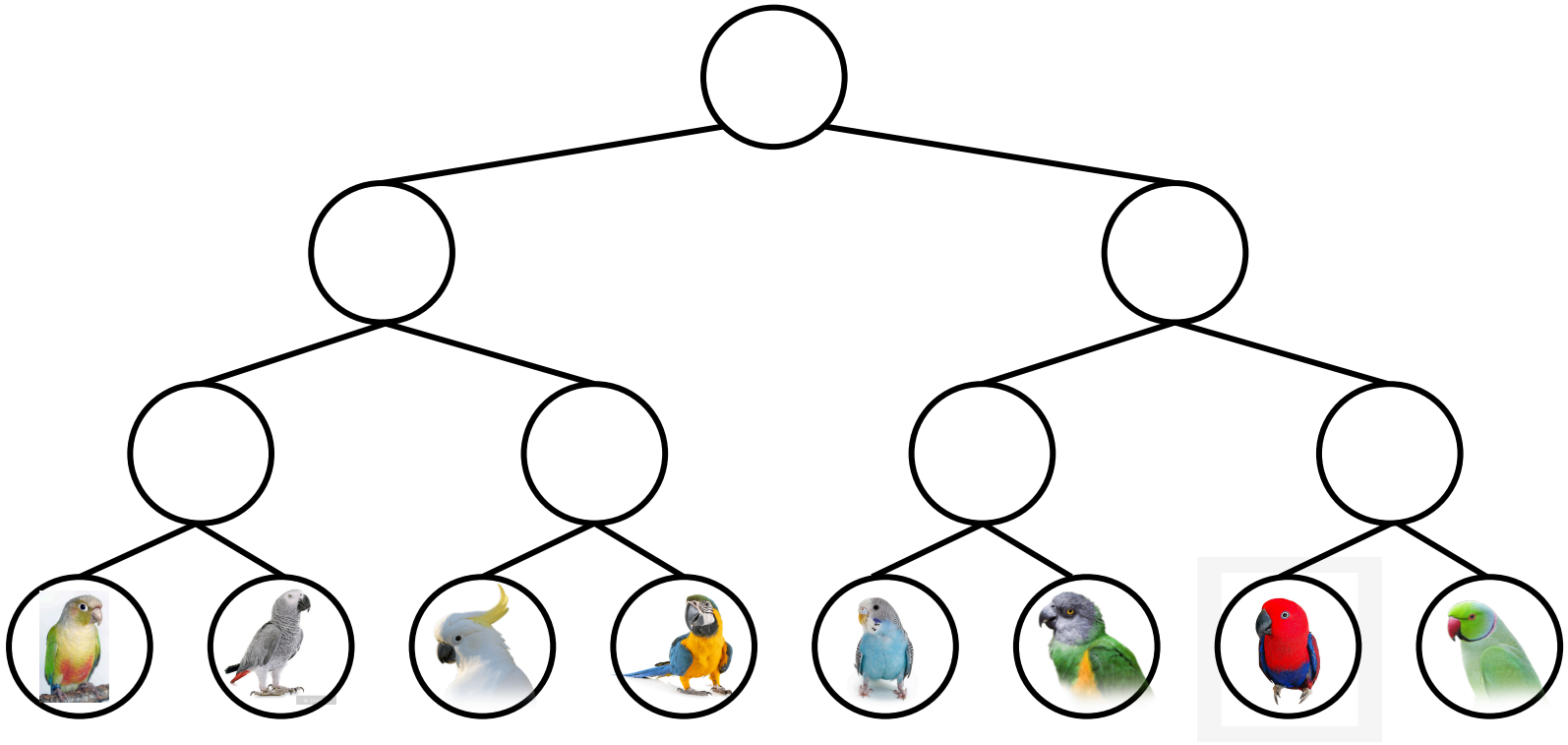
Single-Elimination (or Knockout) Tournament

Competition involving *n* (power of 2) players, conducted as follows:



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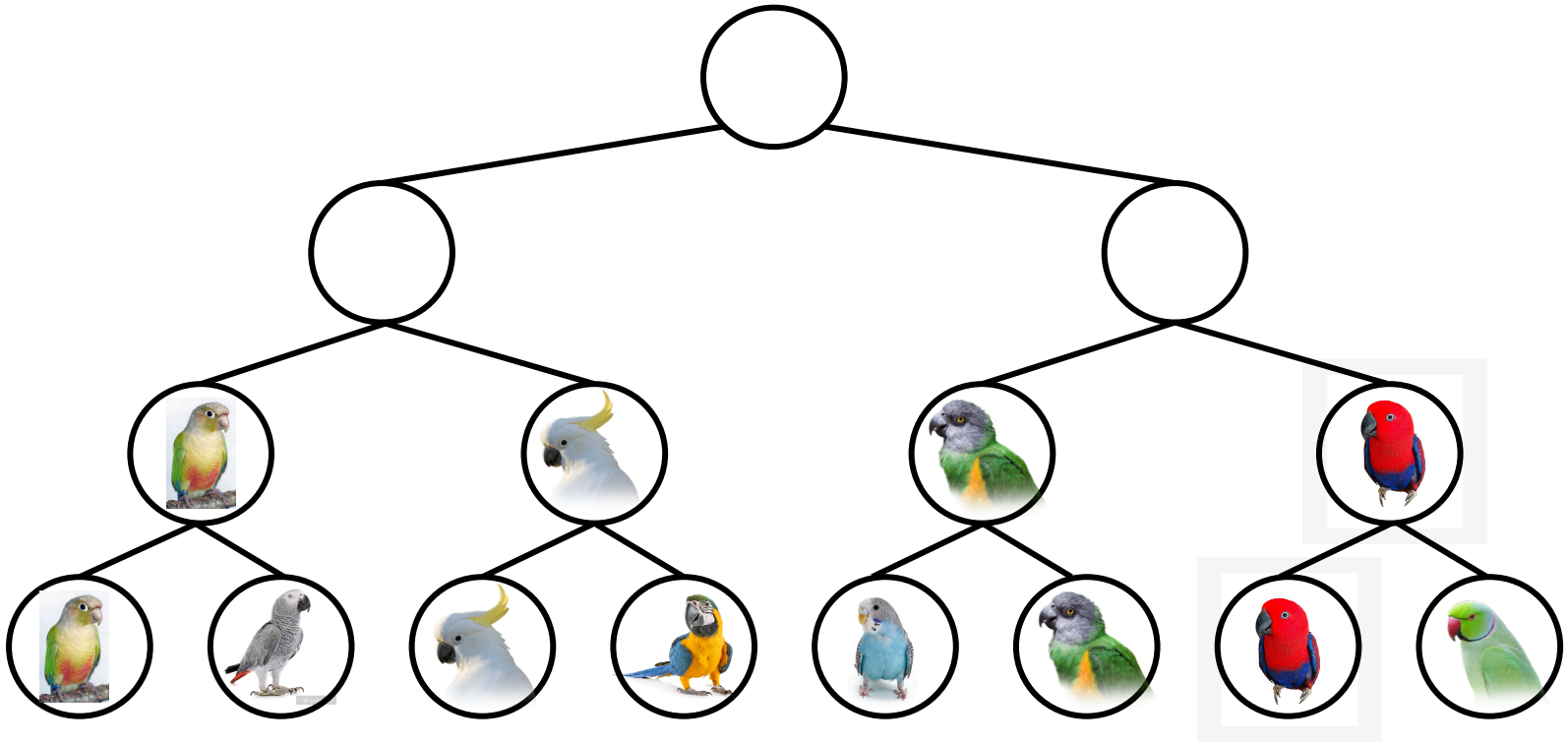

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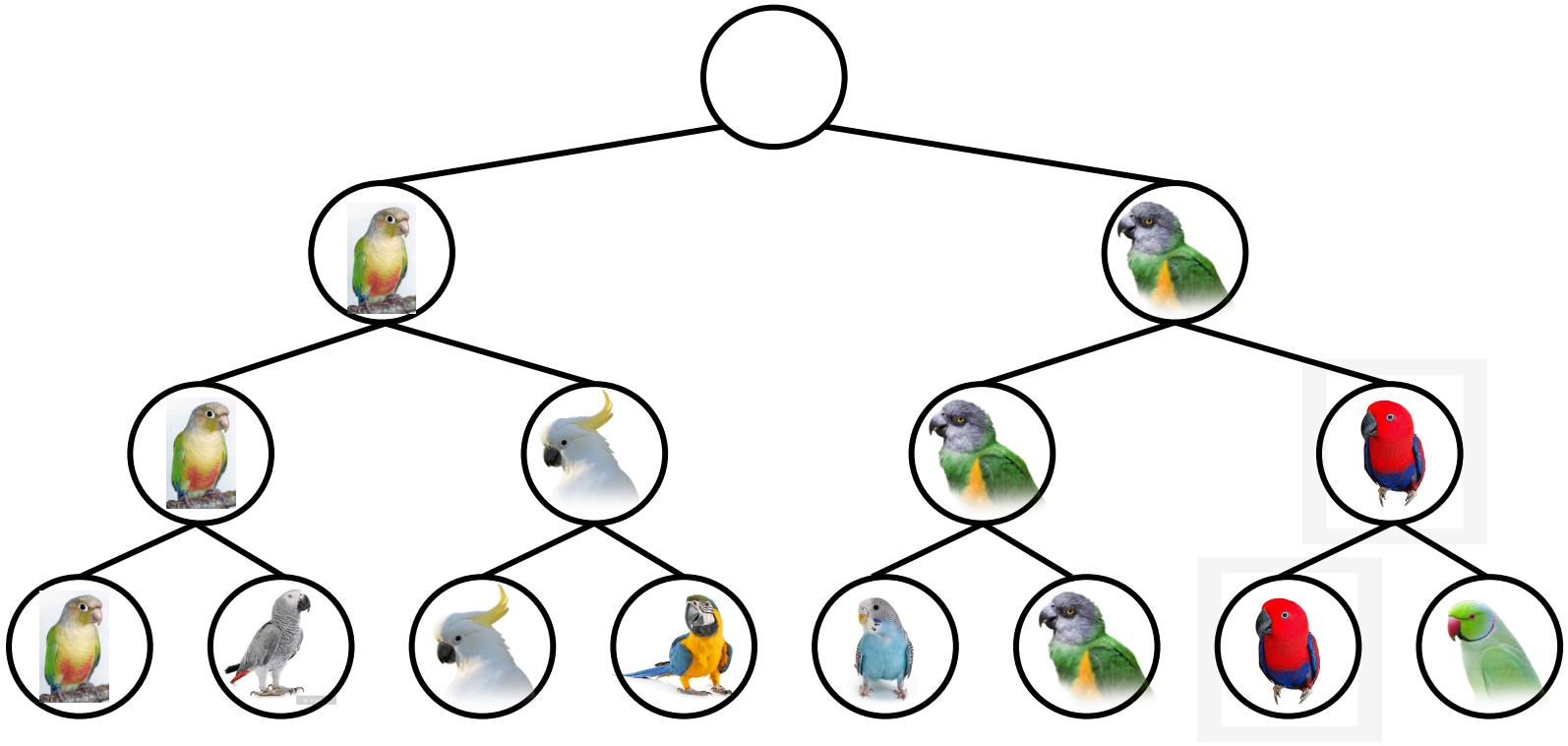

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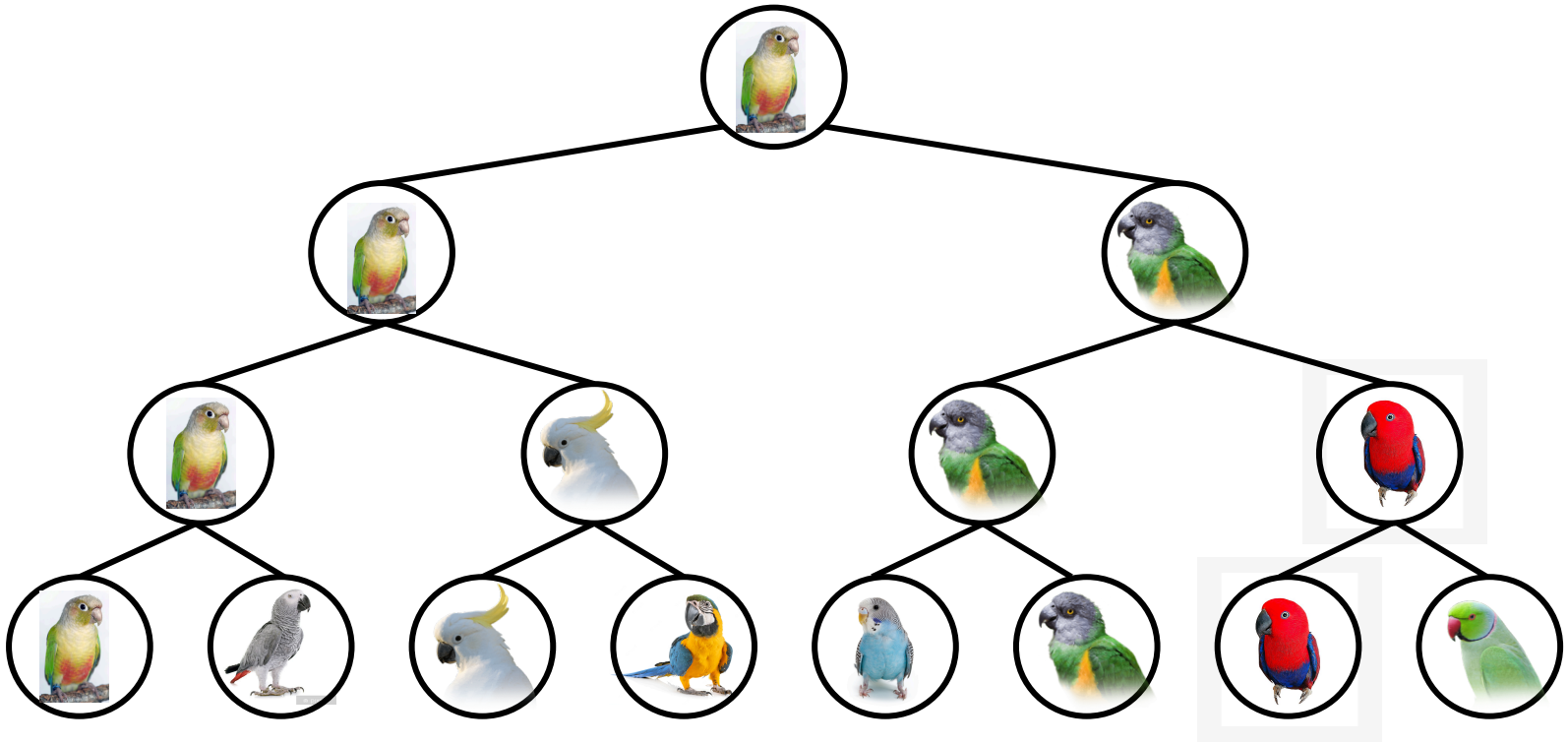
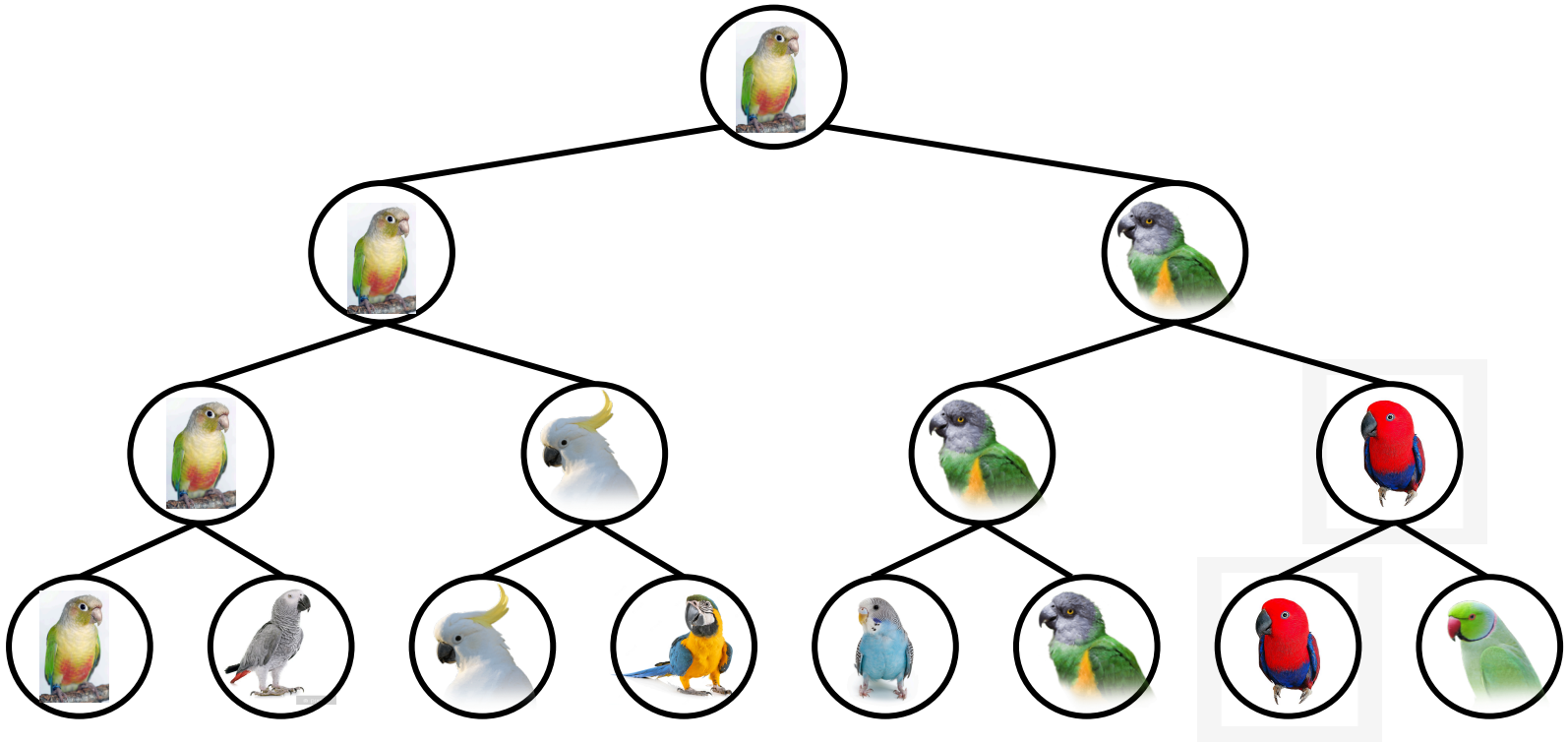


Background

Single-Elimination (or Knockout) Tournament

Competition involving *n* (power of 2) **players**, conducted as follows:

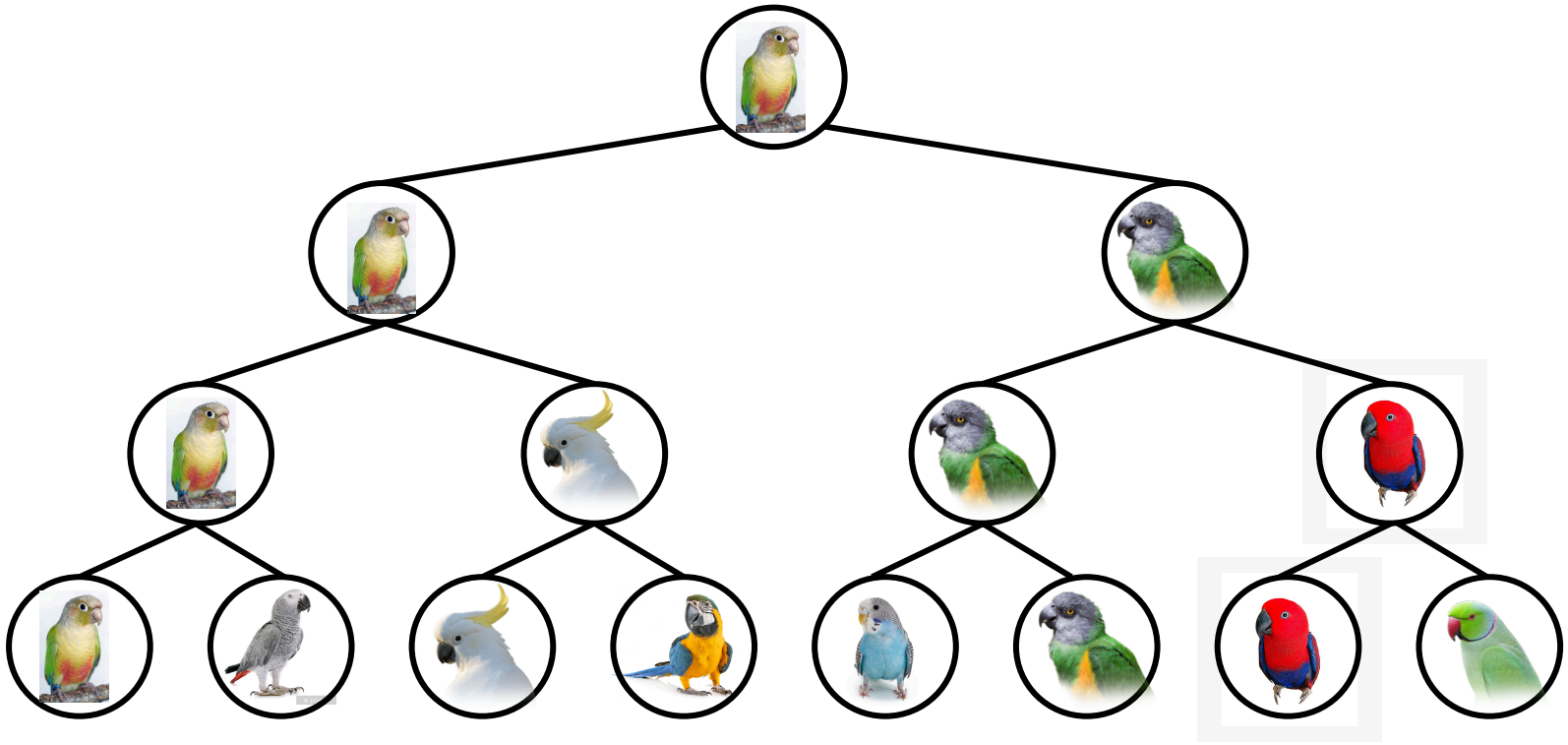
People, companies, ideas, … (Essentially, anything.)



Background

Single-Elimination (or Knockout) Tournament

Motivation: Competitions (sports, music, …), voting, webpage ranking, biological interactions, …



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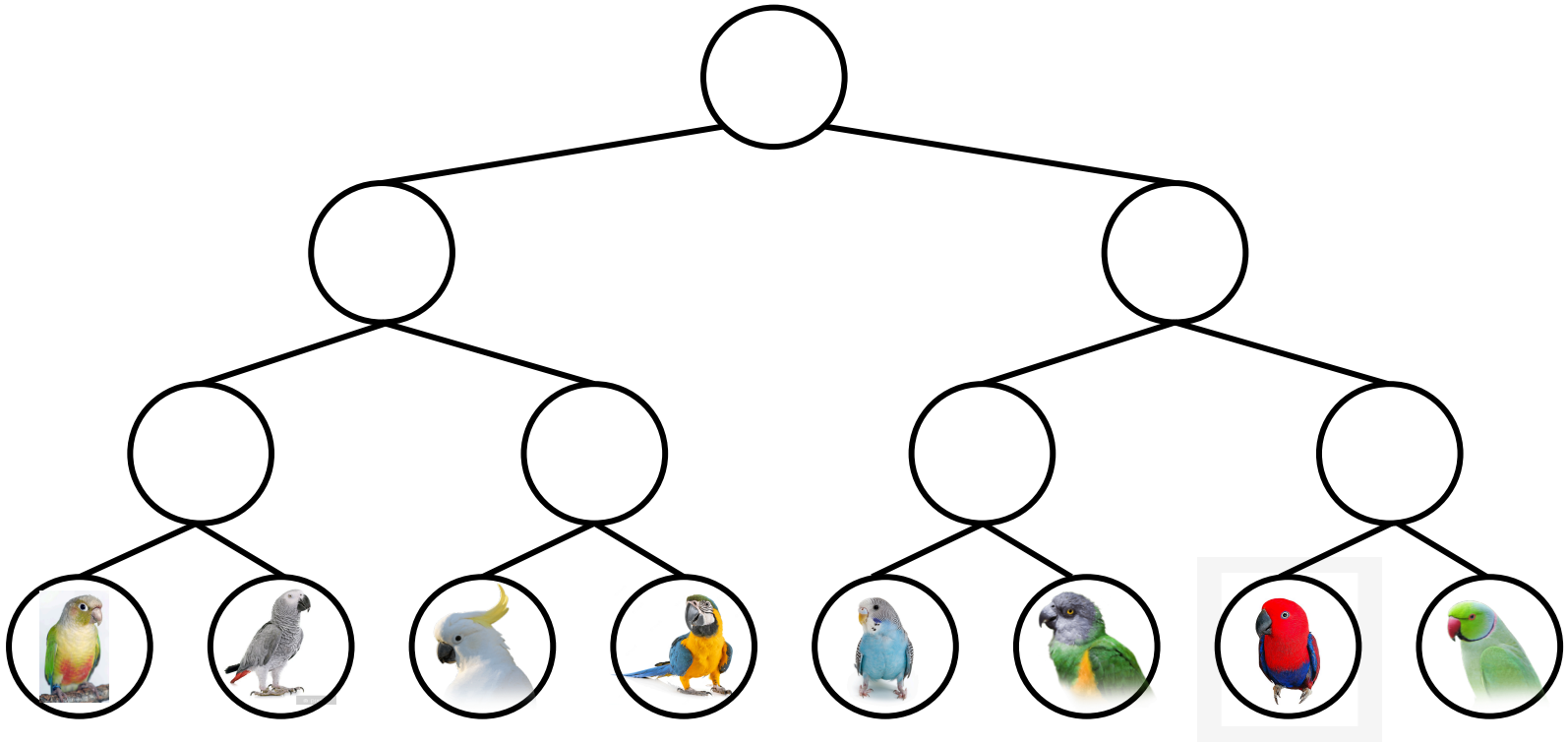
Single-Elimination (or Knockout) Tournament

Seeding (or bracket): Assignment of players to the leaves.



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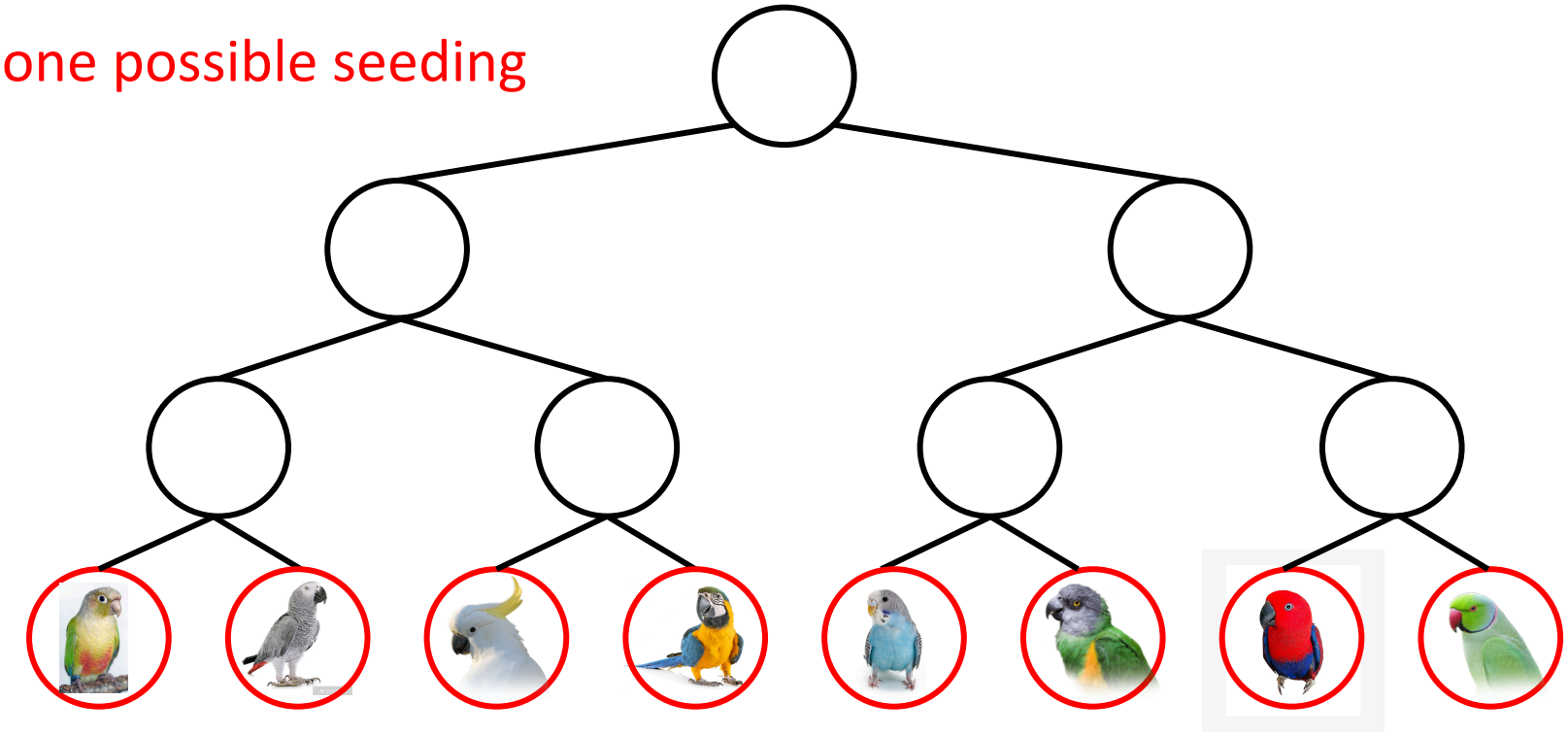
Single-Elimination (or Knockout) Tournament

Seeding (or bracket): Assignment of players to the leaves.

one possible seeding



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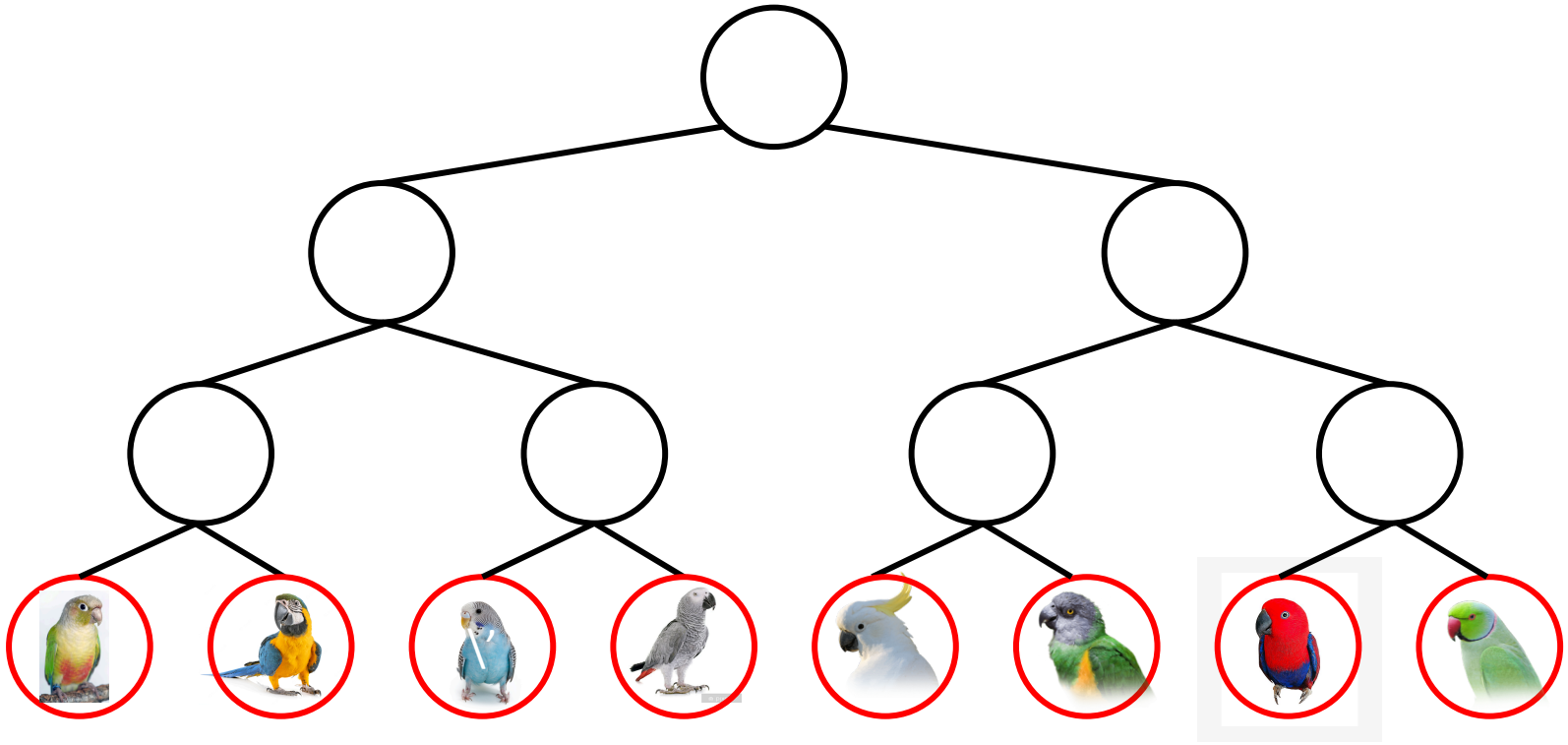
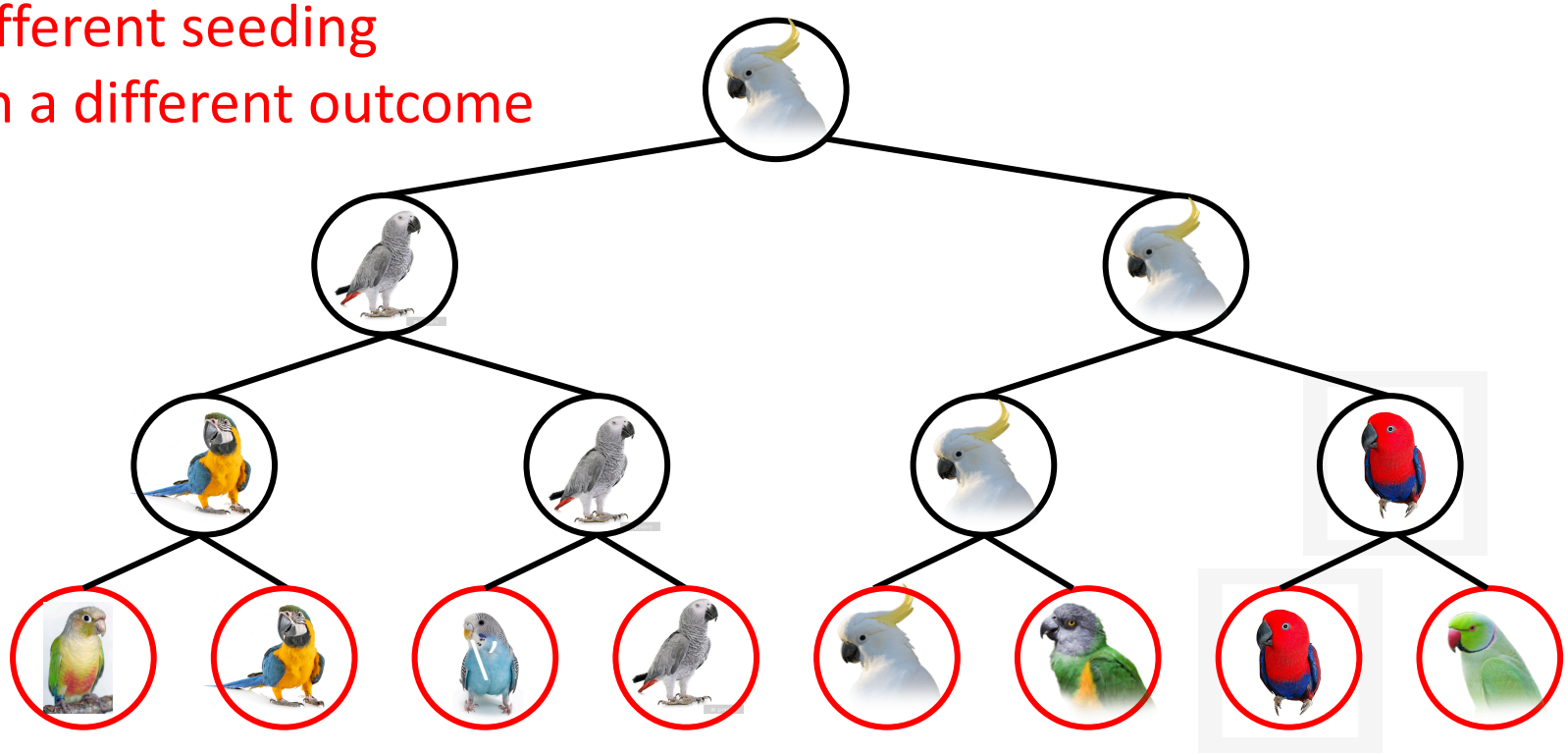


Background

Single-Elimination (or Knockout) Tournament

Seeding (or bracket): Assignment of players to the leaves.

a different seeding
with a different outcome



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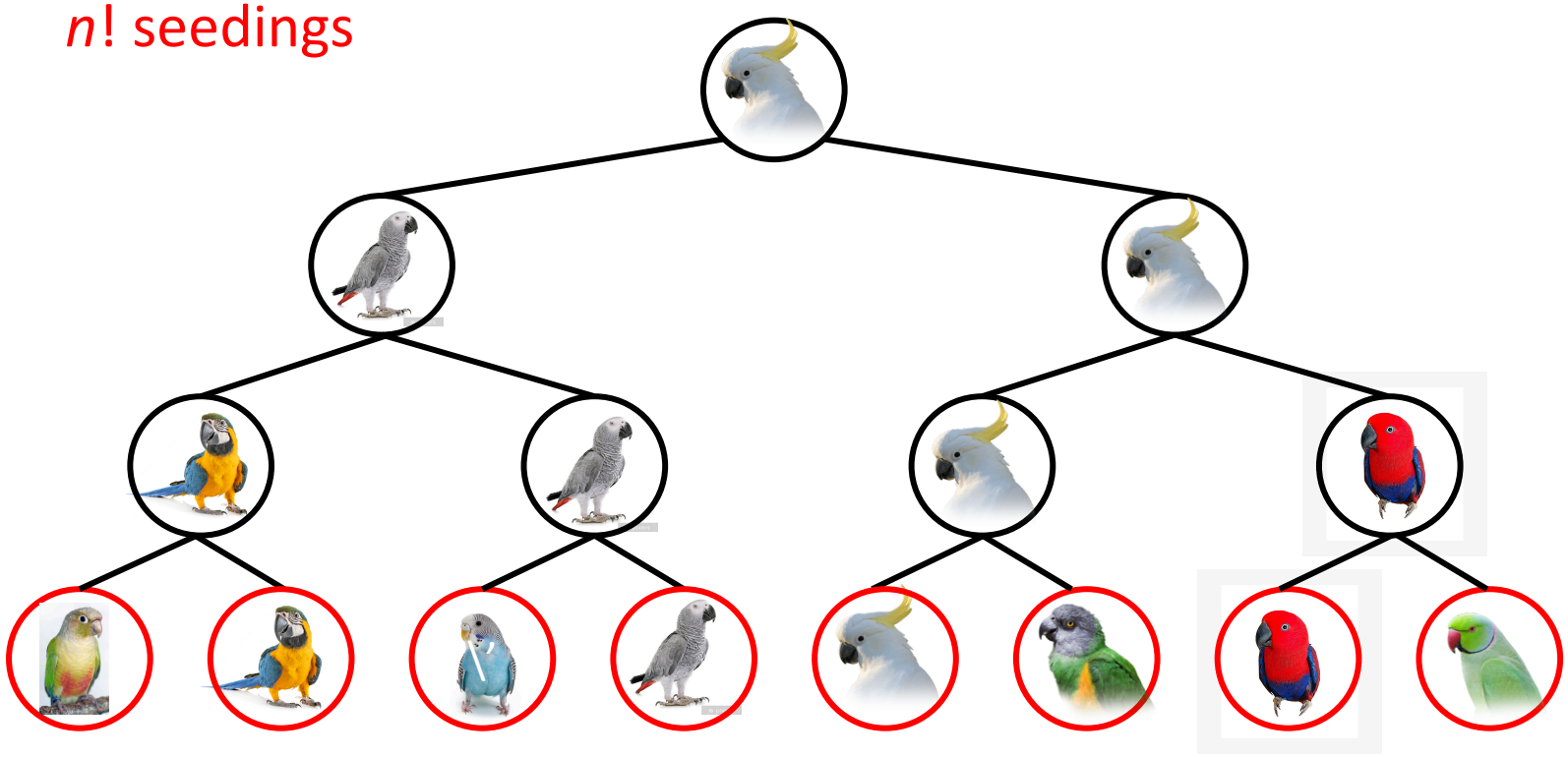
Single-Elimination (or Knockout) Tournament

Seeding (or bracket): Assignment of players to the leaves.

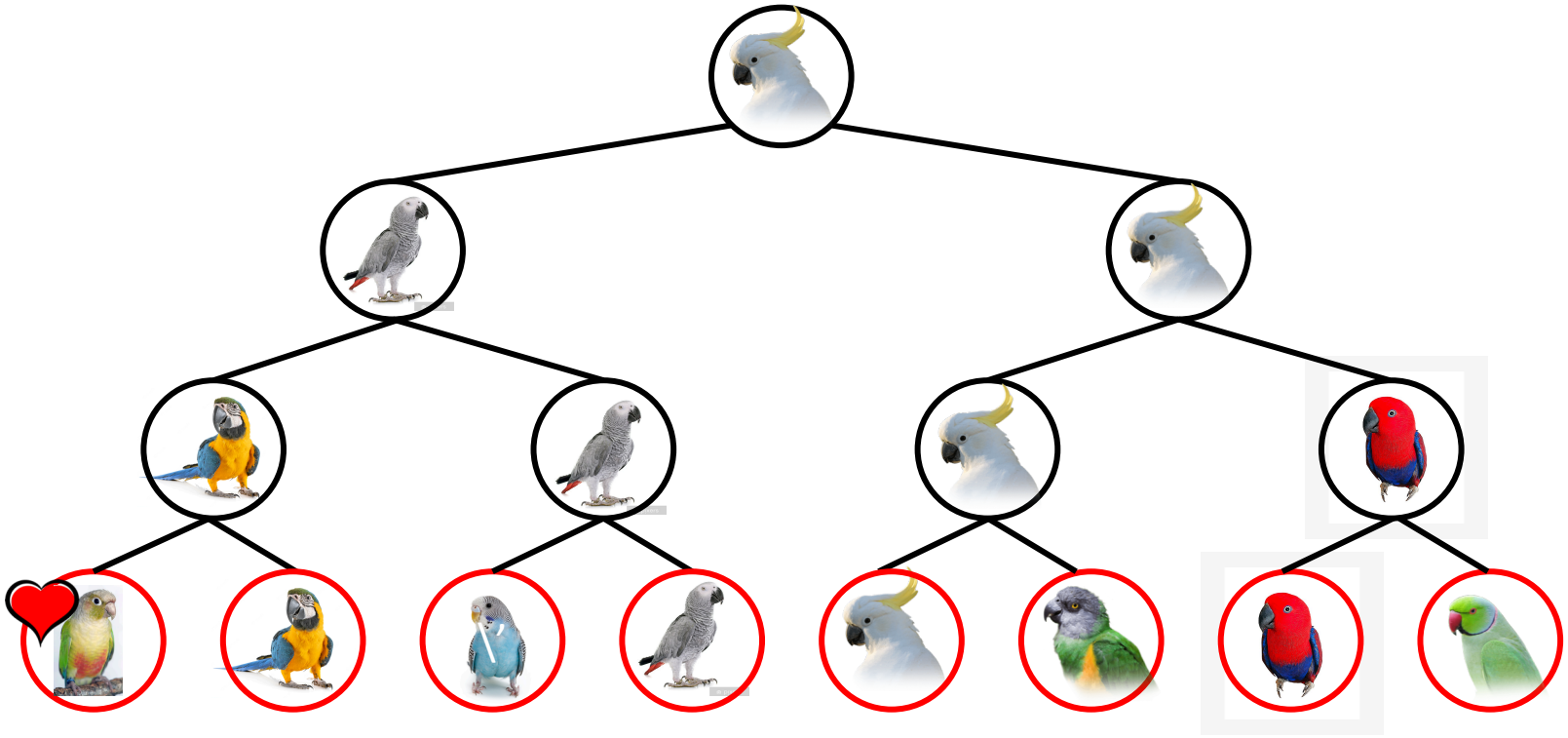
n! seedings



Background

Single-Elimination (or Knockout) Tournament

Objective: Make our favorite player win!



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Single-Elimination (or Knockout) Tournament

Prediction of outcomes of matches: Tournament graph on the set of players.

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Single-Elimination (or Knockout) Tournament

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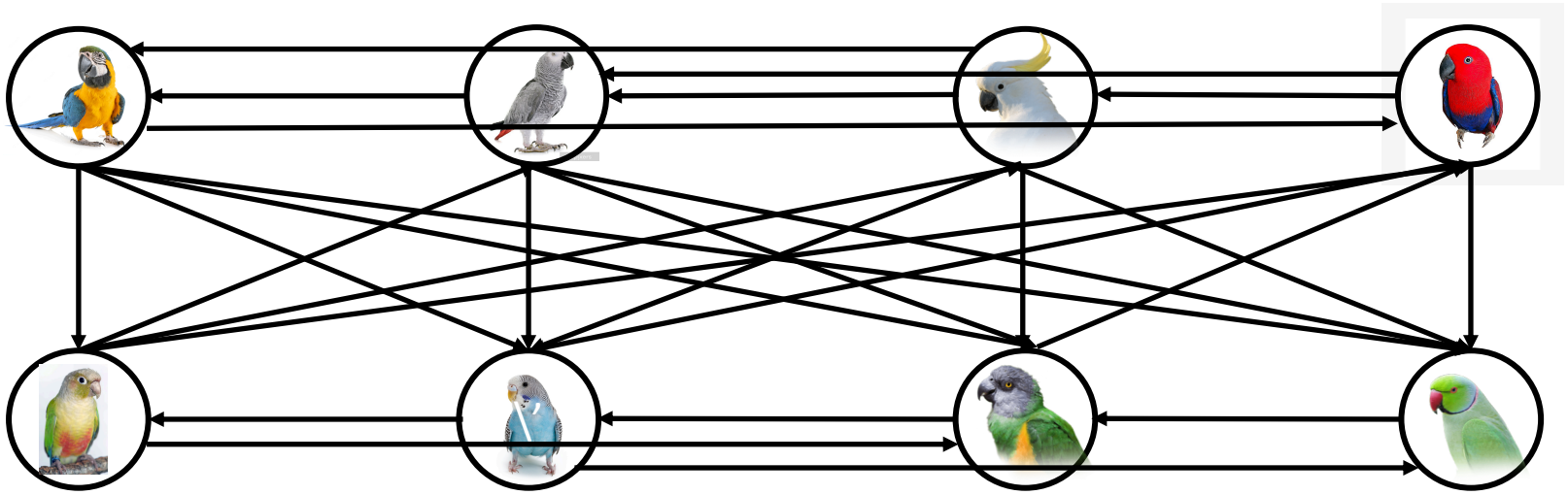
Digraph containing, for every pair of vertices *u,v*, either (*u,v*) or (*v,u*).



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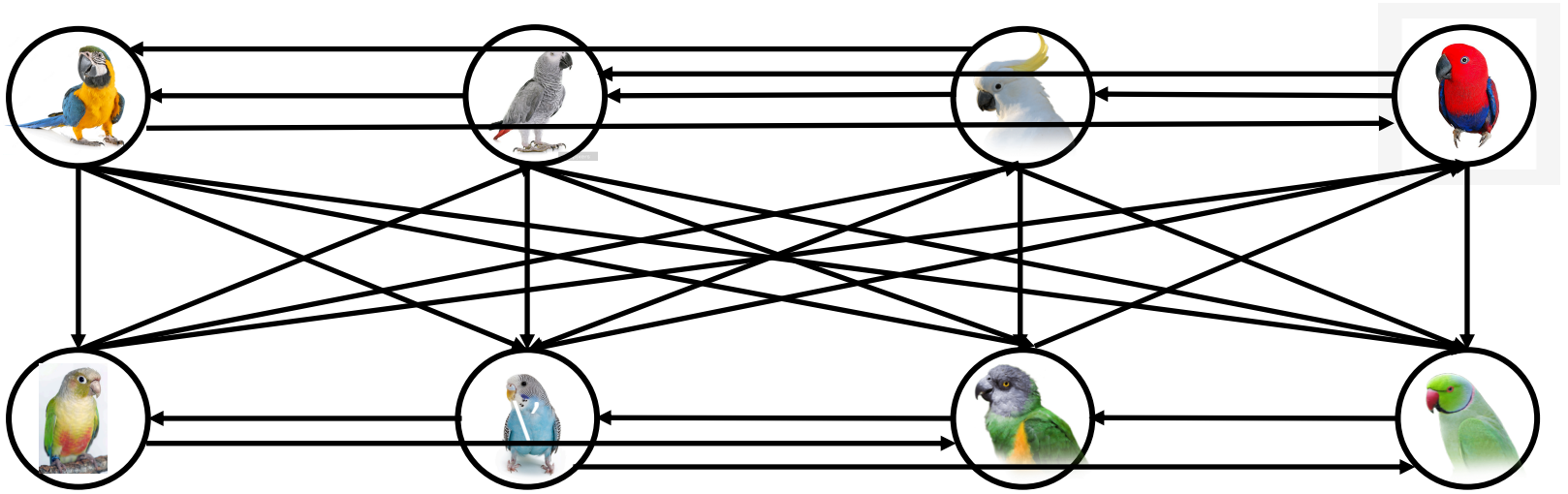
Single-Elimination (or Knockout) Tournament

Prediction of outcomes of matches: Tournament graph on the set of players.

Certainty? Probabilistic models (not discussed here).



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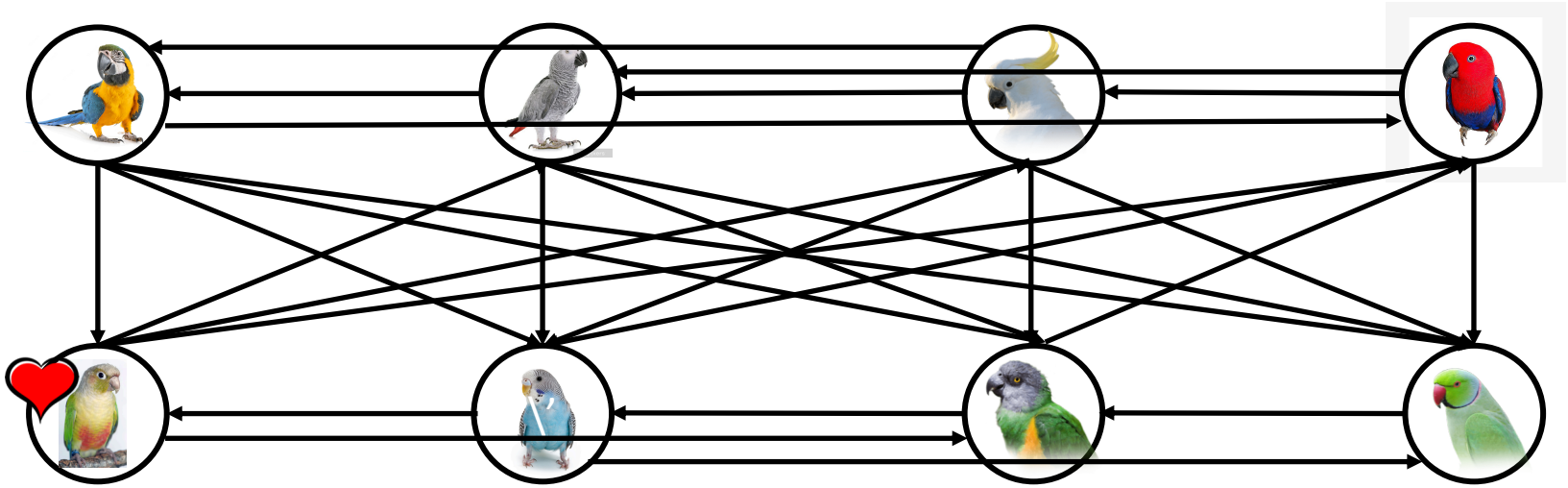
Tournament Fixing Problem

Input: Tournament graph on the set of players; favorite player *w*.



Question: Does there exist a seeding that makes *w* win?

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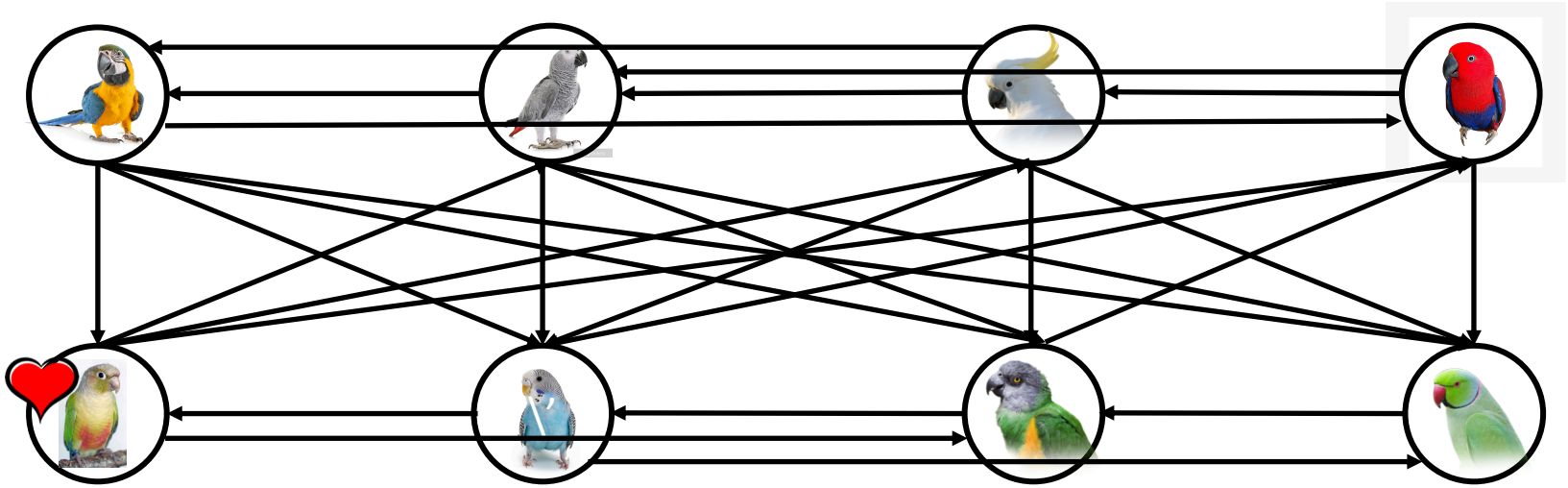

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Counting Version: How many seedings make *w* win?

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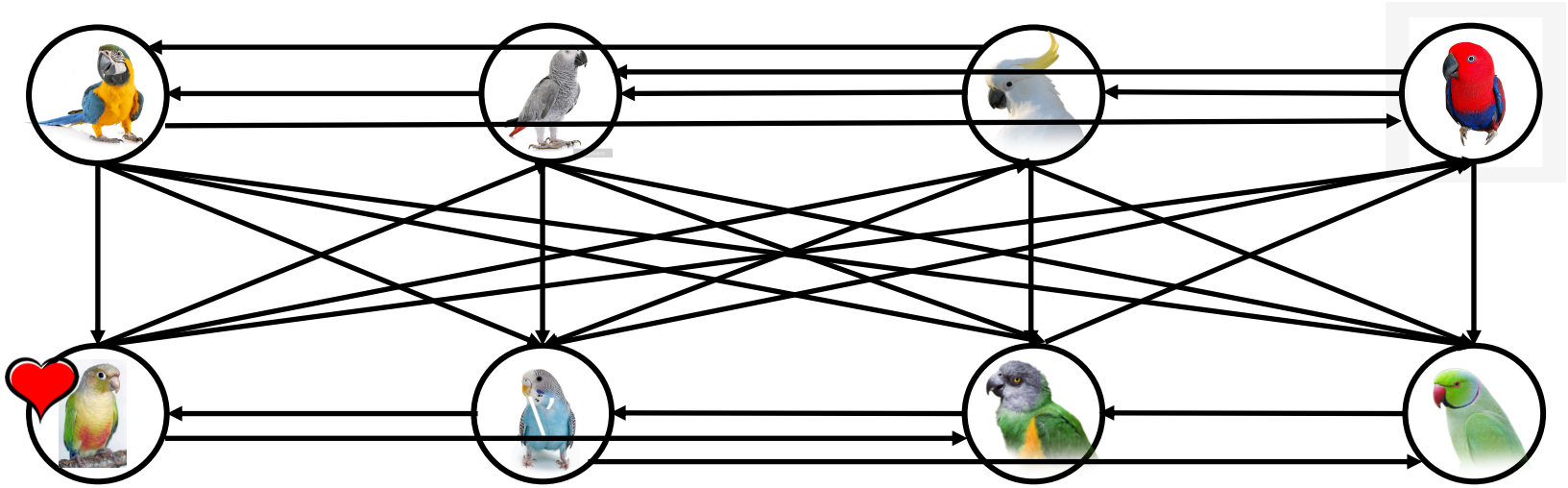

Question: Does there exist a seeding that makes *w* win?

Counting Version: How many seedings make *w* win?

Surveys: [Suksompong, IJCAI'21], [Williams, Handbook of Computational Social Choice, 2016]

Background

Tournament Fixing Problem: Complexity

Theorem. TFP is NP-hard. [AGMMSW, AAAI'14]

- TFP is NP-hard even if *w* is a "king" that beats at least *n*/4 other players. [KW, IJCAI'15]
- #TFP does not admit a fully polynomial-time randomized approximation scheme (FPRAS) unless RP=NP. [AGMMSW, AAAI'14]

For every player *p*, *w* beats *p* or *w* beats a player that beats *p*.

Background

Tournament Fixing Problem: Exact Exponential-Time Algorithms

Theorem. #TFP is solvable in $2^n n^{O(1)}$ time and space. [KW, IJCAI'15]

- Earlier time-space tradeoff by [AGMMSW, AAAI'14]: $O(2.83^n)$ time and $O(1.75^n)$ space, $4^{n+o(n)}$ time and polynomial-space.

Theorem. (Bribery) TFP is solvable in $2^n n^{O(1)}$ time and polynomial-space. [GRSZ, IJCAI'18a]

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Theorem. (Bribery) TFP is solvable in $2^n n^{O(1)}$ time and polynomial-space. [GRSZ, IJCAI'18a]

Question: Can TFP be solved in time $1.999^n n^{O(1)}$?

Background

Tournament Fixing Problem: Parameterized Complexity

Parameter *k*: Feedback arc set number (fas).

(The minimum number of arcs to remove from the tournament graph to obtain a DAG.)

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Tournament Fixing Problem: Parameterized Complexity

Parameter *k*: Feedback arc set number (fas).
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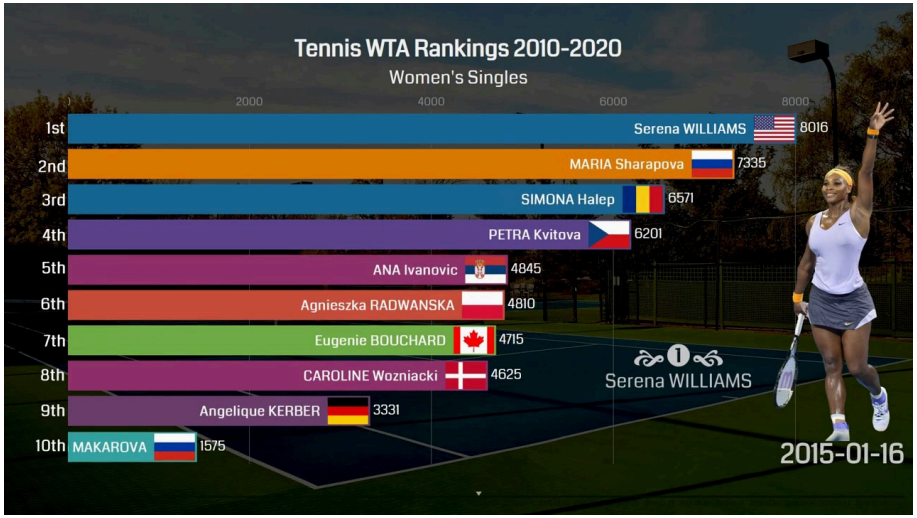
Motivation: We know of a (rough) ranking of the players.

- ``Expect'' only few occurrences of a player that beats a player ranked higher than it.



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Tournament Fixing Problem: Parameterized Complexity

Parameter *k*: Feedback arc set number (fas).
(The minimum number of arcs to remove from the tournament graph to obtain a DAG.)

Theorem. TFP is solvable in $2^{O(k\log k)}n^{O(1)}$ time. [GRSZ, IJCAI'18b]

- Previously: $2^{O(k^2\log k)}n^{O(1)}$ time. [RS, AAAI'17]

Theorem. TFP admits a polynomial kernel. [GRSZ, IJCAI'19]

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Question. Better (or best) running time? (**Subexponential?**)

Question. Is TFP in FPT parameterized by the feedback **vertex** set number?

Background

Tournament Fixing Problem: Some Structural Results

- ***w* is a ``superking'':** For every player *p*, *w* beats *p* or *w* beats at least $\log_2 n$ players that beat *p*. [W, AAAI'10]
- ***w* is a king of ``high-degree'':** A king with outdegree *d* that loses to fewer than *d* players with outdegree >*d*. [SW, IJCAI'11]

➢ Generalization by [KSW, AAAI'16]

- ***w* is a 3-king + other restrictions:** [KW, IJCAI'15; KSW, AAAI'16].

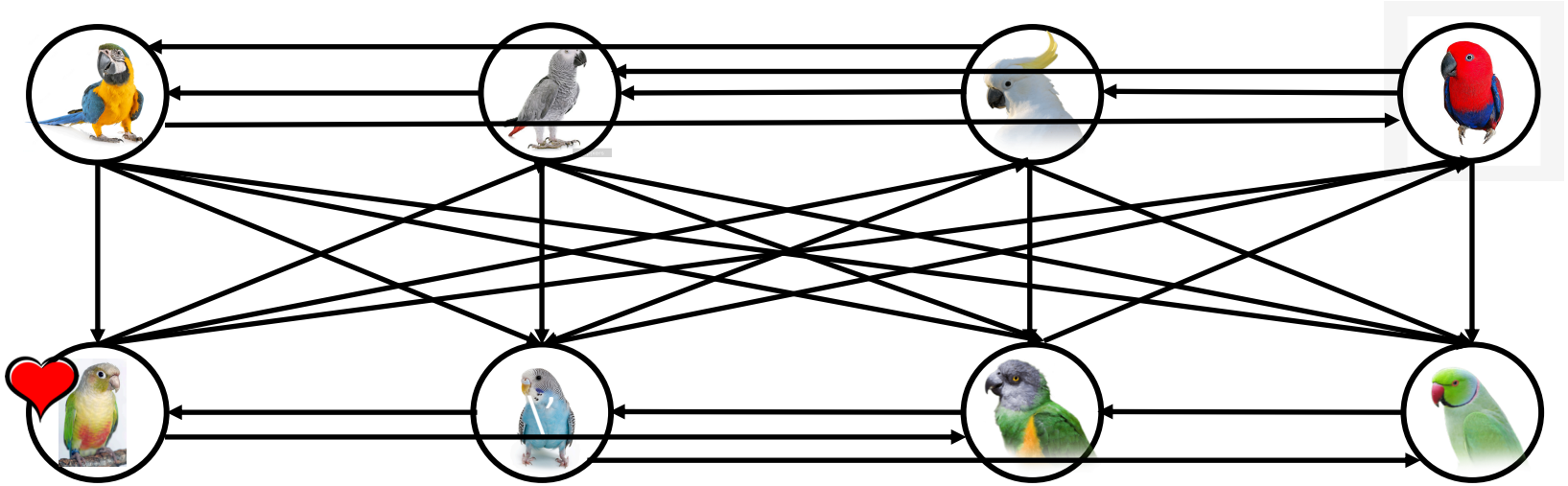
(A 3-king is a player that can reach any other player via a path of length 3 in the tournament graph.)

- **Note:** A 3-king may not be able to win even if it beats *n*-3 other players.

Background

Bribery Tournament Fixing Problem

Input: Tournament graph on the set of players; favorite player *w;* budget *b*.



Question: Does there exist a seeding such that, after changing the outcomes of at most *b* matches, *w* wins?

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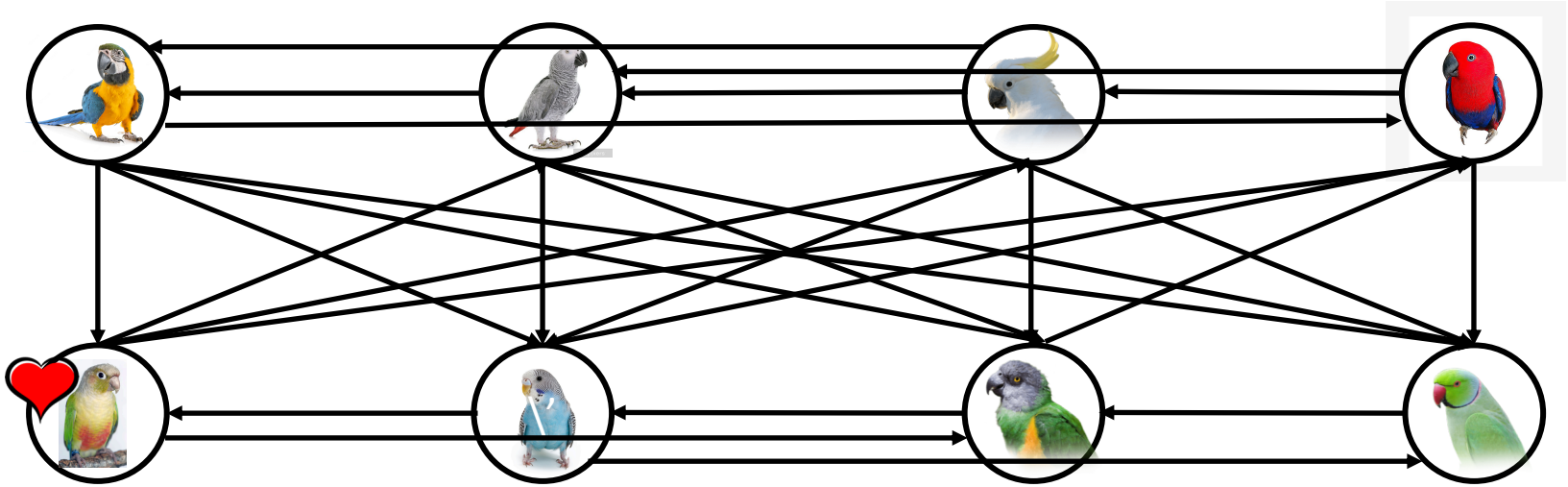

Question: Does there exist a seeding such that, after changing the outcomes of at most *b* matches, *w* wins?

Note: TFP is the special case of BTFP where *b*=0.

Background

Bribery Tournament Fixing Problem: Complexity

Corollary. BTFP is NP-hard when b=0.

Observation. BTFP is polynomial-time solvable when $b \geq \log_2 n$. (Output Yes.)

Theorem. For any fixed ε>0, BTFP is NP-hard when $b \leq (1- \varepsilon)\log_2 n$. [KW, IJCAI'15]

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Question(s). Non-trivial results for the parameter $\log_2 n$-b?

Background

Bribery Tournament Fixing Problem: Exponential-Time Algorithms and Parameterized Complexity

Parameter *k*: Feedback arc set number (fas).

Not harder than TFP:

Theorem. I. BTFP is solvable in $2^n n^{O(1)}$ time and polynomial-space.
II. BTFP is solvable in $2^{O(k\log k)} n^{O(1)}$ time. [GRSZ, IJCAI'18b]

In the manuscript: $2^{O(k^2\log k)} n^{O(1)}$.

Background

Bribery Tournament Fixing Problem: Structural Results

Theorem (informal). If there is a solution, then there is one manipulating only edges $\{w,v\}$ where v belongs to an "elite club". [GRSZ, IJCAI'18b]

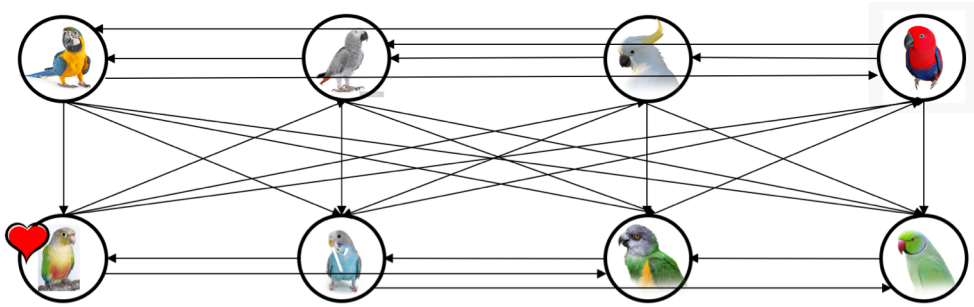
Also in [GRSZ, IJCAI'18b]:

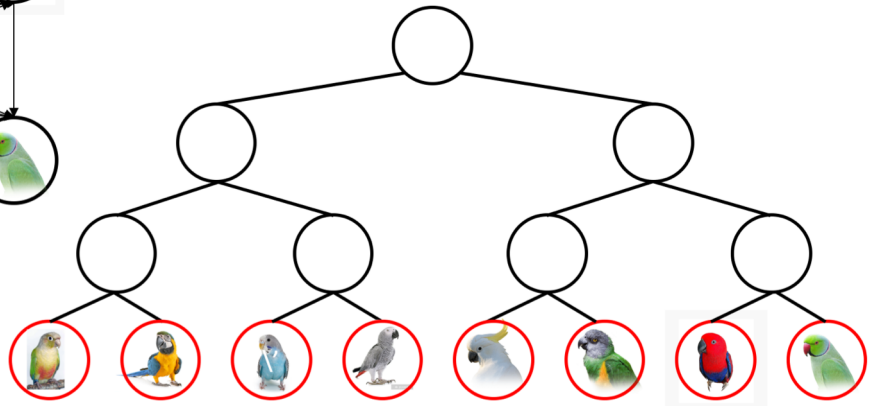
- Characterization of Yes-instances. (Yields a simple formula for DAGs.)
- ``Obfuscation'' operations.

Background

Bribery Tournament ~~Fixing~~ Problem

Input: Tournament graph on the set of players; favorite player *w;* budget *b*; seeding.





Question: Is it possible to change the outcomes of at most *b* matches so that *w* wins?

Background

Bribery Tournament ~~Fixing~~ Problem: Complexity

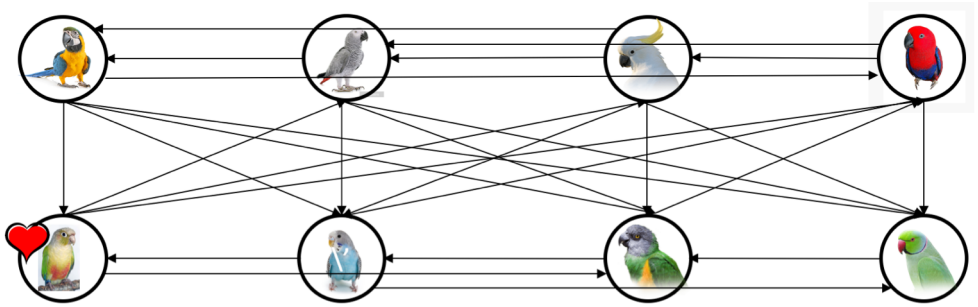
Theorem. BTP is solvable in polynomial time.[RW, ADT'09]

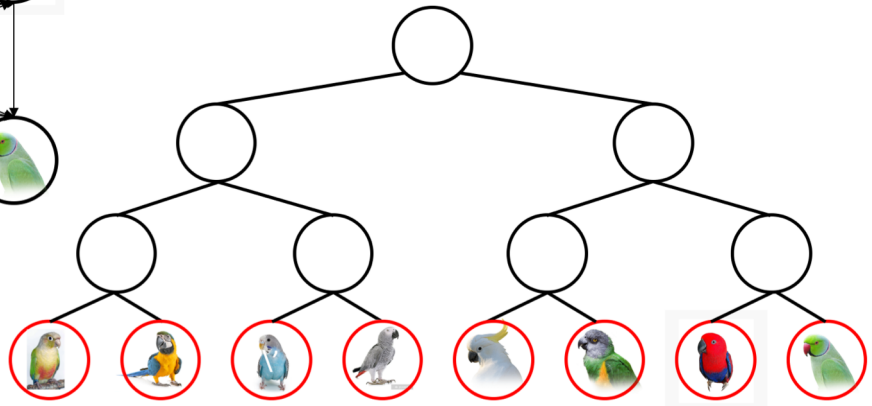
- **Also:** The ``destructive'' version (prevent *w* from winning) is also solvable in polynomial time.

Background

``Closest'' Tournament Fixing Problem

Input: Tournament graph on the set of players; favorite player *w;* seeding; distance threshold *d*.





Question: Does there exist a seeding of Hamming distance at most *d* from the given seeding that makes *w* win?

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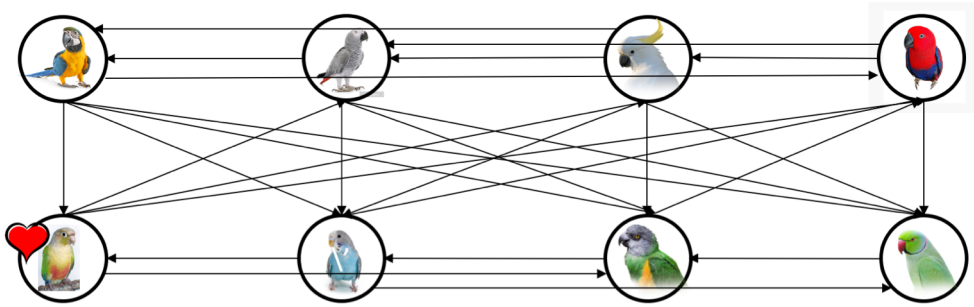
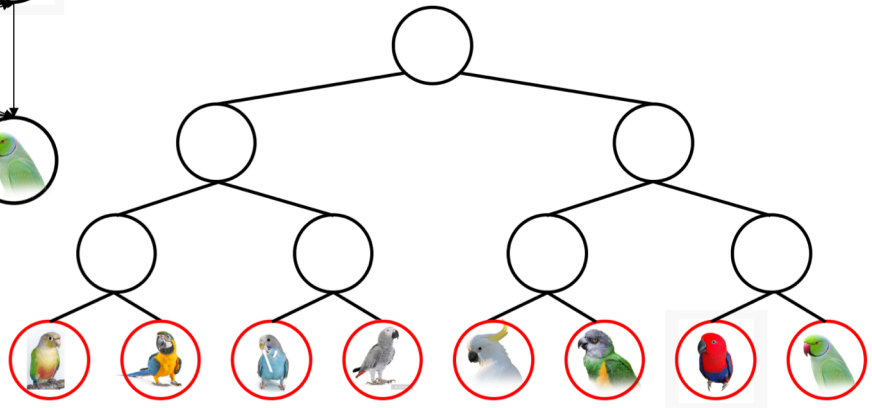




Question: Does there exist a seeding of Hamming distance at most *d* from the given seeding that makes *w* win?

Note: TFP is the special case of CTFP where *d*=*n*.

Background

``Closest'' Tournament Fixing Problem: Motivation

Why is the input seeding attractive? Might have been already agreed upon, might be based on customs or traditions, …

Remark 1: Cyclic shifts. ABCDEFGH is similar to HABCDEFG, but all starting matches are different.

Remark 2: Extreme case. All starting matches are the same in ABCDEFGH and BADCFEHG, but the Hamming distance is *n*.

- Still, the order of each pair might be important. (Who plays first, who plays at what court, …).

Background

Bribery Tournament Fixing Problem: Complexity & Algorithms

Corollary. CTFP is NP-hard when $d=n$.

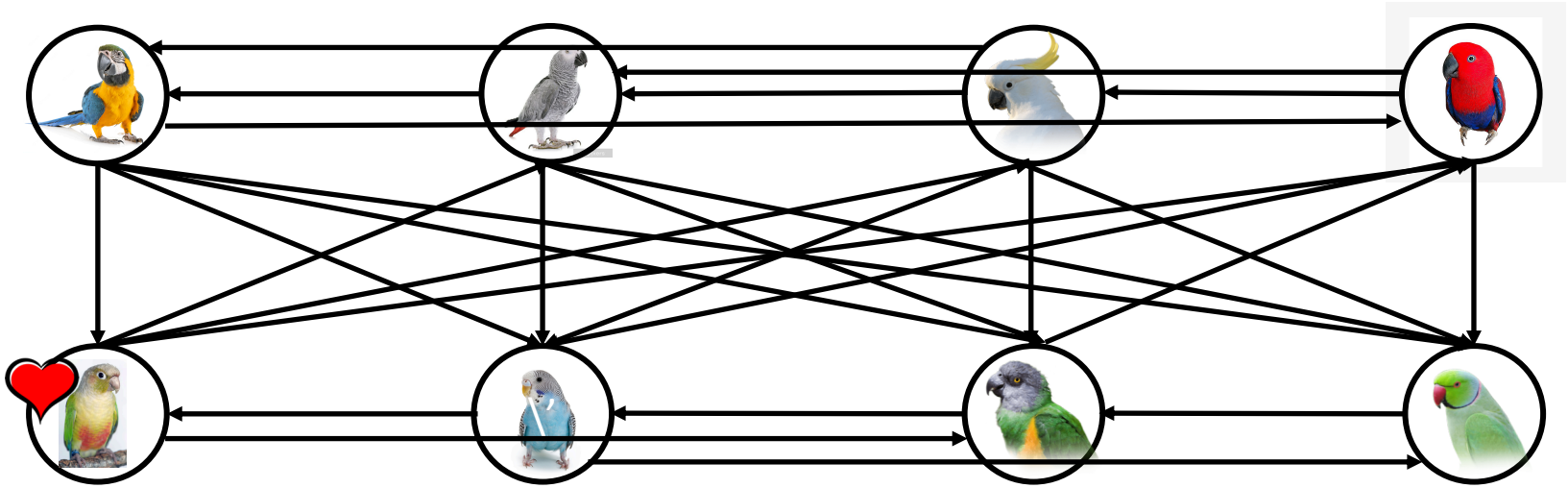
Theorem. CTFP is W[1]-hard when parameterized by d.
[GSZ, Manuscript]

Theorem. CTFP is solvable in $2^{O(n)}$ time. [GSZ, Manuscript]

Background

k-Wins Tournament Fixing Problem

Input: Tournament graph on the set of players; favorite player *w*; *k*.



Question: Does there exist a seeding that makes *w* win at least *k* matches?

Note: TFP is the special case of WTFP where $k=\log_2 n$.

Background

k-Wins Tournament Fixing Problem: Complexity & Algorithms

Corollary. WTFP is NP-hard when $k=\log_2 n$.

Theorem. WTFP is FPT when parameterized by t, the maximum number of players that a player beats.
[ACS, Annals of Mathematics and Artificial Intelligence, 2017]

- Also in [ACS, …'17]: Generalizations of WTFP. One of them is proved to be W[1]-hard wrt k.

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Question. Is WTFP W[1]-hard or in FPT wrt k?

Background

Remark. Knockout tournament is only one out of several formats of competition that were studied in the literature.

Round-Robin Tournament. Every player plays against every other player. Usually, the winner(s) is the player that has won the largest number of matches.

The Fair Non-Eliminating Tournament Problem. Given the infrastructure of the tournament, and the rankings of the players (multiset of integers), determine if there exists an assignment of the players to the vertices so that the sum of the rankings of the neighbors of every vertex is the same.

- Analyzed the parameterized complexity of this problem wrt several parameters (*tw, fvs, vc,* Δ*,* α) and combinations thereof. [GZ, AAMAS'21]

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Kernel for the TF problem

FPT Algorithm for TFP

Theorem. TFP is solvable in $2^{O(k \log k)} n^{O(1)}$ time. [GRSZ, IJCAI'18b]

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Phase I: Guessing

First, compute an fas F of D:

Option 1: Branch in $3^k n^{O(1)}$ time, or use a known $2^{O(\sqrt{k})}n^{O(1)}$-time algorithm.

Option 2: Use a polynomial-time 3-approximation algorithm, or use a known PTAS.

FPT Algorithm for TFP

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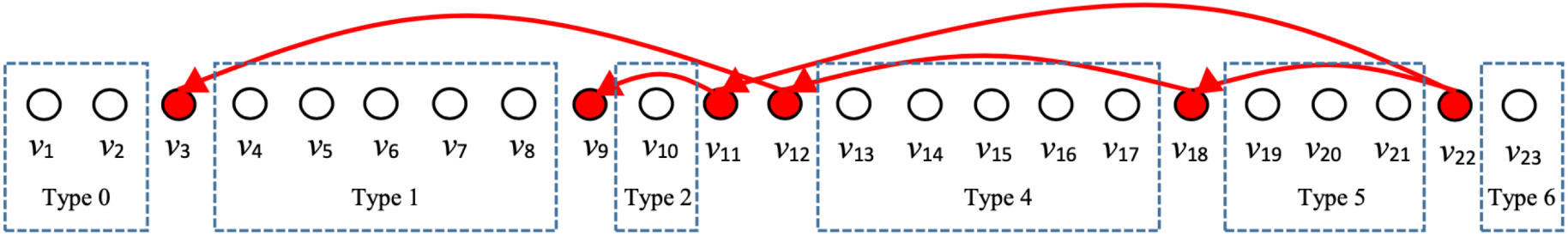
Why is an fas useful?



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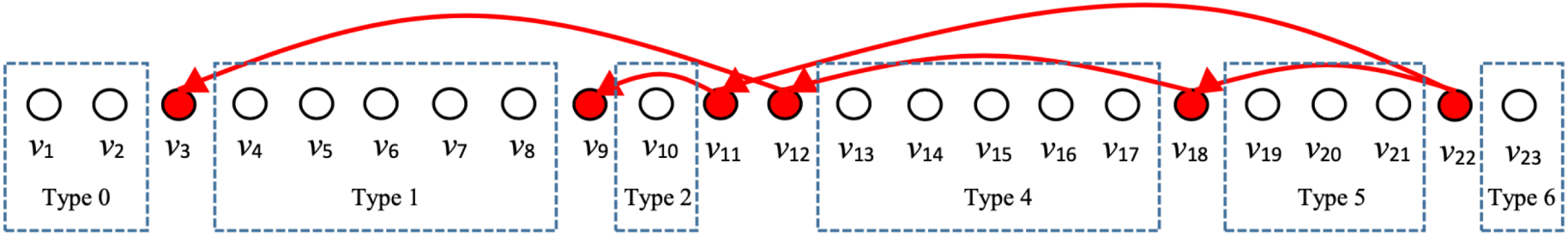


For simplicity, suppose that w belongs to V_F.

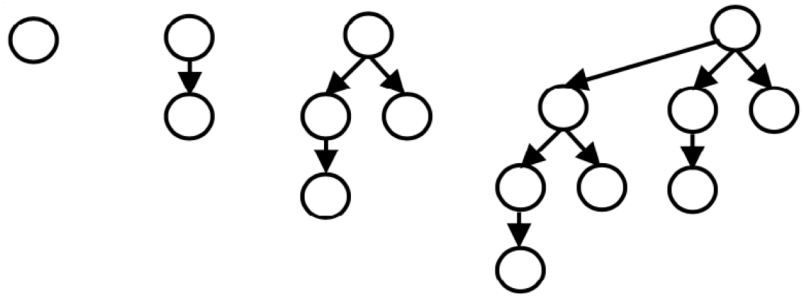
FPT Algorithm for TFP

Phase I: Guessing

Unlabeled Binomial Arborescence (UBA). Rooted tree B, defined recursively:

- A single vertex.
- Given two disjoint UBAs of equals size, B_v and B_u, the addition of an arc from v to u results in a BA rooted at v.

Given a digraph D such that $V(B)=V(D)$, we say that B is a (labeled) spanning binomial arborescence (SBA) of D.



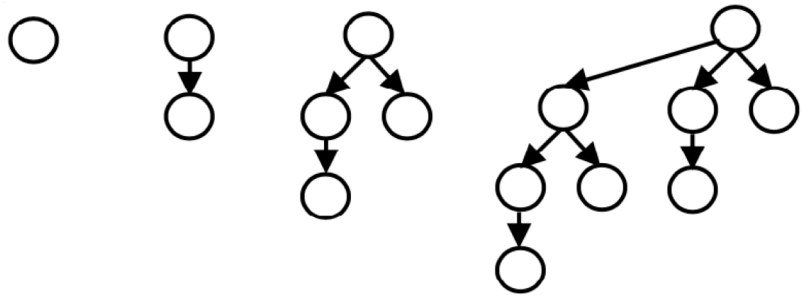
FPT Algorithm for TFP

Phase I: Guessing

Unlabeled Binomial Arborescence (UBA). Rooted tree B, defined recursively:

- A single vertex.
- Given two disjoint UBAs of equals size, B_v and B_u, the addition of an arc from v to u results in a BA rooted at v.

Given a digraph D such that $V(B)=V(D)$, we say that B is a (labeled) spanning binomial arborescence (SBA) of D.



Theorem. (D,w) is a Yes-instance iff D has an SBA rooted at w. [W, AAAI'10]

FPT Algorithm for TFP

Phase I: Guessing

Objective. Guess the relative position of the vertex set of F, V_F, in the (unknown) SBA B rooted at w.

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Folklore. $|\text{LCA}(M)| \leq 2|M|$, and every connected component of T-LCA(M) has at most two neighbors in LCA(M).

FPT Algorithm for TFP

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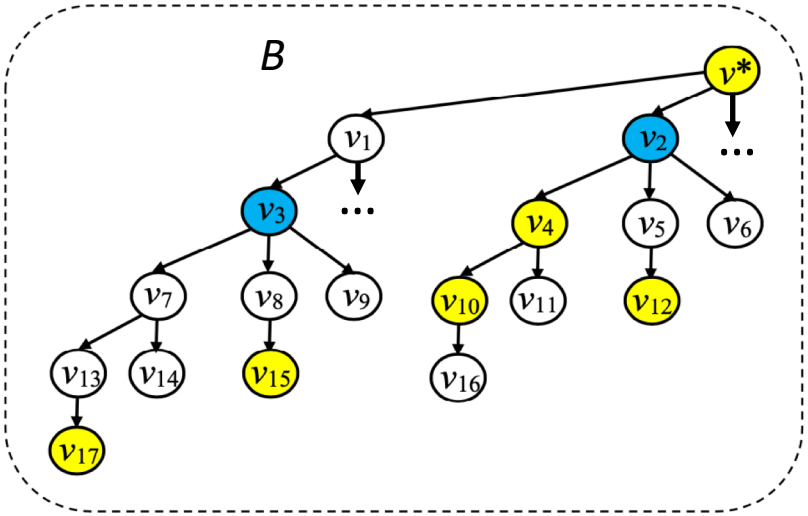

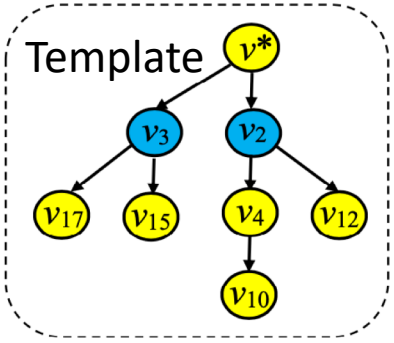

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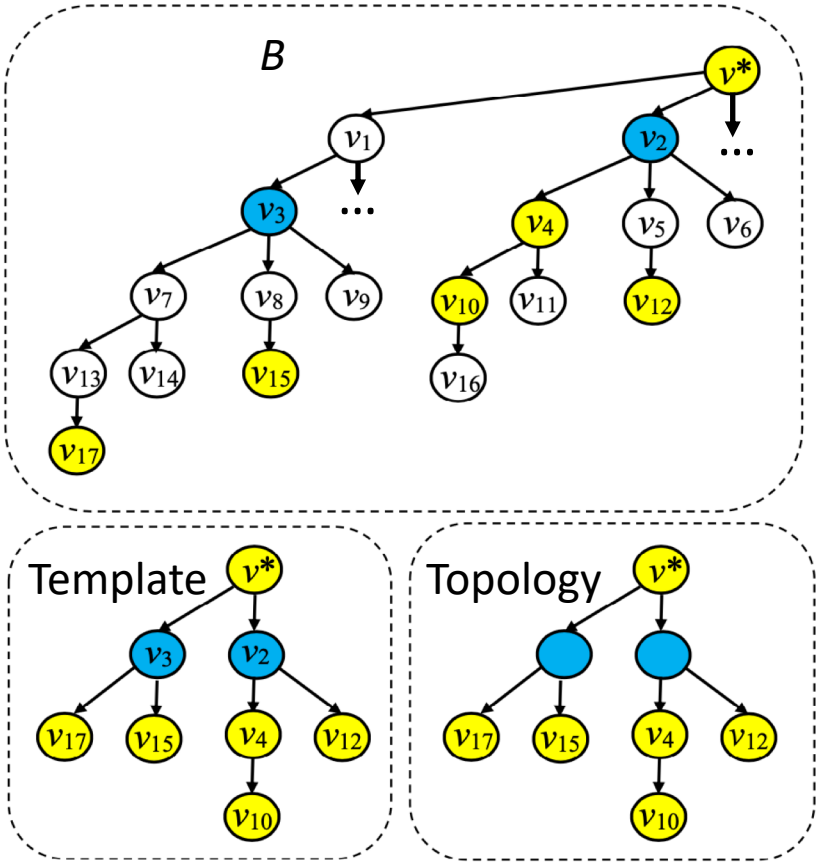

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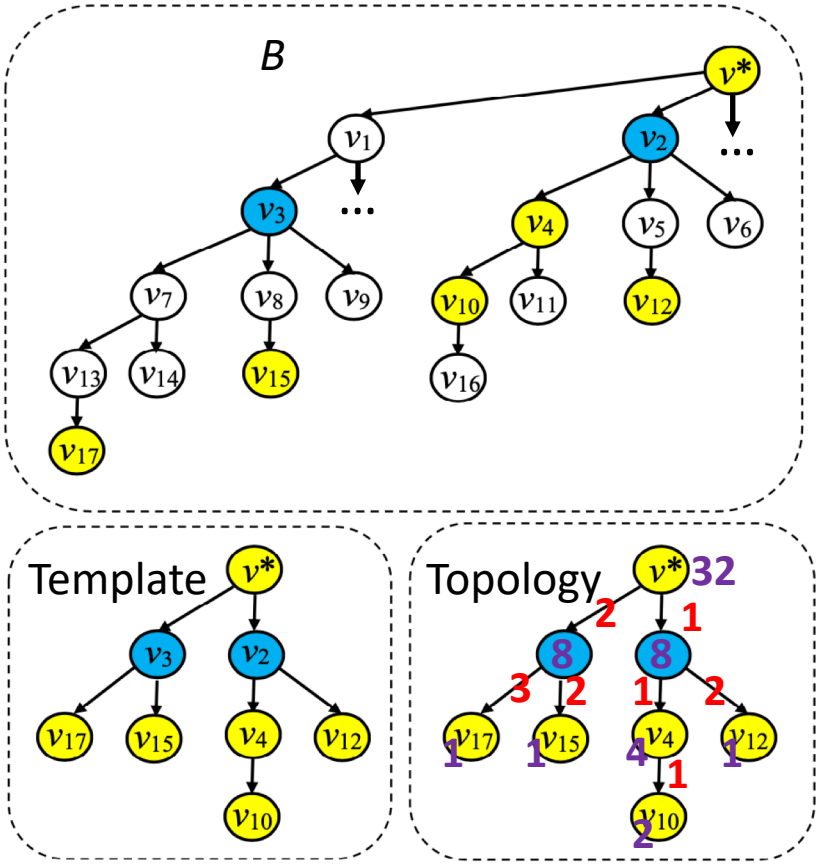
Lengths+Sizes. For every edge: assign the **length** of its path in B. For every vertex: assign the **size** of its subtree in B.



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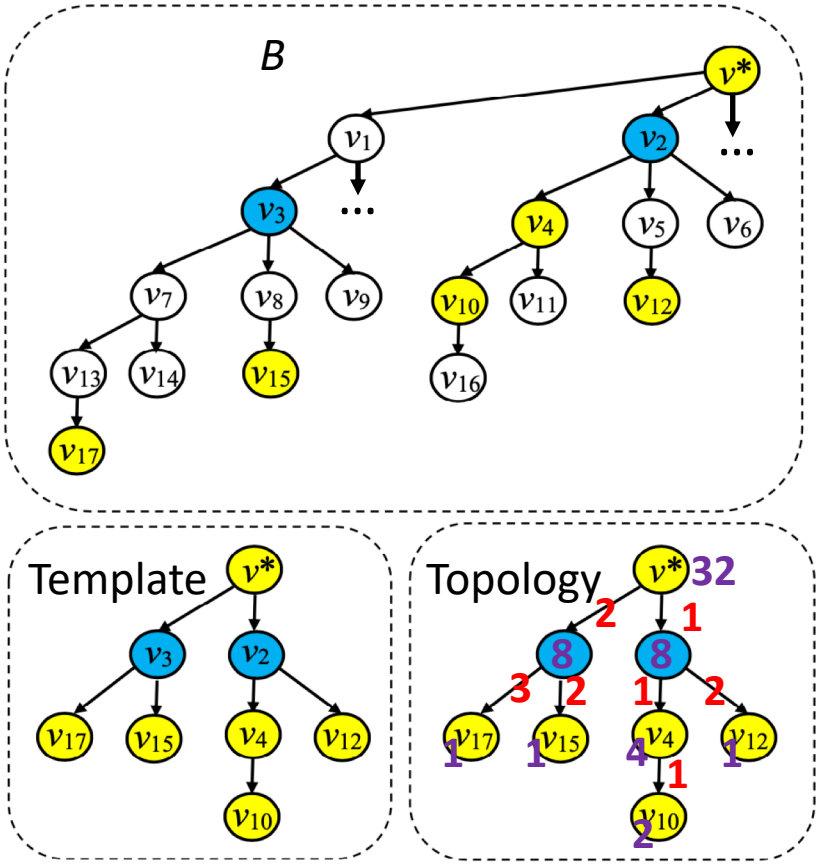

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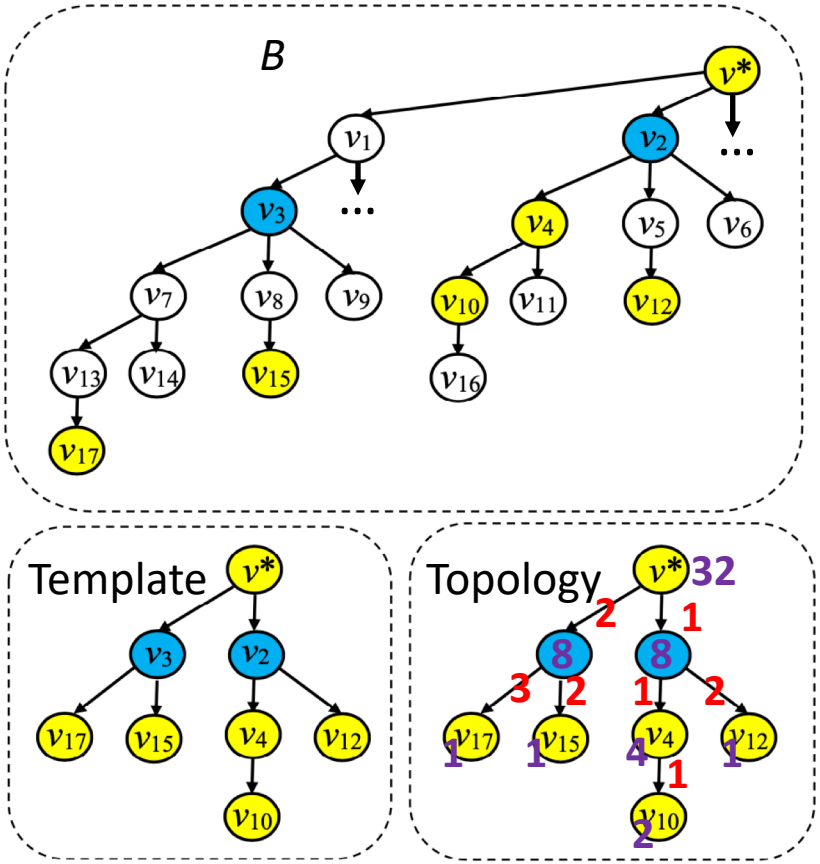
Algorithm. Guess the labeled topology.

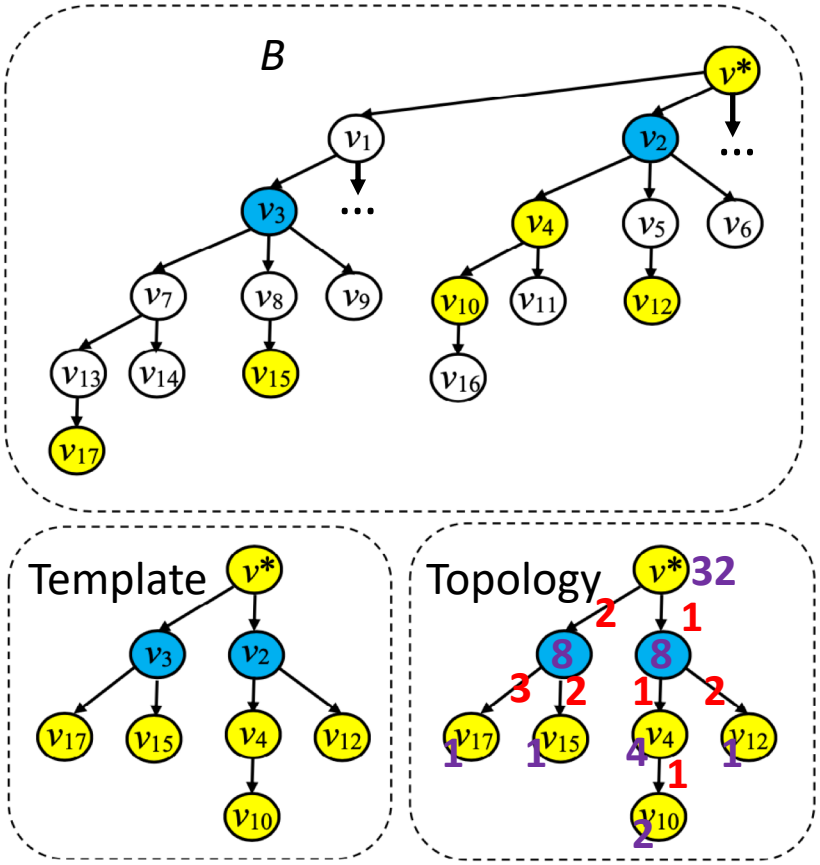
Only $k^{O(k)}$ options! (Suppose log$n$<k.)



FPT Algorithm for TFP

Phase II: Verify Realizability

When is a guess realizable? There exists an unlabeled BA to which we can map the vertices of the topology in compliance with the lengths + sizes.



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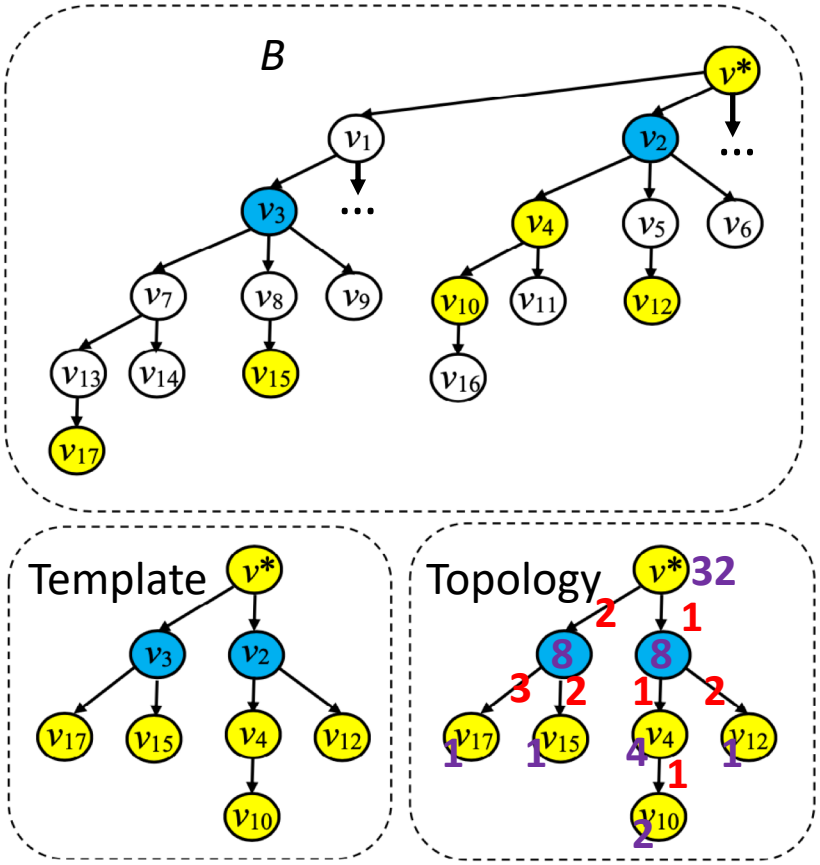

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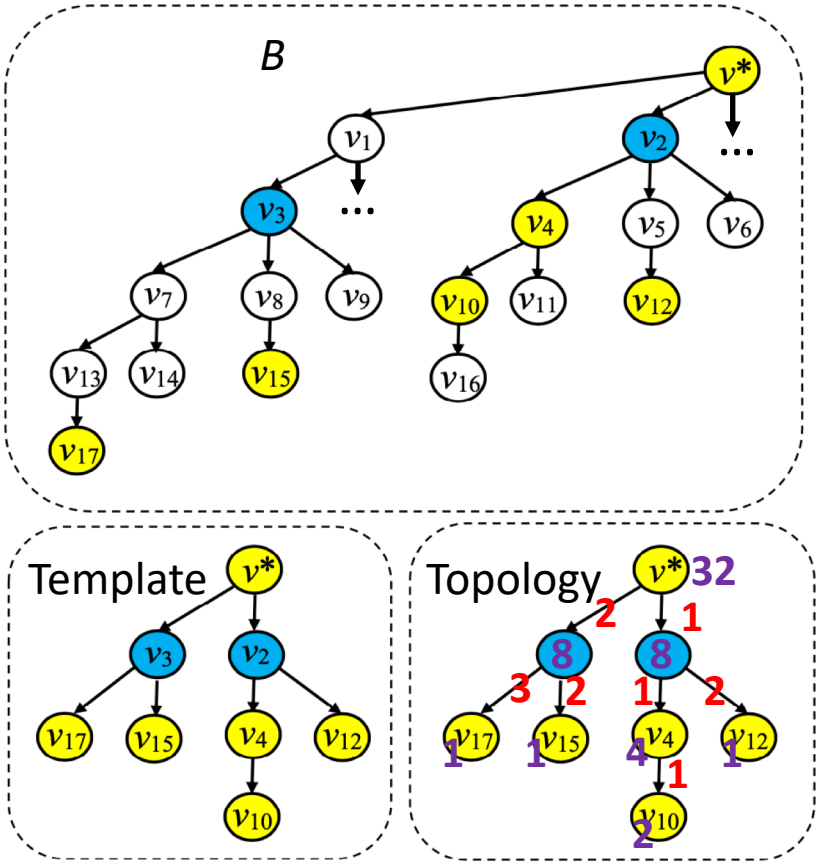
Note. A guess can be realized by many unlabeled BAs.

Surprisingly, we can ``work'' with any one that we want! (Later.)



FPT Algorithm for TFP

Phase III: Greedy Resolution of Paths

- An edge is **resolved** if it was replaced by a degree-2 path of its length.
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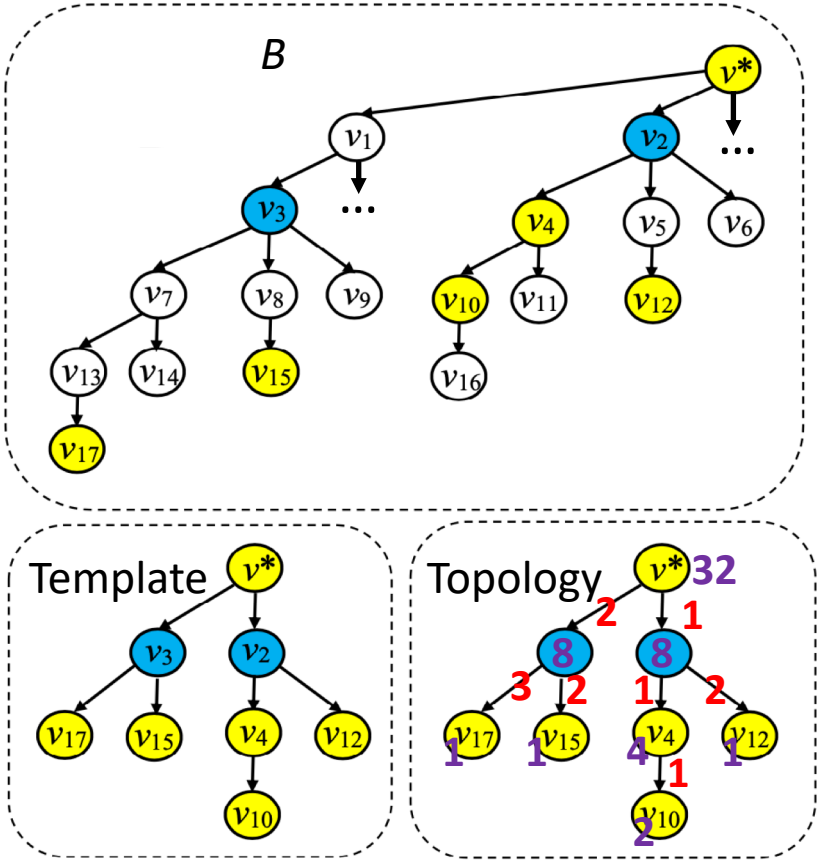

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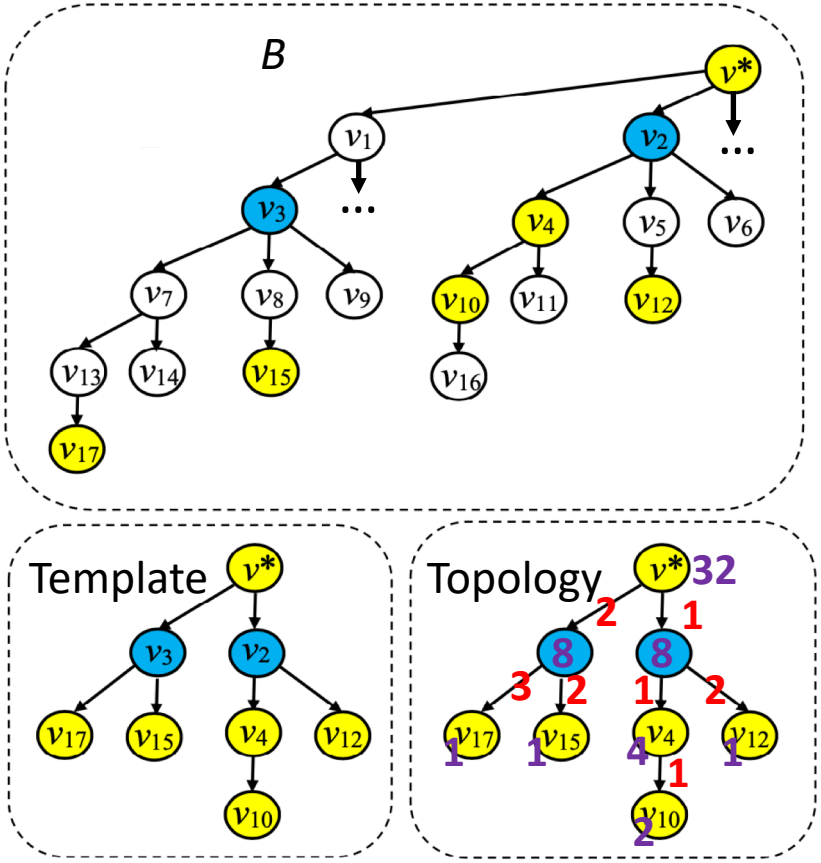

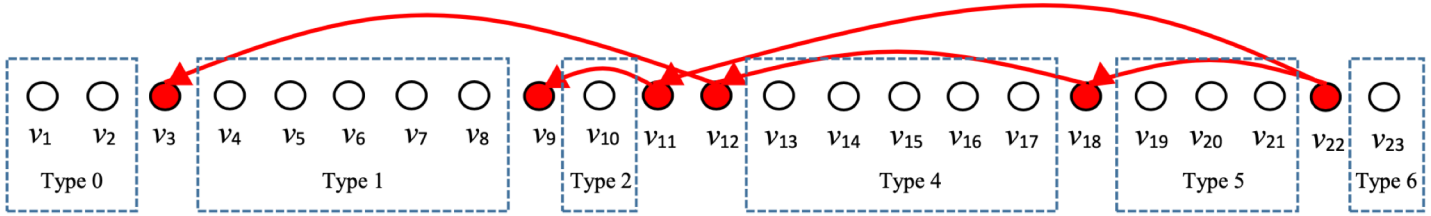

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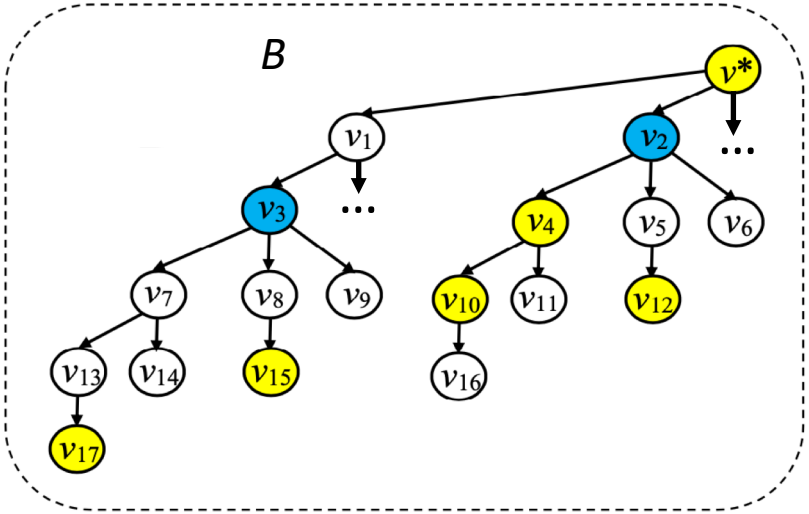

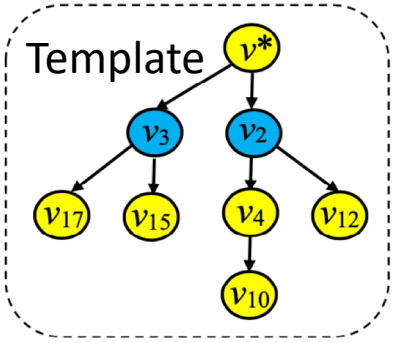

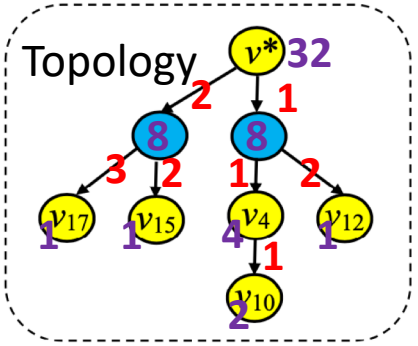

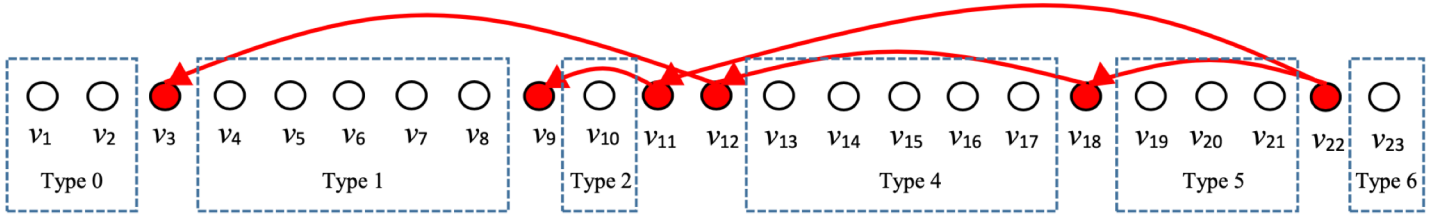

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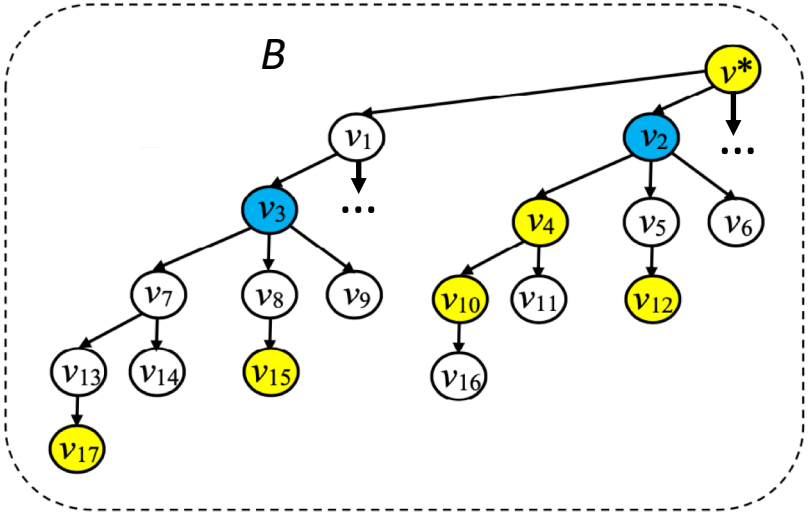

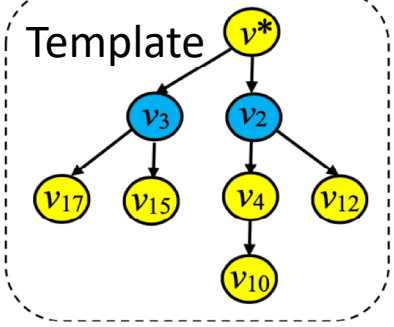

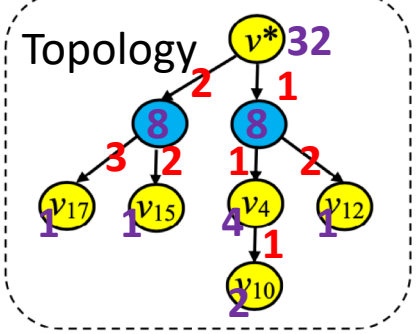

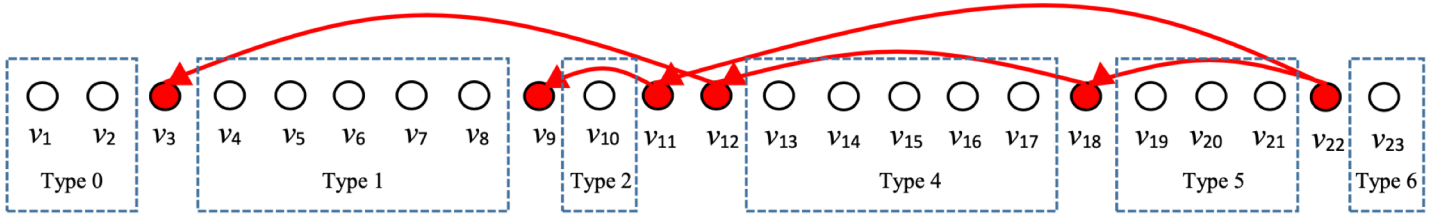

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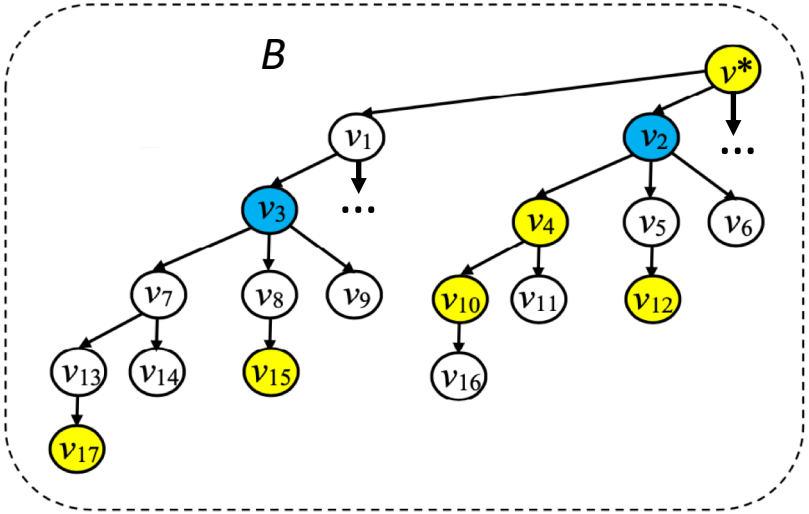

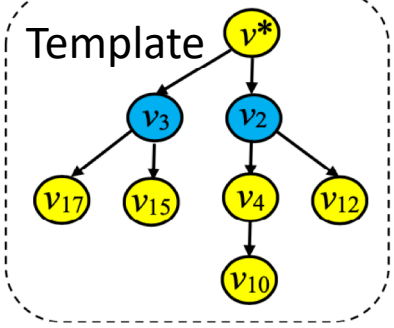

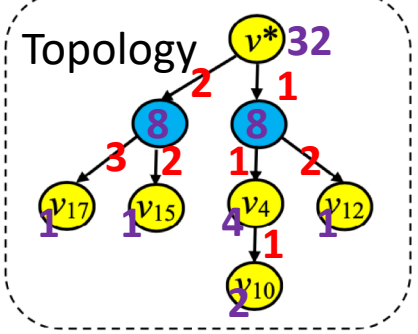

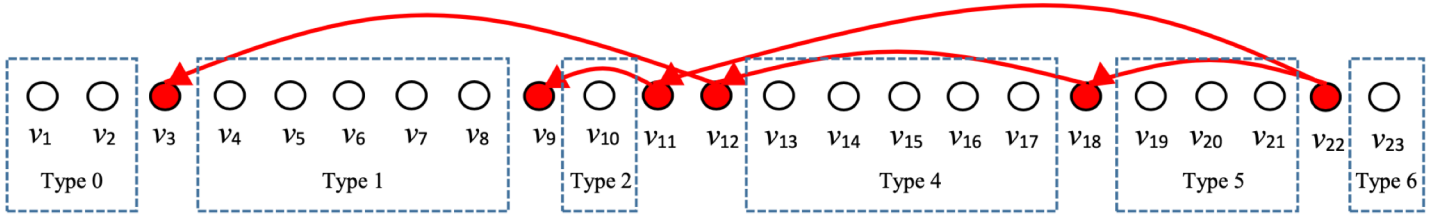

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➢ **Resolution:** Pick the *x* ``free'' vertices between *u* and *v* in closest to *u* in the topo. order of *D* with *F* reversed.
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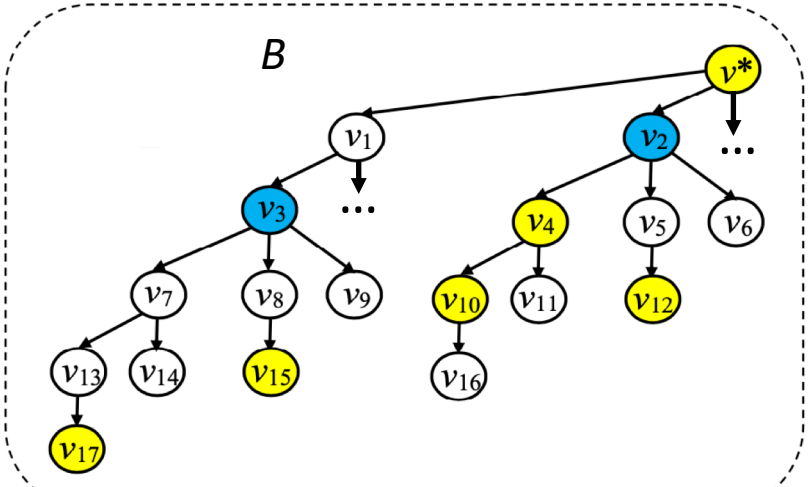

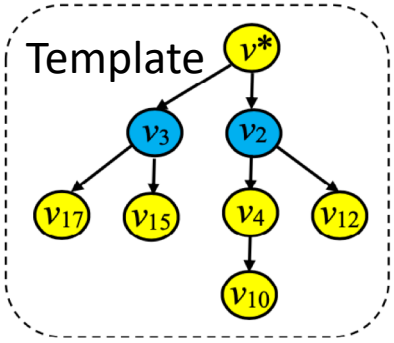

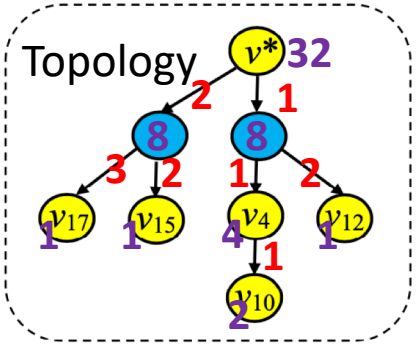

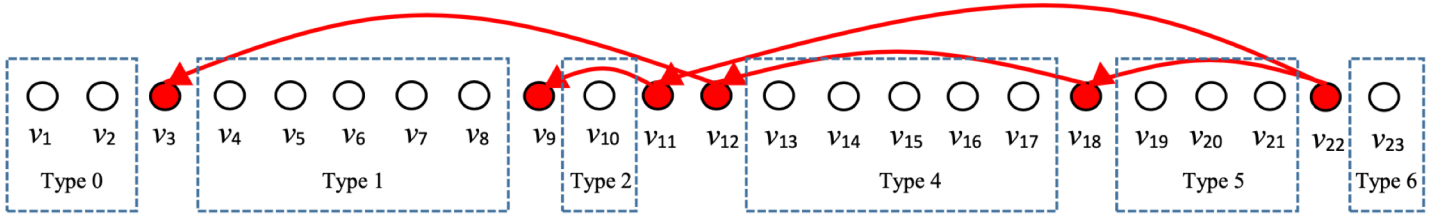

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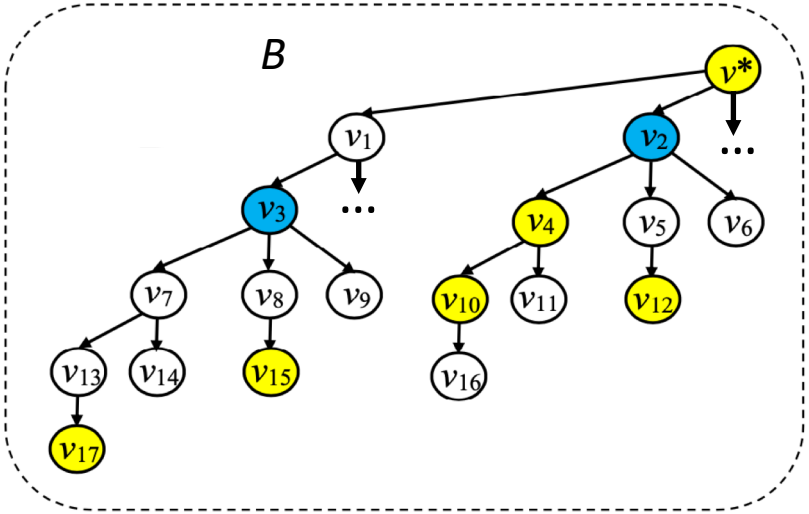
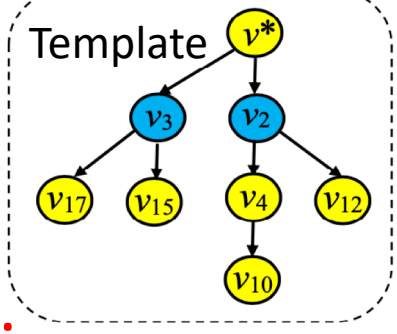
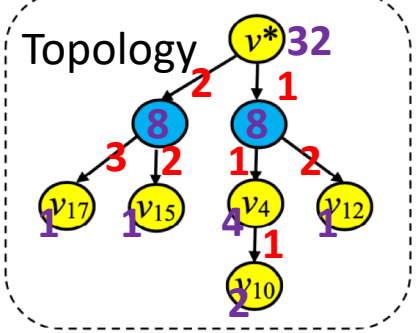
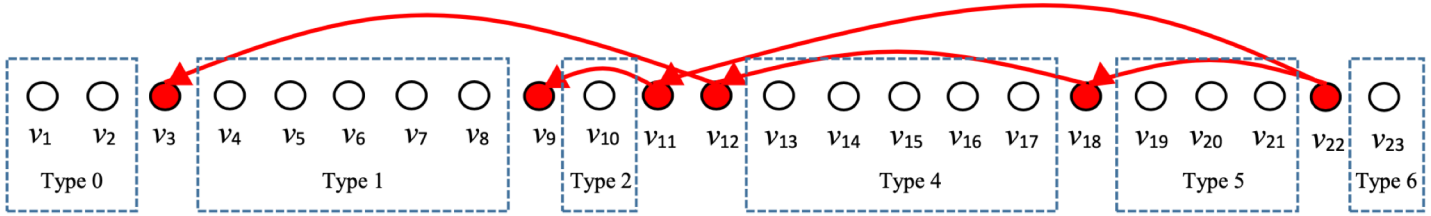
If we do not have enough vertices, answer No.









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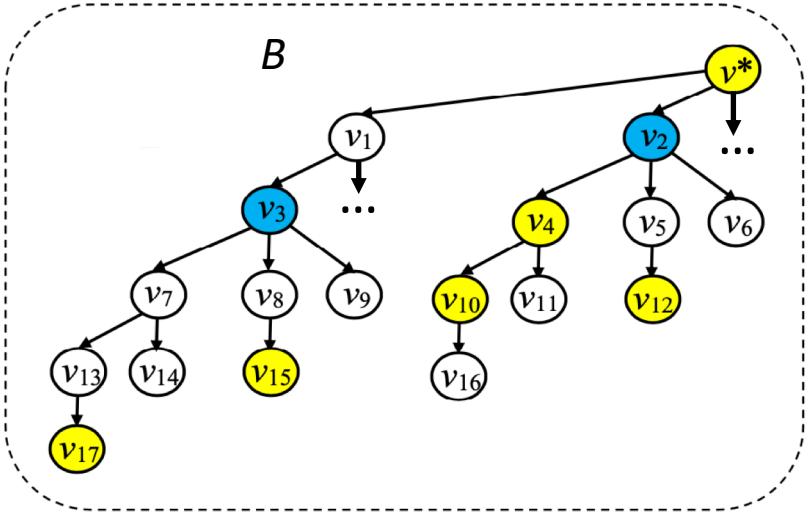
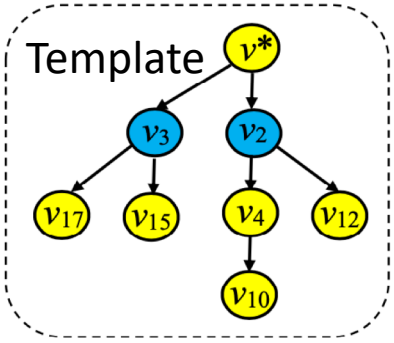
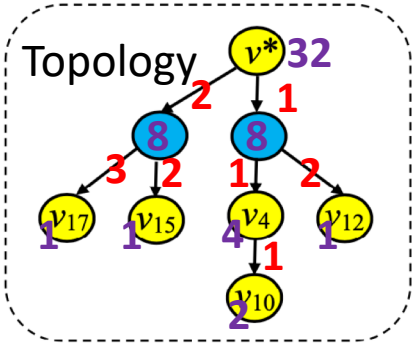
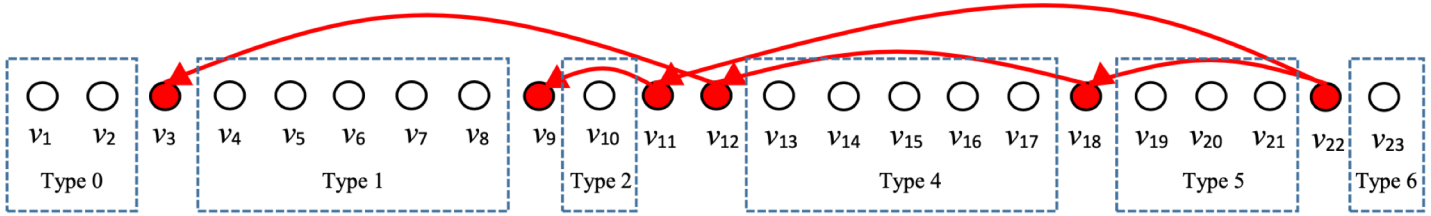




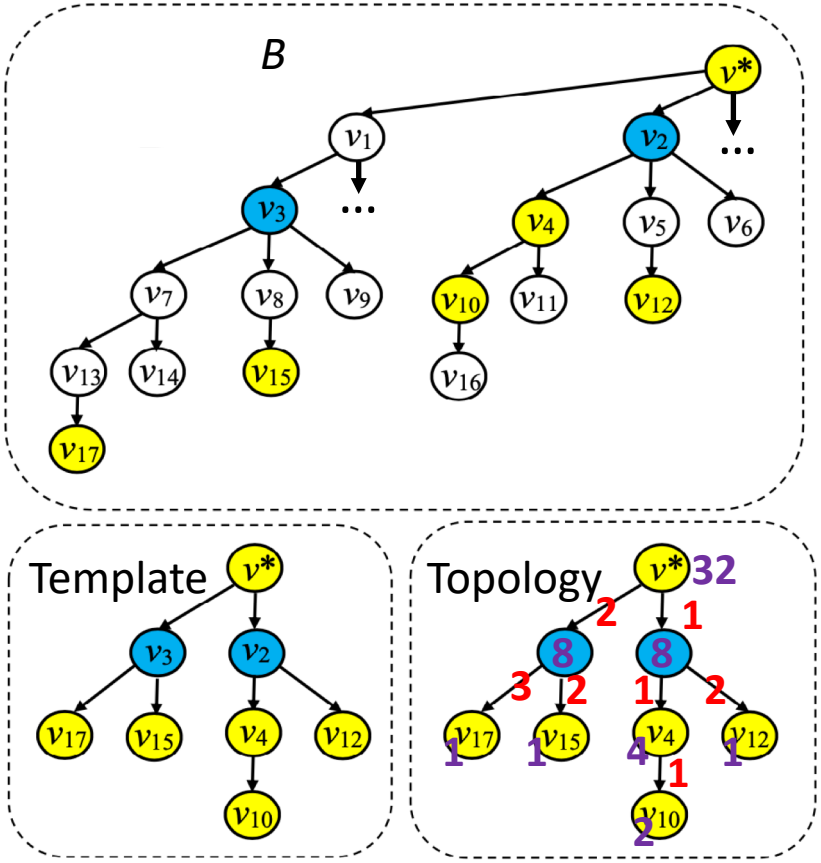


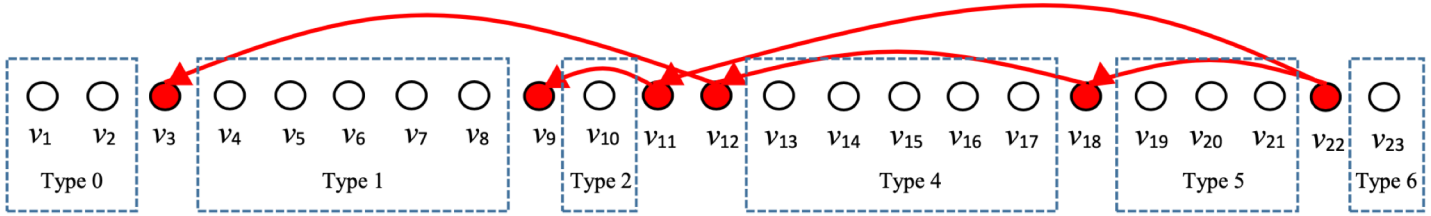
Intuition?



Phase IV: Greedy Resolution of Subtrees

- The **private vertices** of a V_F-vertex are the vertices that belong to its subtree and to none of the subtrees of its V_F-descendants.
- A V_F-vertex is **resolved** if it was assigned its private vertices.





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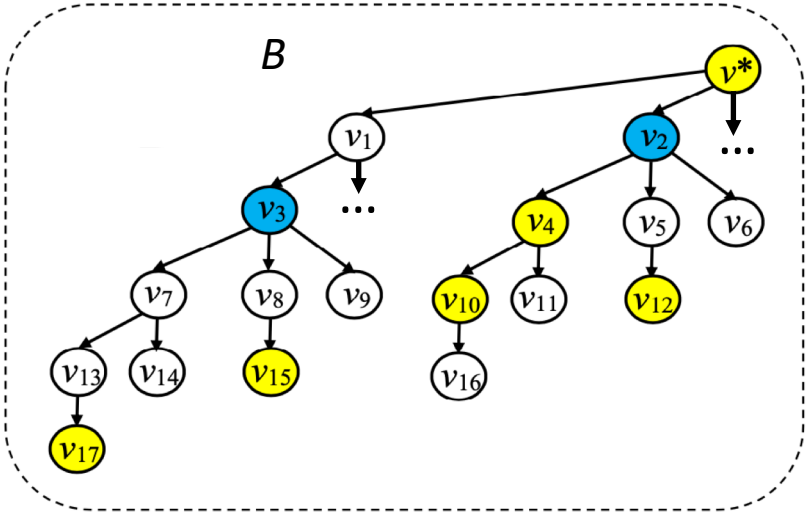

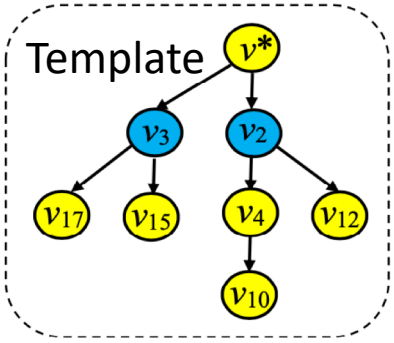

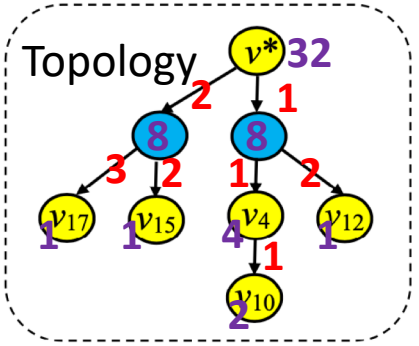

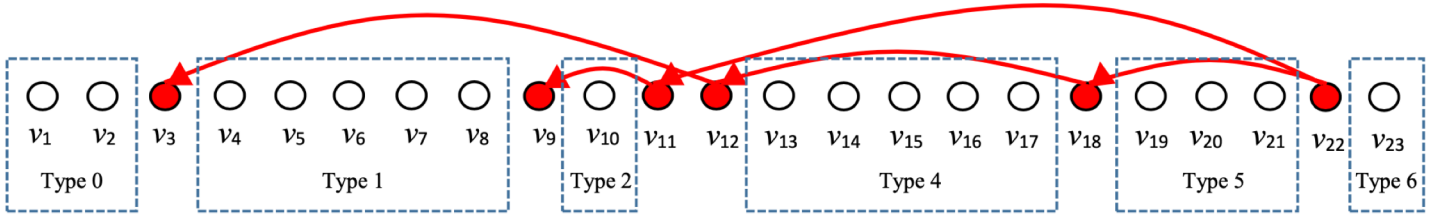

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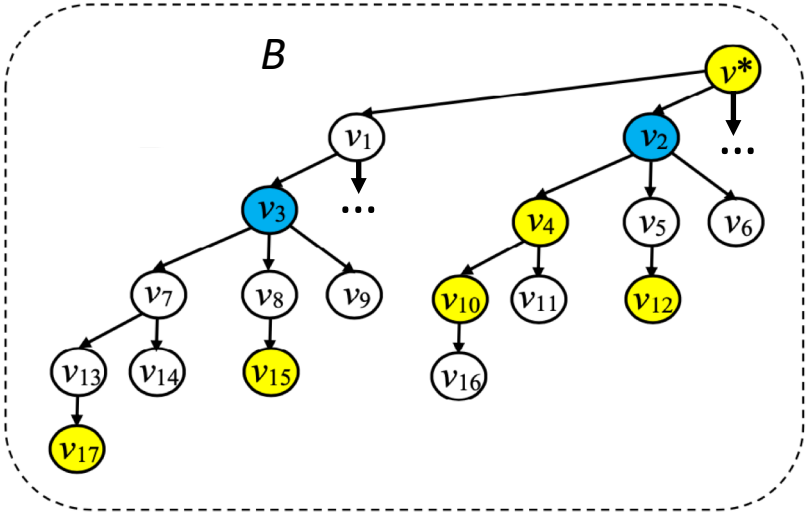

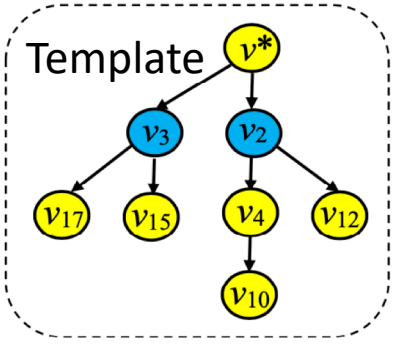

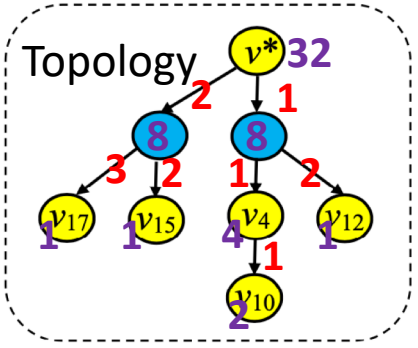

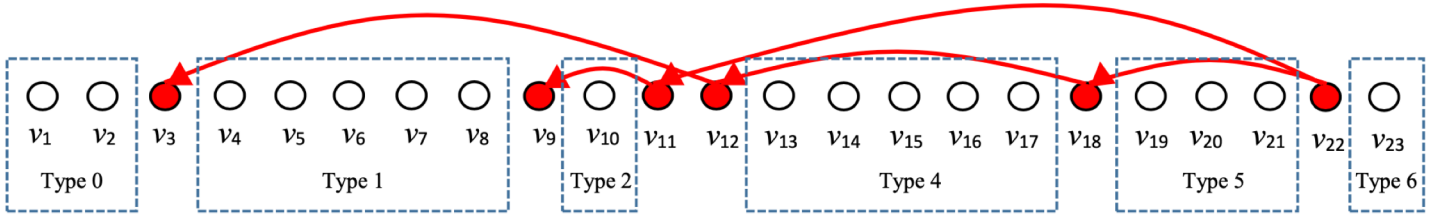

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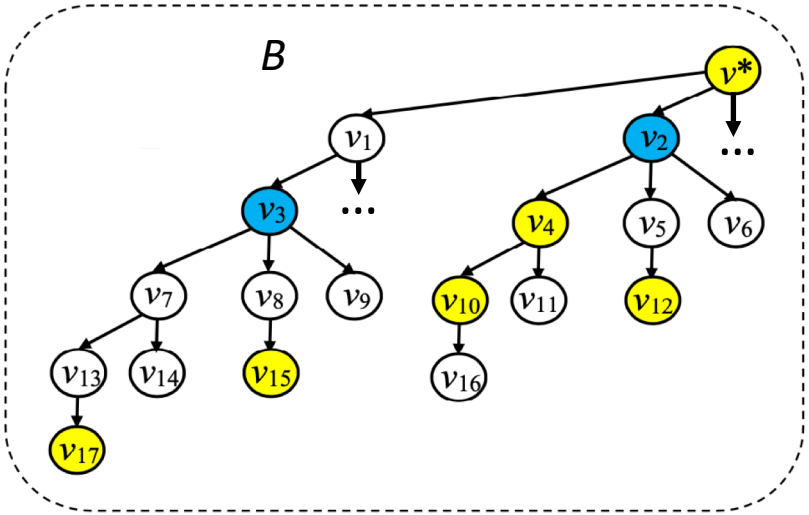

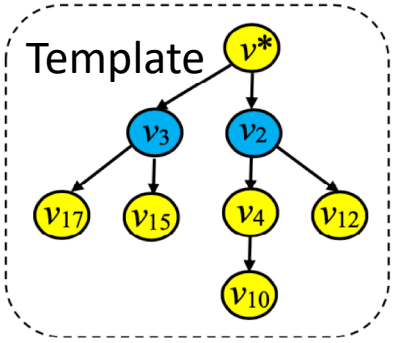

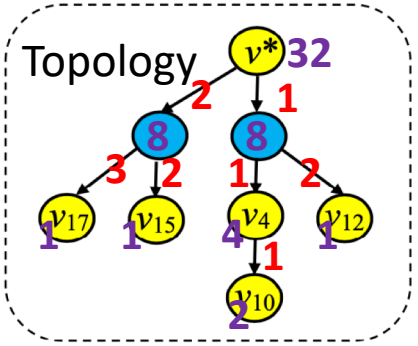

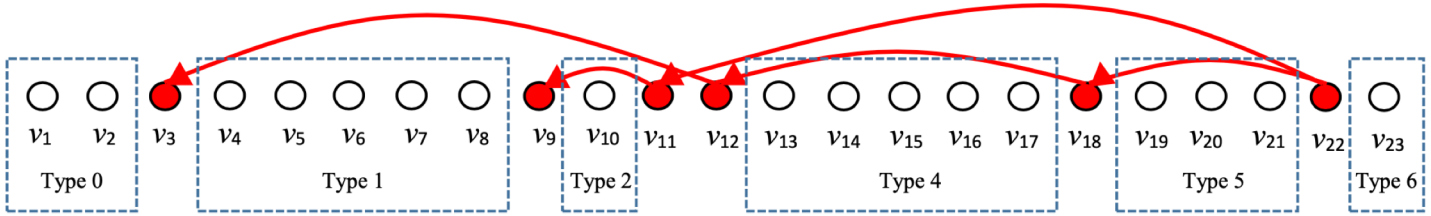

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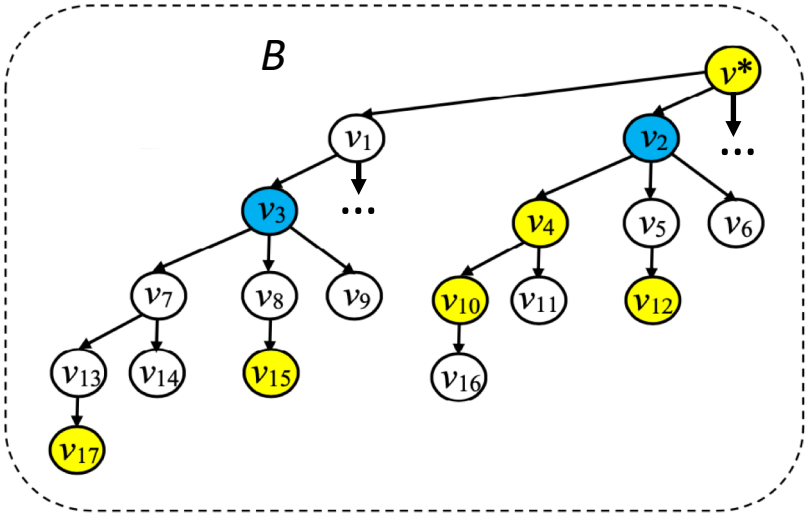

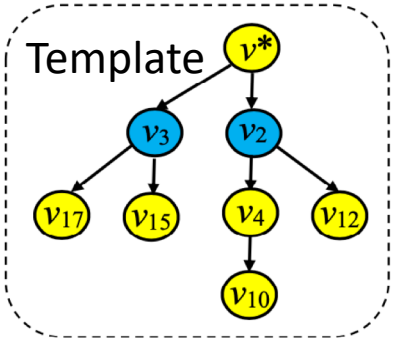

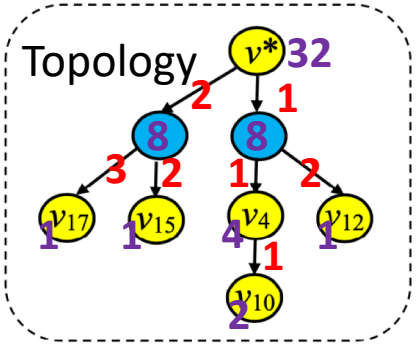

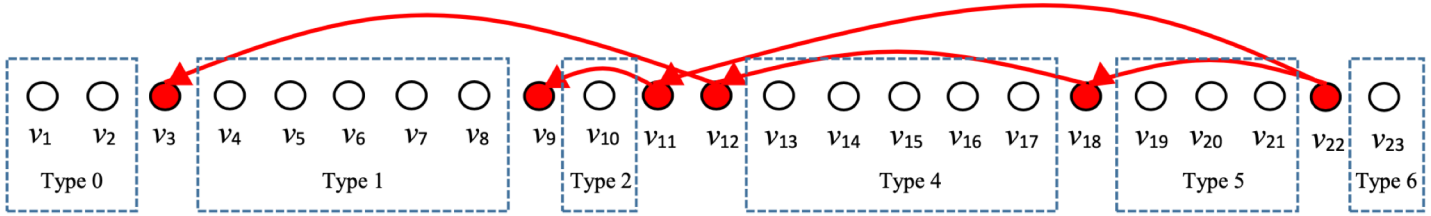

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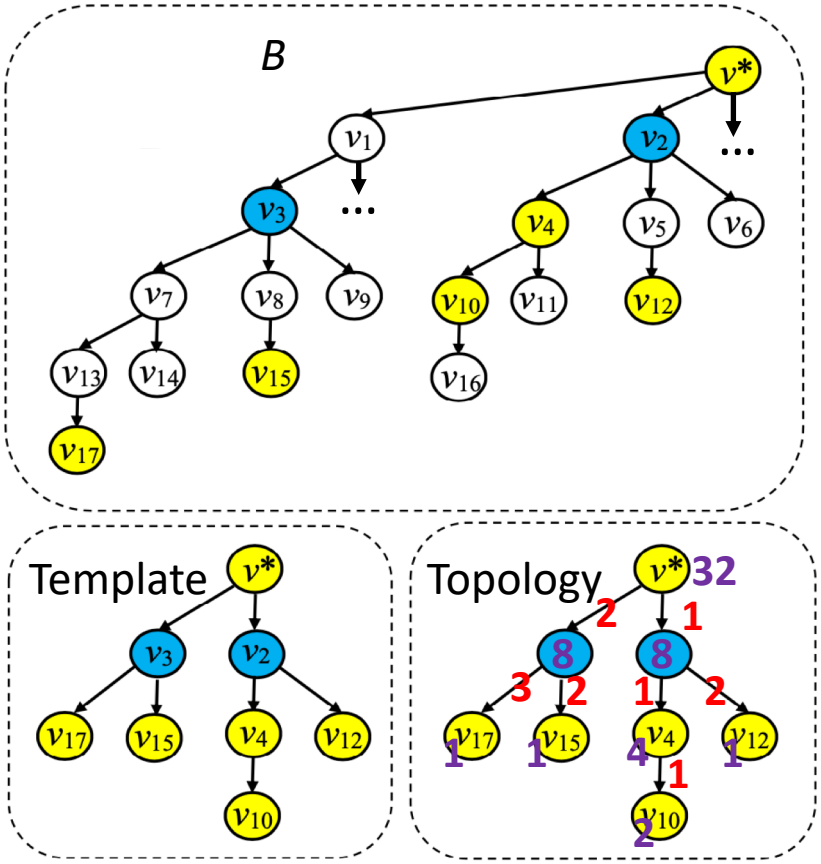
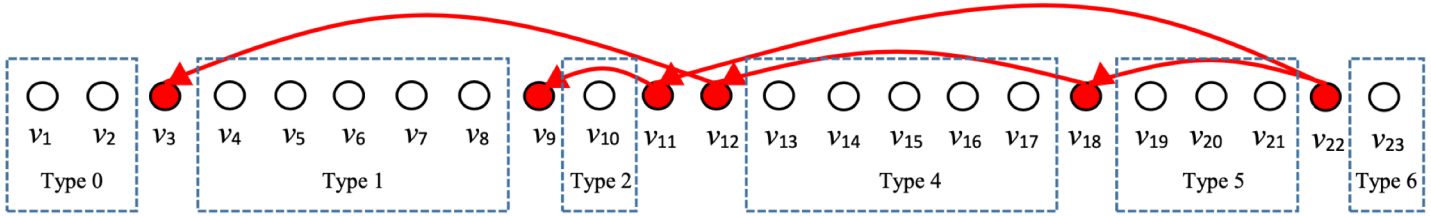
Else, when the algorithm terminates, answer Yes.





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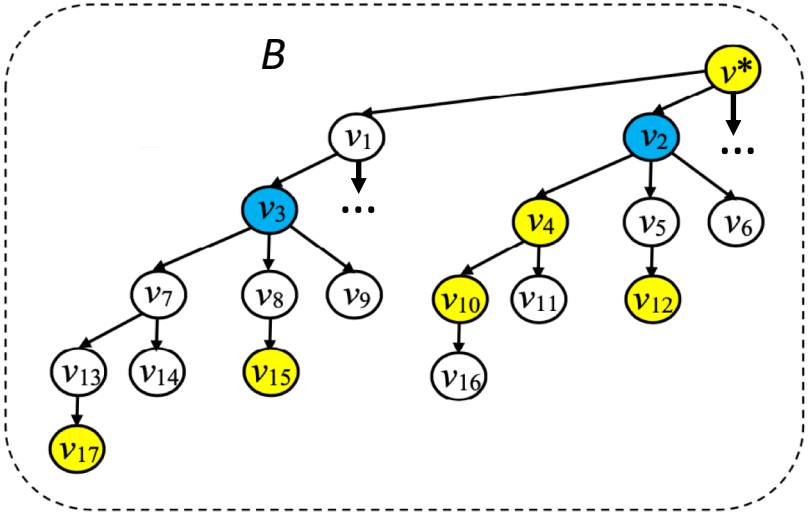

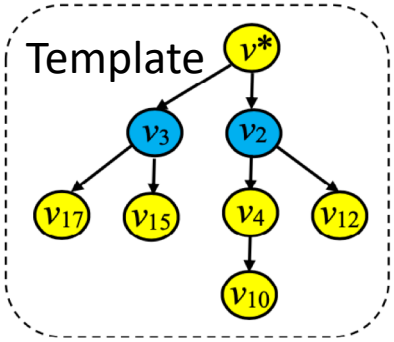

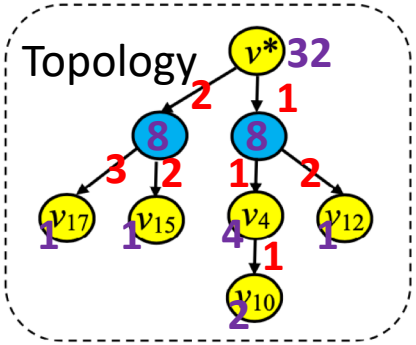

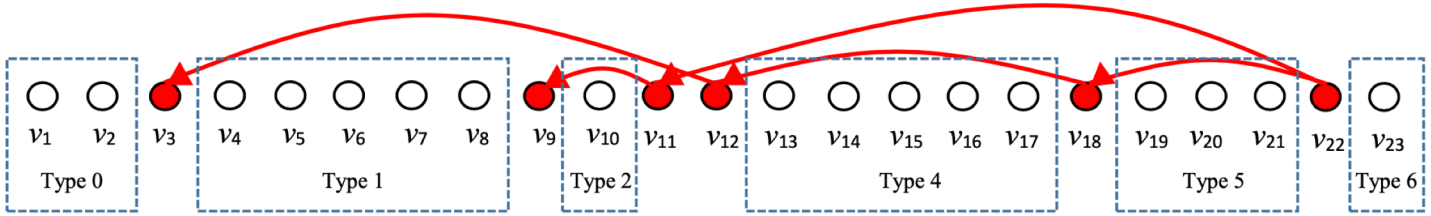

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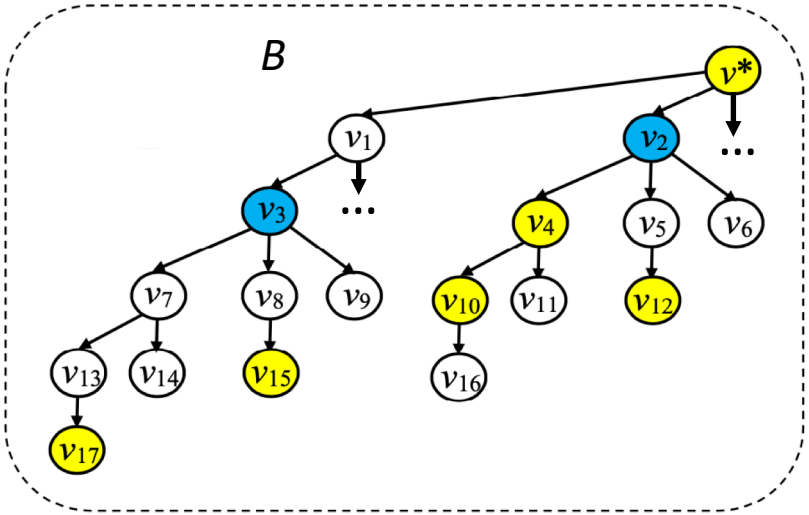
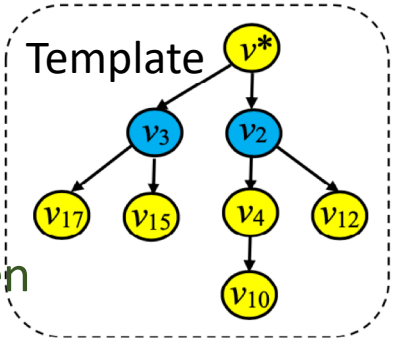
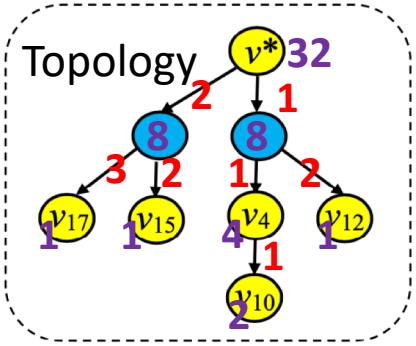






Intuition?

In particular, to argue that if we answer Yes, then it is correct: Use Phase II (Verify Realizability).

Outline

Background

FPT algorithm for the TF problem

Kernel for the TF problem

Kernel for TFP

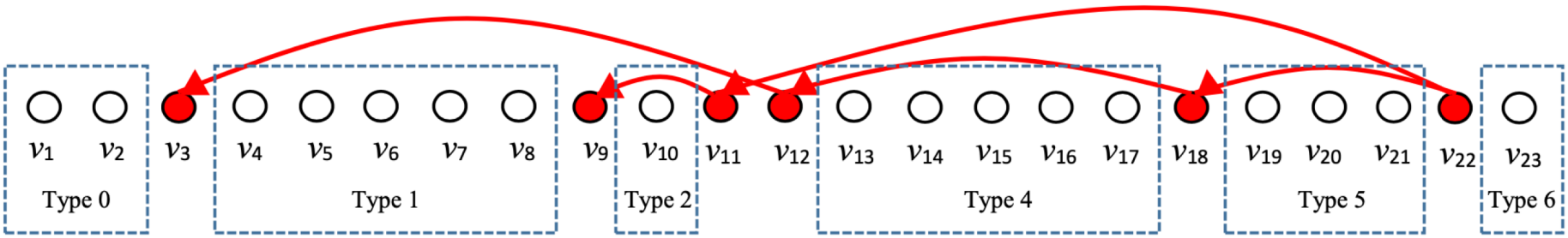
Outline

I. Trivial polynomial compression where witnesses have exponential size.
II. Reinterpretation of the problem as a “packing problem”.
III. Encoding the packing problem using SAT. (Most of the work.)
IV. Sequence of known reductions:
SAT → 3SAT → 3SAT where every literal appears at most twice → TFP.

Kernel for TFP

Trivial Compression

Compute V_F (use an approximation algorithm).



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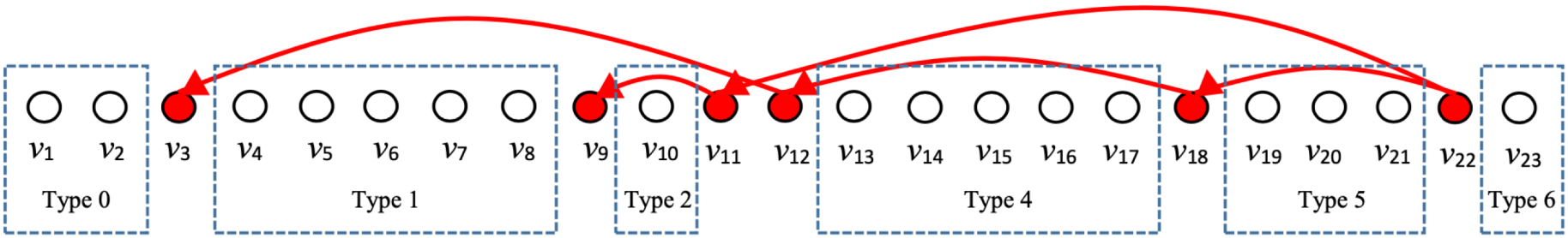

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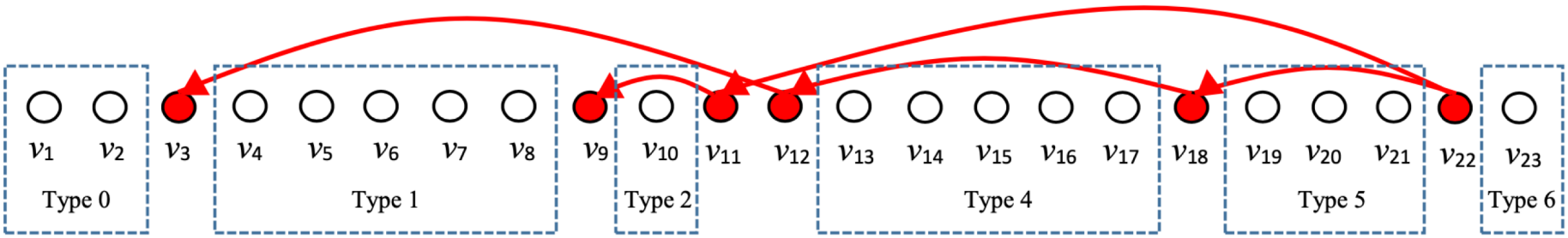

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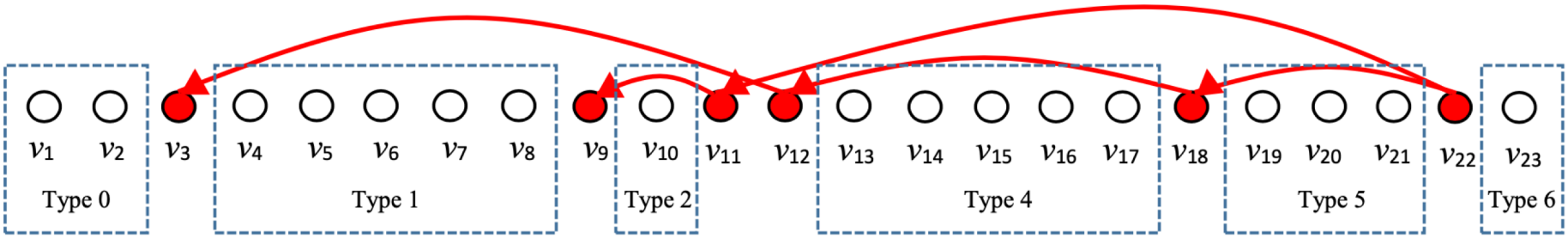
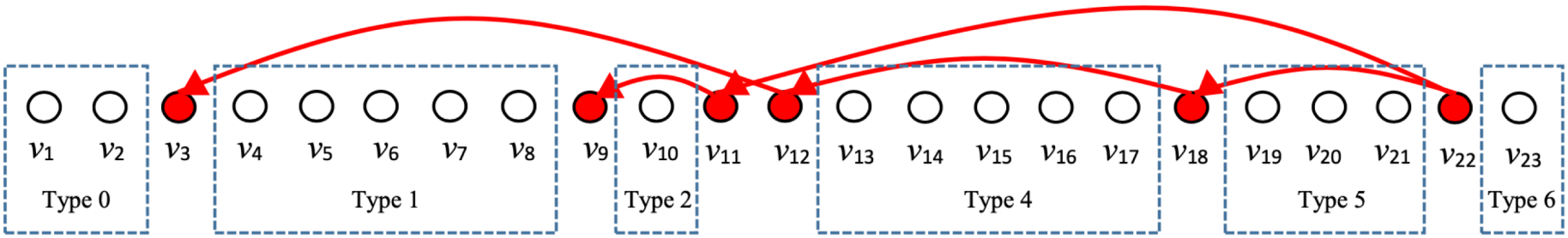
Witness: Single-elimination tournament / SBA: Space $\Omega(n)$ (can be $\Omega(2^{k \log k})$).



Kernel for TFP

Packing-Based Reinterpretation

Guess. Tuple (T, lengths, sizes).



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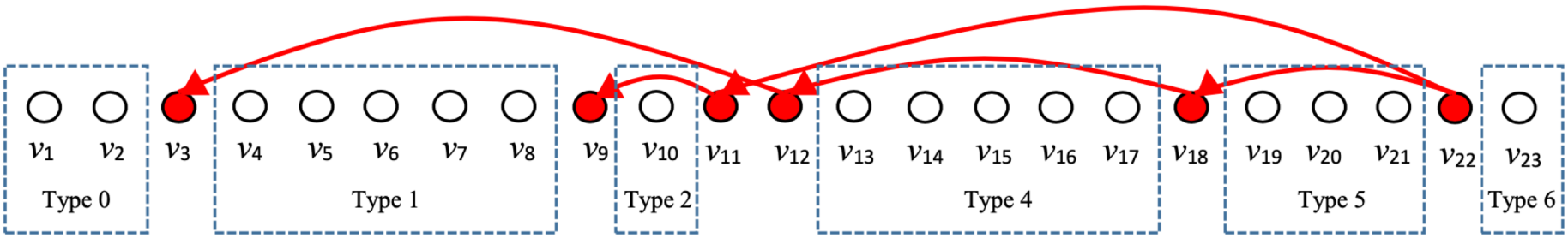

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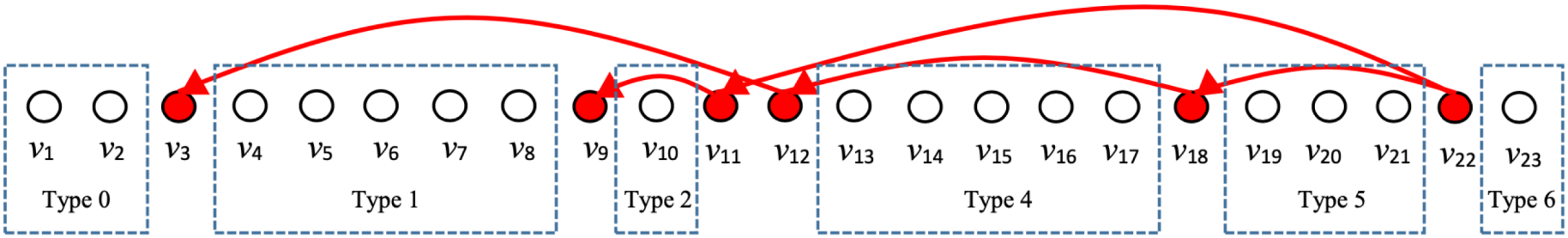

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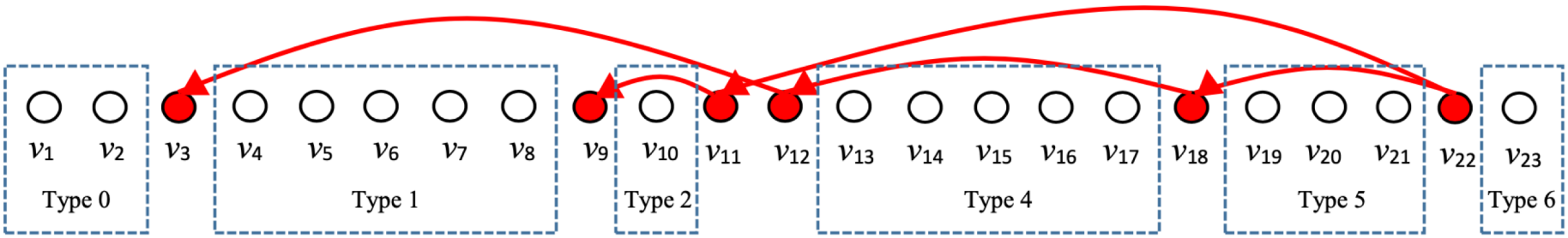

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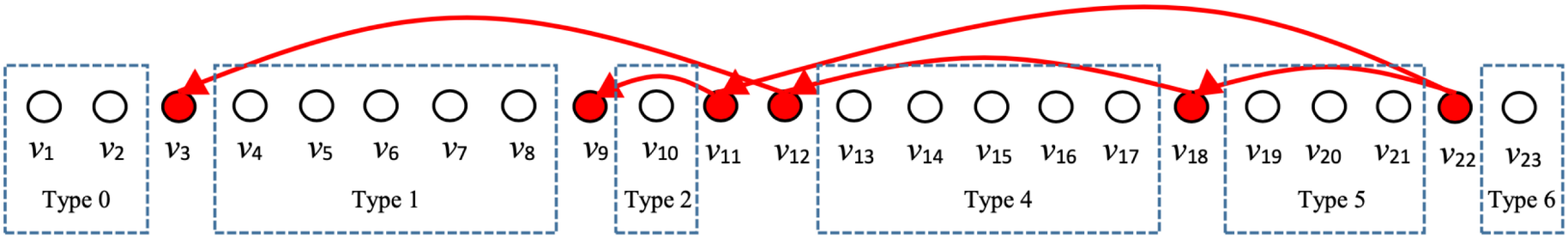

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Still: The ``UBA-part'' of the witness can be of exponential size.

Kernel for TFP

Main Step: SAT-Encoding

We show how to encode the existence of a realizable guess using SAT-constraints.

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Question: ``Direct'' polynomial kernel (where the polynomial is of small degree)?

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Thank you! Questions?