On triangle-free graphs that do not contain a subdivision of the complete graph on four vertices as an induced subgraph

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Abstract

We prove a decomposition theorem for the class of triangle-free graphs that do not contain a subdivision of the complete graph on four vertices as an induced subgraph. We prove that every graph of girth at least 5 in this class is 3-colorable.
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1 Introduction

Here graphs are simple and finite. We say that G *contains* H when H is isomorphic to an induced subgraph of G. We say that a graph G is F*-free* if G does not contain F. For a family of graphs $\mathcal{F}$, we say that G is $\mathcal{F}$*-free* if for every $F \in \mathcal{F}$, G does not contain F. *Subdividing* an edge $e = vw$ of a graph G means deleting e, and adding a new vertex u of degree 2 adjacent to v and w. A *subdivision* of a graph G is any graph H obtained from G by repeatedly subdividing edges. Note that G is a subdivision of G. We say that H is an *ISK4 of a graph* G when H is an induced subgraph of G and H is a subdivision of K_4 (where K_4 denotes the complete graph on four vertices). ISK4 stands for "Induced Subdivision of K_4".

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In [2], a decomposition theorem for ISK4-free graphs is given (see Theorem 2.1) and it is proved that their chromatic number is bounded by a constant c. The proof in [2] follows from a theorem by Kühn and Osthus [1], from which it follows that c is at least 2^{512}. It is conjectured in [2] that every ISK4-free is 4-colorable. The goal of this paper is to prove a stronger decomposition theorem for ISK4-free graphs, under the additional assumption that they are triangle-free (see Theorems 2.2 and 3.8). We also propose the following conjectures, prove that the first one implies the second one (see Theorem 2.7), and prove both of them for graphs of girth at least 5 (see Theorem 5.2). A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$.

Conjecture 1.1 *Every {triangle, ISK4, $K_{3,3}$}-free graph contains a vertex of degree at most 2.*

Conjecture 1.2 *Every {triangle, ISK4}-free graph is 3-colorable.*

Outline of the paper

In Section 2, we state several known decomposition theorems, and derive easy consequences of them for our class. In particular we prove that Conjecture 1.1 implies Conjecture 1.2. In Section 3, we prove our main decomposition theorem. In Section 4, we give some properties needed later for the class of chordless graphs (graphs where all cycles are chordless). In Section 5, we prove Conjecture 1.1 for graphs of girth at least 5. In Section 6, we give structural results for some series-parallel graphs that might be of use to prove Conjecture 1.1.

2 Decomposition theorems

In this section, we provide the notation needed to state decomposition theorems for ISK4-free graphs, and we state them.

A graph G is *series-parallel* if no subgraph (possibly not induced) of G is a subdivision of K_4. Clearly, every series-parallel graph is ISK4-free. When R is a graph, the *line graph* of R is the graph G whose vertex-set is $E(R)$ and such that two vertices of G are adjacent whenever the corresponding edges are adjacent in R.

For a graph G, when C is a subset of $V(G)$, we denote by $G[C]$ the subgraph of G induced by C and we write $G \setminus C$ instead of $G[V(G) \setminus C]$. A *cutset* in a graph G is a set C of vertices such that $G \setminus C$ is disconnected.

A *star cutset* is a cutset C that contains a vertex c, called a *center* of C, adjacent to all other vertices of C. Note that a star cutset may have more than one center, and that a cutset of size 1 is a star cutset. A *double star cutset* is a cutset C that contains two adjacent vertices x and y, such that every vertex of $C \setminus \{x, y\}$ is adjacent to x or y. Note that a star cutset of size at least 2 is a double star cutset.

A path from a vertex a to a vertex b is refered to as an *ab-path.* A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent vertices a, b such that V can be partitioned into non-empty sets X, Y and $\{a, b\}$ so that: there is no edge between X and Y; and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path and neither of $G[X \cup \{a, b\}]$ nor $G[Y \cup \{a, b\}]$ is a chordless path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset. The following is the main decomposition theorem for ISK4-free graphs.

Theorem 2.1 (see [2]) *If G is an ISK4-free graph, then G is series-parallel, or G is the line graph of a graph of maximum degree at most 3, or G has a proper 2-cutset, a star cutset, or a double star cutset.*

Our first goal is to improve this theorem for triangle-free graphs. Some improvements are easy to obtain (they trivially follow from the absence of triangles). The non-trivial one is done in the next two sections: we show that double star cutsets and proper 2-cutsets are in fact not needed. We state the result now, but it follows from Theorem 3.8 that needs more terminology and is slightly stronger.

Theorem 2.2 *If G is a {triangle, ISK4}-free graph, then either G is a series-parallel graph or a complete bipartite graph, or G has a clique cutset of size at most two, or G has a star cutset.*

We state now several lemmas from [2] that we need. A *hole* in a graph is a chordless cycle of length at least 4. A *prism* is a graph made of three vertex disjoint paths of length at least 1, $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$ and $P_3 = a_3 \dots b_3$, with no edges between them except the following: a_1a_2, a_1a_3, a_2a_3, b_1b_2, b_1b_3, b_2b_3. Note that the union of any two of the paths of a prism induces a hole. A *wheel* (H, x) is a graph that consists of a hole H plus a vertex $x \notin V(H)$ that has at least three neighbors on H.

Lemma 2.3 (see [2]) *If G is an ISK4-free graph, then either G is a series-parallel graph, or G contains a prism, or G contains a wheel or G contains $K_{3,3}$.*

A *complete tripartite graph* is a graph that can be partitioned into three stable sets so that every pair of vertices from two different stable sets is an edge of the graph.

Lemma 2.4 (See [2]) *If G is an ISK4-free graph that contains $K_{3,3}$, then either G is a complete bipartite graph, or G is a complete tripartite graph, or G has a clique-cutset of size at most 3.*

We now state the consequences of the lemmas above for triangle-free graphs.

Lemma 2.5 *If G is a {triangle, ISK4}-free graph, then either G is a series-parallel graph, or G contains a wheel or G contains $K_{3,3}$.*

PROOF — Clear from Lemma 2.3 and the fact that every prism contains a triangle. □

Lemma 2.6 *If G is an {ISK4, triangle}-free graph that contains $K_{3,3}$, then either G is a complete bipartite graph, or G has a clique-cutset of size at most 2.*

PROOF — Clear from Lemma 2.4 and the fact that complete tripartite graphs and clique-cutsets of size 3 contain triangles. □

It is now easy to prove the next theorem.

Theorem 2.7 *If Conjecture 1.1 is true, then Conjecture 1.2 is true.*

PROOF — Suppose that Conjecture 1.1 is true. Let G be a {triangle, ISK4}-free graph. We prove Conjecture 1.2 by induction on $|V(G)|$. If $|V(G)| = 1$, the outcome is clearly true.

If G contains $K_{3,3}$, then by Lemma 2.6, either G is bipartite and therefore 3-colorable, or G has a clique-cutset K. In this last case, we recover a 3-coloring of G from 3-colorings of $G[K \cup C_1], \dots, G[K \cup C_k]$ where $C_1, \dots, C_k$ are the connected components of $G \setminus K$.

If G contains no $K_{3,3}$, then by Conjecture 1.1 it has a vertex v of degree at most 2. By the induction hypothesis, $G \setminus \{v\}$ has a 3-coloring, and we 3-color G by giving to v a color not used by its two neighbors. □

3 Proof of the decomposition theorem

Appendices to a hole

When x is a vertex of a graph G, $N(x)$ denotes the neighborhood of x, that is the set of all vertices of G adjacent to x. We set $N[x] = N(x) \cup \{x\}$. When $C \subseteq V(G)$, we set $N(C) = (\cup_{x \in C} N(x)) \setminus C$. When G is a graph, K an induced subgraph of G, and C a set of vertices disjoint from $V(K)$, the *attachment* of C to K is $N(C) \cap V(K)$, that we also denote by $N_K(C)$.

When $P = p_1 \dots p_k$ is a path and $1 \leq i, j \leq k$, we denote by $p_i P p_j$ the $p_i p_j$-subpath of P. Let A and B be two disjoint vertex sets such that no vertex of A is adjacent to a vertex of B. A path $P = p_1 \dots p_k$ *connects A and B* if either $k = 1$ and p_1 has a neighbor in A and a neighbor in B, or $k > 1$ and one of the two endvertices of P is adjacent to at least one vertex in A and the other is adjacent to at least one vertex in B. P is a *direct connection between A and B* if in $G[V(P) \cup A \cup B]$ no path connecting A and B is shorter than P. The connection P is said to be *from A to B* if p_1 is adjacent to a vertex of A and p_k is adjacent to a vertex of B.

Let H be a hole. A chordless path $P = p_1 \dots p_k$ in $G \setminus H$ is an *appendix* of H if no vertex of $P \setminus \{p_1, p_k\}$ has a neighbor in H, and one of the following holds:

(i) $k = 1$, $N(p_1) \cap H = \{u_1, u_2\}$ and $u_1 u_2$ is not an edge, or

(ii) $k > 1$, $N(p_1) \cap H = \{u_1\}$, $N(p_k) \cap H = \{u_2\}$ and $u_1 \neq u_2$.

So $\{u_1, u_2\}$ is an attachment of P to H. The two $u_1 u_2$-subpaths of H are called the *sectors* of H w.r.t. P.

Let Q be another appendix of H, with attachment $\{v_1, v_2\}$. Appendices P and Q are *crossing* if one sector of H w.r.t. P contains v_1, the other contains v_2 and $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$.

Lemma 3.1 *If G is an {ISK4, $K_{3,3}$}-free graph, then no two appendices of a hole of G can be crossing.*

PROOF — Let $P = p_1 \dots p_k$ and $Q = q_1 \dots q_l$ be appendices of a hole H of G, and suppose that they are crossing. Let $\{u_1, u_2\}$ be the attachment of P to H, and let $\{v_1, v_2\}$ be the attachment of Q to H. So $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$ and w.l.o.g. u_1, v_1, u_2, v_2 appear in this order when traversing H. W.l.o.g. u_1 is adjacent to p_1, and v_1 to q_1.

A vertex of P must be adjacent to or coincident with a vertex of Q, since otherwise $H \cup P \cup Q$ induces an ISK4. Note that $\{p_1, p_k\} \cap \{q_1, q_l\} = \emptyset$. Let

p_i be the vertex of P with lowest index that has a neighbor in Q, and let q_j (resp. $q_{j'}$) be the vertex of Q with lowest (resp. highest) index adjacent to p_i. Note that p_i is not coincident with a vertex of Q.

First suppose that $i = k$. If $j \neq l$ then $H \cup P \cup \{q_1, \dots, q_j\}$ induces an ISK4. So $j = l$. In particular, no vertex of P is coincident with a vertex of Q, and $p_k q_l$ is the only edge between P and Q. If $k \neq 1$ then $H \cup Q \cup \{p_k\}$ induces an ISK4. So $k = 1$ and by symmetry $l = 1$. Since $H \cup \{p_1, q_1\}$ cannot induce a $K_{3,3}$, w.l.o.g. $u_1 v_1$ is not an edge. Let H' be the $u_1 v_1$-subpath of H that does not contain u_2 and v_2. Then $(H \setminus H') \cup \{u_1, v_1, p_1, q_1\}$ induces an ISK4. Therefore $i < k$.

If p_i has a unique neighbor in Q, then $H \cup Q \cup \{p_1, \dots, p_i\}$ induces an ISK4. Let H'_Q be the sector of H w.r.t. Q that contains u_1. If p_i has exactly two neighbors in Q, then $H'_Q \cup Q \cup \{p_1, \dots, p_i\}$ induces an ISK4. So p_i has at least three neighbors in Q. In particular, $j' \notin \{j, j+1\}$. But then $H \cup \{p_1, \dots, p_i, q_1, \dots, q_j, q_{j'}, \dots, q_l\}$ induces an ISK4. □

Lemma 3.2 *Let G be an {triangle, ISK4, $K_{3,3}$}-free graph. Let H be a hole and $P = p_1 \dots p_k$, $k > 1$, a chordless path in $G \setminus H$. Suppose that $|N(p_1) \cap H| = 1$ or 2, $|N(p_k) \cap H| = 1$ or 2, no vertex of $P \setminus \{p_1, p_k\}$ has a neighbor in H, $N(p_1) \cap H \not\subseteq N(p_k) \cap H$ and $N(p_k) \cap H \not\subseteq N(p_1) \cap H$. Then P is an appendix of H.*

PROOF — Assume not. Then at least one of p_1, p_k has two neighbors in H. If one of p_1 or p_k has one neighbor in H and the other one has two neighbors in H, then $H \cup P$ induces an ISK4. So $|N(p_1) \cap H| = |N(p_k) \cap H| = 2$, and hence (since G is triangle-free) both p_1 and p_k are appendices of H. By Lemma 3.1, p_1 and p_k cannot be crossing. So for a sector H' of H w.r.t. p_1, $N(p_k) \cap H \subseteq H'$. But then $H' \cup P$ induces an ISK4. □

Wheels

Let (H, x) be a wheel contained in a graph G. A *sector* is a subpath of H whose endvertices are adjacent to x and interior vertices are not. Two sectors are *consecutive* or *adjacent* if they have an endvertex in common.

Throughout this section we use the following notation for a wheel (H, x). We denote by $x_1, \dots, x_n$ the neighbors of x in H, appearing in this order when traversing H. In this case, we also say that (H, x) is an *n-wheel.* For $i = 1, \dots, n$, S_i denotes the sector of (H, x) whose endvertices are x_i and

x_{i+1} (here and throughout this section we assume that indices are taken modulo n). For $i = 1, \dots, n$, edges xx_i are refered to as *spokes*.

A path P is an *appendix* of a wheel (H, x) if the following hold:

(i) P is an appendix of H,

(ii) each of the sectors of H w.r.t. P properly contains a sector of (H, x), and

(iii) x has at most one neighbor in P.

Lemma 3.3 *Let G be an ISK4-free graph. Let P be an appendix of a wheel (H, x) of G, and let H'_P be a sector of H w.r.t. P. Then H'_P contains at least three neighbors of x. In particular, H'_P contains at least two sectors of (H, x).*

PROOF — Let $\{u_1, u_2\}$ be the attachment of P to H. Since P is an appendix of (H, x), H'_P contains at least two neighbors of x. Suppose H'_P contains exactly two neighbors of x. If x has a neighbor in P, then $H'_P \cup P \cup \{x\}$ induces an ISK4. So x does not have a neighbor in P. Since P is an appendix of (H, x), H'_P properly contains a sector of (H, x) and so w.l.o.g. x is not adjacent to u_2. Let H''_P be the other sector of H w.r.t. P, and let x' be the neighbor of x in H''_P that is closest to u_2. Note that since $H \cup \{x\}$ cannot induce an ISK4, $n \geq 4$, and hence $x' \neq u_1$ and $x'u_1$ is not an edge. Let H' be the $x'u_2$-subpath of H''_P. Then $H'_P \cup H' \cup P \cup \{x\}$ induces an ISK4. □

A wheel (H, x) of G is *proper* if vertices $u \in G \setminus (H \cup N[x])$ are one of the following types:

- type 0: $|N(u) \cap H| = 0$;
- type 1: $|N(u) \cap H| = 1$;
- type 2: $|N(u) \cap H| = 2$ and for some sector S_i of (H, x), $N(u) \cap H \subseteq S_i$.

Lemma 3.4 *Let G be a $\{$triangle, ISK4$\}$-free graph. If (H, x) is a wheel of G with fewest number of vertices, then (H, x) is a proper wheel.*

PROOF — Let $u \in G \setminus (H \cup N[x])$. It follows from the following two claims that u is of type 0, 1 or 2 w.r.t. (H, x), and hence that (H, x) is proper.

(1) For every sector S_i of (H, x), $|N(u) \cap S_i| \leq 2$.

Otherwise, $S_i \cup \{x, u\}$ induces a wheel with fewer vertices than (H, x), a contradiction. This proves (1).

(2) For some sector S_i of (H, x), $N(u) \cap H \subseteq N(u) \cap S_i$.

Assume otherwise, and choose $i, j \in \{1, \ldots, n\}$ so that u has a neighbor in $S_i \setminus S_j$ and in $S_j \setminus S_i$, $N(u) \cap (S_i \cup S_j)$ is not contained in a sector of (H, x) and $|j - i|$ is minimized. W.l.o.g. $i = 1$ and $1 < j < n$ (since the case when $j = n$ is symmetric to the case when $j = 2$). Let u' (resp. u'') be the neighbor of u in S_1 that is closest to x_1 (resp. x_2). Let u_j (resp. u_{j+1}) be the neighbor of u in S_j that is closest to x_j (resp. x_{j+1}). Let P be the $u''u_j$-subpath of H that contains x_2. Note that by the choice of i, j, vertex u has no neighbor in the interior of P and $u_j \neq x_n$. Since G is triangle-free, $x_{j+1}x_1$ is not an edge, and if $j \neq 2$ then x_2x_j is not an edge.

First suppose that u has at least two neighbors in S_1. Then by (1), u has exactly two neighbors in S_1. If $u_{j+1} = x_j$ then $j \neq 2$ (by the choice of i, j), and hence $S_1 \cup \{x, u, x_j\}$ induces an ISK4. So $u_{j+1} \neq x_j$. If $u_{j+1}x_2$ is an edge, then ux_2 is not an edge (since G is triangle-free) and hence $S_1 \cup \{x, u, u_{j+1}\}$ induces an ISK4. So $u_{j+1}x_2$ is not an edge. But then $S_1 \cup \{x, u\}$ together with $u_{j+1}S_jx_{j+1}$ induces an ISK4.

Therefore u has exactly one neighbor in S_1, and by symmetry it has exactly one neighbor in S_j. If $j = 2$ then $S_1 \cup S_2 \cup \{x, u\}$ induces an ISK4. So $j > 2$. If P contains at least three neighbors of x, then (since $j < n$ and $u_j \neq x_n$) $P \cup \{x, u\}$ induces a wheel with center x that has fewer vertices than (H, x), a contradiction. Therefore P contains exactly two neighbors of x. But then $j = 3, u_j \neq x_4$ and hence $S_1 \cup S_2 \cup \{x, u\}$ together with $x_3S_3u_j$ induces an ISK4. This proves (2).

□

Lemma 3.5 *Let G be a {triangle, ISK4, $K_{3,3}$}-free graph. Let (H, x) be a proper wheel of G with fewest number of spokes. If (H, x) has an appendix, then (H, x) is a 4-wheel.*

PROOF — Assume (H, x) has an appendix. Note that by Lemma 3.3, if P is an appendix of (H, x) then each of the sectors of H w.r.t. P contains at least two sectors of (H, x). If (H, x) has an appendix such that one of the sectors of H w.r.t. this appendix is $S_i \cup S_{i+1}$ for some $i \in \{1, \ldots, n\}$, then let P be such an appendix and $H'_P = S_i \cup S_{i+1}$. Otherwise, let P be an appendix of (H, x) such that for a sector H'_P of H w.r.t. P, there is no appendix Q of (H, x) such that H'_P properly contains a sector of H w.r.t. Q. Assume further that such a P is chosen so that $|V(P)|$ is minimized. Let H''_P be the other sector of H w.r.t. P. W.l.o.g. we may assume that H'_P contains $x_1, \ldots, x_l$, $l \geq 3$, and does not contain $x_{l+1}, \ldots, x_n$. Let $\{y_1, y_2\}$ be the attachment of P to H such that $y_1 \in S_n \setminus x_n$ and $y_2 \in S_l \setminus x_{l+1}$. Let $P = p_1 \dots p_k$ and w.l.o.g. assume that p_1 is adjacent to y_1, and p_k to y_2. Let S'_n be the $x_1 y_1$-subpath of S_n, and S'_l the $x_l y_2$-subpath of S_l. Let H' be the hole induced by $H'_P \cup P$. Note that since $l \geq 3$, (H', x) is a wheel.

(1) (H', x) is a proper wheel.

Let $u \in G \setminus (H' \cup N[x])$ and assume that u is not of type 0, 1 or 2 w.r.t. (H', x). Note that $u \notin H$. Since (H, x) is a proper wheel, u must have a neighbor in P. Let $P' = y_1 P y_2$. Let u_1 (resp. u_2) be the neighbor of u in P that is closest to y_1 (resp. y_2). Note that sectors $S_1, \dots, S_{l-1}$ of (H, x) are also sectors of (H', x).

First suppose that $N(u) \cap H' \subseteq S'_n \cup S'_l \cup P$. Since (H, x) is proper, u cannot have a neighbor in both S'_n and S'_l. If u has a neighbor in $S'_n \setminus y_1$, then by Lemma 3.2 and since u has a neighbor in P, $P \cup u$ contains an appendix Q of (H, x) such that a sector of H w.r.t. Q is properly contained in H'_P, contradicting our choice of P. So u does not have a neighbor in $S'_n \setminus y_1$, and by symmetry it does not have a neighbor in $S'_l \setminus y_2$. Since u is not of type 0, 1 or 2 w.r.t. (H', x), $u_1 \neq u_2$, and since G is triangle-free, $u_1 u_2$ is not an edge. Let P'' be the chordless path from y_1 to y_2 in $P \cup u$ that contains u. Since the length of P'' cannot be less than the length of P, by the choice of P, there is a vertex $p \in P$ such that $u_1 p$ and $u_2 p$ are edges. Since G is triangle-free, u is not adjacent to p, and hence u has exactly two neighbors in P'. Since u is not of type 2 w.r.t. (H', x), x must be adjacent to p. But then $S'_n \cup S'_l \cup P \cup \{x, u\}$ induces an ISK4.

Therefore u must have a neighbor in $(S_1 \cup \dots \cup S_{l-1}) \setminus \{x_1, x_l\}$. Since (H, x) is proper, for some $i \in \{1, \dots, l-1\}$, $N(u) \cap H \subseteq S_i$. Suppose that u is of type 2 w.r.t. (H, x). W.l.o.g. u is not adjacent to x_1. Let p_j be the vertex of P with lowest index adjacent to u. If $j = k$ then, since G is triangle-free, not both p_k and u can be adjacent to x_l, and hence $H \cup \{u, p_k\}$ induces an ISK4. So $j < k$, and hence $H \cup \{u, p_1, \dots, p_j\}$ induces an ISK4. Therefore u is of type 1 w.r.t. (H, x). Recall that by our assumption that u has a neighbor in $(S_1 \cup \dots \cup S_{l-1}) \setminus \{x_1, x_l\}$, u is not adjacent to x_1 nor x_l. But then $H \cup P \cup \{u\}$ contains an ISK4. This proves (1).

So (H', x) is a proper wheel. Since it cannot have fewer sectors than (H, x) and by Lemma 3.3, it follows that $y_1 = x_1$, $y_2 = x_l$ and $l = n - 1$. But then by the choice of P, $l = 3$, and hence (H, x) is a 4-wheel. □

A *short connection* between sectors S_i and S_{i+1} of (H, x) is a chordless path $P = p_1 \dots p_k$, $k > 1$, in $G \setminus (H \cup N[x])$ such that the following hold:

(i) $N(p_1) \cap (H \setminus \{x_{i+1}\}) = \{u_1\}$, $u_1 \in S_i \setminus \{x_{i+1}\}$,

(ii) $N(p_k) \cap (H \setminus \{x_{i+1}\}) = \{u_2\}$, $u_2 \in S_{i+1} \setminus \{x_{i+1}\}$, and

(iii) the only vertex of H that may have a neighbor in $P \setminus \{p_1, p_k\}$ is x_{i+1}.

Lemma 3.6 *Let G be a {triangle, ISK4, $K_{3,3}$}-free graph. Let (H, x) be a proper wheel of G with fewest number of spokes. Then (H, x) has no short connection.*

PROOF — Suppose (H, x) has a short connection $P = p_1 \dots p_k$. Assume that (H, x) and P are chosen so that $|V(P)|$ is minimized. W.l.o.g. p_1 is adjacent to $u_1 \in S_1 \setminus x_2$ and p_k to $u_2 \in S_2 \setminus x_2$. Let S_1' be the u_1x_1-subpath of S_1, and let S_2' be the u_2x_3-subpath of S_2. Let S_P be the u_1u_2-subpath of H that contains x_2. Let H' be the hole induced by $(H \setminus S_P) \cup P \cup \{u_1, u_2\}$.

(1) $n \geq 5$

If $n = 3$ then $H \cup \{x\}$ induces an ISK4. If $n = 4$ then $H' \cup \{x\}$ induces an ISK4. Therefore, $n \geq 5$. This proves (1).

(2) (H, x) has no appendix.

Follows from (1) and Lemma 3.5. This proves (2).

(3) Vertex x_2 has a neighbor in $P \setminus \{p_1, p_k\}$.

Assume not. If x_2 has no neighbor in P, then $S_1 \cup S_2 \cup P \cup \{x\}$ induces an ISK4. So w.l.o.g. x_2 is adjacent to p_1. Then u_1x_2 is not an edge, since G is triangle-free. If x_2 is not adjacent to p_k, then $S_1' \cup S_2 \cup P \cup \{x\}$ induces an ISK4. So x_2 is adjacent to p_k, and hence u_2x_2 is not an edge. But then $S_1' \cup S_2' \cup P \cup \{x, x_2\}$ induces an ISK4. This proves (3).

(4) (H', x) is a proper wheel.

By (1) (H', x) is a wheel. Assume it is not proper and let $u \in G \setminus (H' \cup N[x])$ be such that it is not of type 0, 1 or 2 w.r.t. (H', x). Note that $u \notin H$, and hence u is of type 0, 1 or 2 w.r.t. (H, x). It follows that u must have a neighbor in P. Let p_i (resp. p_j) be the vertex of P with lowest (resp. highest) index adjacent to u.

First suppose that $N(u) \cap H' \subseteq S_1' \cup S_2' \cup P$. Then $|N(u) \cap H'| \geq 3$. If u does not have a neighbor in $(S_1 \cup S_2) \setminus \{x_2\}$, then u has at least three neighbors in P and hence $p_1Pp_iup_jPp_k$ is a short connection of (H, x) that contradicts our choice of P. So u has a neighbor in $(S_1 \cup S_2) \setminus \{x_2\}$. W.l.o.g. $N(u) \cap (S_1 \setminus \{x_2\}) \neq \emptyset$. Since (H, x) is proper, $N(u) \cap (S_2 \setminus \{x_2\}) = \emptyset$. If

u has a unique neighbor in S_1, then $j > 2$ (since G is triangle-free and u has at least two neighbors in P) and hence up_jPp_k is a short connection of (H, x) that contradicts our choice of P. So u is of type 2 w.r.t. (H, x). If $j = 1$ then $|N(u) \cap H'| = 3$ and hence $H' \cup \{u\}$ induces an ISK4. So $j > 1$. If ux_2 is an edge then $j > 2$ (since G is triangle-free and u has at least three neighbors in H') and hence up_jPp_k is a short connection of (H, x) that contradicts our choice of P. So ux_2 is not an edge. If x_2 has no neighbor in p_jPp_k and $u_2 = x_3$, then let $Q = p_jPp_kx_3$. Otherwise let Q be the chordless path from p_j to x_2 in $(S_2 \setminus S'_2) \cup \{p_j, \ldots, p_k, u_2\}$. Then $S_1 \cup Q \cup \{x, u\}$ induces an ISK4.

Therefore, for some $l \in \{3, \ldots, n\}$, u has a neighbor in $S_l \setminus \{x_1, x_3\}$. Since (H, x) is proper, $N(u) \cap H \subseteq S_l$. If $j = 1$ let $Q = up_1$, if $i = k$ let $Q = up_k$, and otherwise let Q be a chordless path in $(P \setminus \{p_1, p_k\}) \cup \{u\}$ from u to a vertex of $P \setminus \{p_1, p_k\}$ that is adjacent to x_2 (note that such a vertex exists by (3)). By Lemma 3.2 Q is an appendix of H. In particular, u is of type 1 w.r.t. (H, x). Let u' be the neighbor of u in H. Since by (2) Q cannot be an appendix of (H, x), w.l.o.g. $j = 1$ and $l = n$. Note that $u' \neq x_1$ and $u_1 \neq x_2$. Let p_t be the vertex of P with lowest index adjacent to x_2 (such a vertex exists by (2)). If $u' = x_n$ then $S_1 \cup \{x, u, u', p_1, \ldots, p_t\}$ induces an ISK4. So $u' \neq x_n$. If If $u_1 \neq x_1$ then $S_1 \cup \{x, u, p_1, \ldots, p_t\}$ together with the x_1u'-subpath of S_1 induces an ISK4. So $u_1 = x_1$. But then $S_n \cup \{x, u, p_1, \ldots, p_t, x_2\}$ induces an ISK4. This proves (4).

By (4) (H', x) is a proper wheel that has fewer sectors than (H, x), a contradiction. □

Lemma 3.7 *Let G be a {triangle, ISK4, $K_{3,3}$}-free graph. Let (H, x) be a proper wheel of G with fewest number of spokes. Then for every $i \in \{1, \ldots, n\}$, $(N[x] \setminus H) \cup \{x_i, x_{i+1}\}$ is a star cutset separating S_i from $H \setminus S_i$.*

PROOF — Assume not. Then w.l.o.g. there is a direct connection $P = p_1 \ldots p_k$ from $S_1 \setminus \{x_1, x_2\}$ to $H \setminus S_1$ in $G \setminus ((N[x] \setminus H) \cup \{x_1, x_2\})$. Note that the only vertices of H that may have a neighbor in the interior of P are x_1 and x_2. Since (H, x) is proper, $k > 1$ and p_1 and p_k are of type 1 or 2 w.r.t. (H, x). Let $i \in \{2, \ldots, n\}$ be such that $N(p_k) \cap H \subseteq S_i$.

First suppose that no vertex of $\{x_1, x_2\}$ has a neighbor in $P \setminus \{p_1, p_k\}$. By Lemma 3.2, P is an appendix of H. In particular, p_1 and p_k are both of type 1 w.r.t. (H, x). If $i \notin \{2, n\}$ then P is an appendix of (H, x). It follows from Lemma 3.5 that $n = 4$ and $i = 3$. But then, since p_1 has a neighbor in $S_1 \setminus \{x_1, x_2\}$, (H, x) and P contradict Lemma 3.3. So $i \in \{2, n\}$, and hence

P is a short connection, contradicting Lemma 3.6. Therefore, a vertex of $\{x_1, x_2\}$ has a neighbor in $P \setminus \{p_1, p_k\}$.

Let p_j be the vertex of $P \setminus p_1$ with highest index adjacent to a vertex of $\{x_1, x_2\}$. W.l.o.g. $p_j x_2$ is an edge. We now show that if p_1 has two neighbors in $S_1 \setminus x_2$, then x_1 has a neighbor in $P \setminus p_1$. Assume not. Then p_1 has two neighbors in $S_1 \setminus x_2$ and x_1 does not have a neighbor in $P \setminus p_1$. Since p_1 is of type 2 w.r.t. (H, x), $p_1 x_2$ is not an edge. But then H and $p_1 \dots p_{j'}$ (where $p_{j'}$ is the vertex of P with lowest index adjacent to x_2) contradict Lemma 3.2.

Suppose $i = 2$. If p_k has two neighbors in $S_2 \setminus x_2$, then $S_2 \cup \{x, p_j, \dots, p_k\}$ induces an ISK4. So, p_k has a unique neighbor in $S_2 \setminus x_2$. If x_1 does not have a neighbor in $P \setminus p_1$, then p_1 has a unique neighbor in $S_1 \setminus x_2$ and hence P is a short connection of (H, x), contradicting Lemma 3.6. So x_1 has a neighbor in $P \setminus p_1$. Let p_t be such a neighbor with highest index. Then $p_t P p_k$ is a short connection of (H, x), contradicting Lemma 3.6. So $i \neq 2$. If $i = n$ then either $p_j P p_k$ is a short connection of (H, x) contradicting Lemma 3.6, or $S_n \cup \{x, x_2, p_j, \dots, p_k\}$ contains an ISK4.

Therefore, $i \in \{3, \dots, n-1\}$ and p_k has a neighbor in $H \setminus (S_1 \cup S_2 \cup S_n)$. By Lemma 3.2 applied to H and $p_j P p_k$, vertex p_k is of type 1 w.r.t. (H, x) and $x_1 p_j$ is not an edge. But then $p_j P p_k$ is an appendix of (H, x). By Lemma 3.5 it follows that $n = 4$ and $i = 3$. But then, since p_k has a neighbor in $S_3 \setminus \{x_3, x_4\}$, (H, x) and $p_j P p_k$ contradict Lemma 3.3. □

We say that a graph G has a *wheel decomposition* if for some wheel (H, x), for every $i \in \{1, \dots, n\}$, $(N[x] \setminus H) \cup \{x_i, x_{i+1}\}$ is a cutset separating S_i from $H \setminus S_i$. We say that such a wheel decomposition is *w.r.t. wheel* (H, x). Note that if a graph has a wheel decomposition, then it has a star cutset.

Theorem 3.8 *If G is a $\{$triangle, ISK4$\}$-free graph, then either G is a series-parallel graph or a complete bipartite graph, or G has a clique cutset of size at most two, or G has a wheel decomposition.*

PROOF — Assume G is not series-parallel nor a complete bipartite graph. By Lemma 2.5 G contains a wheel or $K_{3,3}$. By Lemma 2.6 if G contains a $K_{3,3}$ then it has a clique cutset of size at most two. So we may assume that G does not contain a $K_{3,3}$. So G contains a wheel. By Lemma 3.4 G contains a proper wheel, and hence by Lemma 3.7 G has a wheel decomposition. □

Theorem 3.9 *If G is a {triangle, ISK4, $K_{3,3}$}-free graph, then either G is series-parallel or G has a wheel decomposition.*

PROOF — Assume G is not series-parallel. By Lemma 2.5 G contains a wheel. By Lemma 3.4 G contains a proper wheel, and hence by Lemma 3.7 G has a wheel decomposition. □

The following corollary is needed in the next section.

Corollary 3.10 *If G is an ISK4-free graph of girth at least 5, then either G is series-parallel or G has a star cutset.*

PROOF — Follows directly from Theorem 3.9 because $K_{3,3}$ contains a cycle of length 4. □

4 Chordless graphs

A graph G is *chordless* if no cycle in G has a chord. Chordless graphs were introduced in [2] as roots of wheel-free line graphs, and it is a surprise to us that we need them here for a completely different reason in a very similar class. A graph is *sparse* if every edge is incident to at least one vertex of degree at most 2. A sparse graph is chordless because any chord of a cycle is an edge between two vertices of degree at least three. Recall that proper 2-cutsets are defined in Section 2.

Theorem 4.1 (see [3]) *If G is a 2-connected chordless graph, then either G is sparse or G admits a proper 2-cutset.*

The following theorem is mentioned in [3] without a proof, and we need it in the next section. So, we prove it for the sake of completeness.

Theorem 4.2 *In every cycle of a 2-connected chordless graph that is not a cycle, there exist four vertices a, b, c, d that appear in this order and such that a, c have degree 2 and b, d have degree at least 3.*

PROOF — We prove the result by induction on $|V(G)|$. If G is sparse (in particular, if $|V(G)| = 3$), then it is enough to check that every cycle of G contains at least two vertices of degree at least 3, because these vertices cannot be adjacent in a sparse graph. But this true, because a cycle with all vertices of degree 2 must be the whole graph (since G is connected), and a cycle with a unique vertex of degree 3 cannot exists in a 2-connected graph (the vertex of degree at least 3 would be a cut-vertex).

So, by Theorem 4.1 we may assume that G has a proper 2-cutset with split (X, Y, u, v). We now build two blocks of decompositions of G as follows. The block G_X is obtained from $G[X \cup \{u, v\}]$ by adding a marker vertex m_Y adjacent to u and v, and the block G_X is obtained from $G[Y \cup \{u, v\}]$ by adding a marker vertex m_X adjacent to u and v. By the definition of proper 2-cutsets, $|V(G_X)|, |V(G_Y)| \leq |V(G)|$. Also, G_X and G_Y are chordless and 2-connected. So we may apply the induction hypothesis to the blocks of decomposition.

Let C be a cycle of G. If $V(C) \subseteq X \cup \{u, v\}$, then C is a cycle of G_X, so by the induction hypothesis we get four vertices a, b, c, d in C. We now check that a vertex $w \in V(C)$ has degree 2 in G if and only if it has degree 2 in G_X. This is obvious, except if $w \in \{u, v\}$. But in this case, because of m_Y and because w lies in a cycle of G_X that does not contain m_Y, w has degree at least 3 in both G and G_X. This proves our claim. It follows that we obtain by the induction hypothesis the condition that we need for the degrees of a, b, c and d. The proof is similar when $V(C) \subseteq Y \cup \{u, v\}$.

We may now assume that C has vertices in X and Y. It follows that C edge wise partitions into a path $P = u \dots v$ whose interior is in X and a path $Q = u \dots v$ whose interior is in Y. We apply the induction hypothesis to G_X and $C_X = uPvm_Yu$. So, we get four vertices a, b, c and d in C_X and they have degree 2, ≥ 3, 2, ≥ 3 respectively (in G_X). These four vertices are in C and have the degrees we need (in G), except possibly when $|\{a, c\} \cap \{u, m_Y, v\}| = 1$. In this case, we may assume w.l.o.g. that $a = u$ or $a = m_Y$, and we find in place of a a vertex of degree 2 in $Y \cap V(C_Y)$, where $C_Y = uQvm_Xu$. This vertex exists by the induction hypothesis applied to G_Y and C_Y. □

5 Degree 2 vertices

Lemma 5.1 *Let G be a 2-connected ISK4-free graph of girth at least 5. Then for every pair $\{x, y\}$ of vertices of G such that $x = y$ or $xy \in E(G)$, $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree 2 in G.*

PROOF — We prove the statement by induction on $|V(G)|$. The statement is true for the smallest 2-connected graph of girth at least 5: C_5. We now prove the induction step. There are two cases.

Case 1: G has no star cutset. In particular, G has no clique cutset, and it is series-parallel by Corollary 3.10. We claim that G is chordless. Otherwise, some cycle C of G has a chord ab. Thereofore, C is formed of two ab-paths R and R', both of length at least 2. Since $\{a, b\}$ is not a clique cutset of G, some path S of G is disjoint from $\{a, b\}$, has one end in R, the other one in R', and is internally disjoint from C. Therefore, C, ab and S form a subdivision of K_4 (that is in G as a subgraph), and this contradicts G being series-parallel. This proves our claim.

If $G \setminus (N[x] \cup N[y])$ contains a cycle, then by Theorem 4.2, it contains a vertex of degree 2 (in G). It follows that we may assume that $G \setminus (N[x] \cup N[y])$ is a forest F. So, let u be a leaf (that is a vertex of degree at most 1) of F. We may assume that u has at least two neighbors v, w outside of F (since otherwise, u has degree 2). These two neighbors must be adjacent to x or y, but since G has no C_4, we have $x \neq y$, and we may assume that v is adjacent to x and w is adjacent to y. Also, u has neighbors only in $V(F) \cup \{v, w\}$. Hence u has degree 2, unless it has a neighbor in F. In this last case, F has a second leaf u' and there is a path P in F from u to u'. By the same proof as we did for u, u' has two neighbors v', w' not in F and $v'x, w'y \in E(G)$. Now, we see that G contains a cycle with a chord, a contradiction.

Case 2: G has a star cutset C. We suppose that C is inclusionwise minimal among all possible star cutsets and is centered at c. W.l.o.g. we suppose that x and y are both in $G[C \cup X]$ where X is a component of $G \setminus C$, and we consider another component Y.

We claim that $G[C \cup Y]$ is 2-connected. Indeed, suppose for a contradiction that $G[C \cup Y]$ has a cutvertex v. Since G is 2-connected, $|C| \geq 2$. By the minimality of C, every vertex of $C \setminus \{c\}$ has a neighbor in Y and if $|C| = 2$, then c also has neighbors in Y. It follows that $v \notin C$. So, v is in fact a cutvertex of G, a contradiction. We proved that $G[C \cup Y]$ is 2-connected.

We now apply the induction hypothesis to $\{x, y, c\} \setminus X$ in the graph $G[C \cup Y]$. This gives a vertex in $G \setminus (N[x] \cup N[y])$ that has degree 2 in G. □

Theorem 5.2 *Every ISK4-free graph of girth at least 5 contains a vertex of degree at most 2 and is 3-colorable.*

PROOF — It is enough to prove that every ISK4-free graph of girth at least 5 contains a vertex of degree at most 2. For the sake of induction, we prove by induction on $|V(G)|$ a slightly stronger statement: every ISK4-free graph of girth at least 5 on at least two vertices contains at least two vertices of degree at most 2. If $|V(G)| = 2$, this is clearly true. If G is 2-connected, it follows from Lemma 5.1 applied twice (once to find a vertex x of degree 2, and another time to find the second one in $G \setminus N[x]$). So, we may assume that G is not 2-connected and has at least 3 vertices, so it has a cutvertex v. The result follows from the induction hypothesis applied to $G[X \cup \{v\}]$ and to $G[Y \cup \{v\}]$ where X and Y are connected components of $G \setminus \{v\}$. □

6 Series-parallel graphs

A graph G together with two of its vertices x and y such that $xy \in E(G)$ or $x = y$, have the *(x, y)-property* if $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree 2 in G. Instead of (x, x)-property, we simply write *x-property*. The (x, y)-property is very convenient for us, because it ensures the existence of vertices of degree 2, and also because, as shown in the previous section, it is well preserved in proofs by induction. When squares are allowed, there are graphs in our class that do not have the (x, y)-property, for intance the graphs represented in Fig. 1 do not have the x-property when x is a vertex with maximum degree in the graph. In order to prove Conjecture 1.1, it is therefore of interest to understand fully the triangle-free series-parallel graphs that do not have the (x, y)-property. This is what we do in the next lemmas.

Lemma 6.1 *Let T be a tree, and suppose that the vertices of T are labelled with labels x and y (each vertex may receive one label, both labels, or no label). One and only one of the following situations occurs.*

- *In T, there exist two vertex-disjoint paths P and Q, and each of them is from a vertex with label x to a vertex with label y (possibly, P and/or Q have length 0).*
- *There exists $v \in V(T)$ and two subtrees of T, T_x and T_y such that:*
  1. *$V(T_x) \cup V(T_y) = V(T)$;*
  2. *$V(T_x) \cap V(T_y) = \{v\}$;*
  3. *T_x contains all vertices of T with label x and T_y contains all vertices of T with label y.*

PROOF — It is clear that not both outcomes hold, because if the second holds, then v must be on every path from a vertex with label x to a vertex with label y, so no two such paths can be vertex-disjoint.

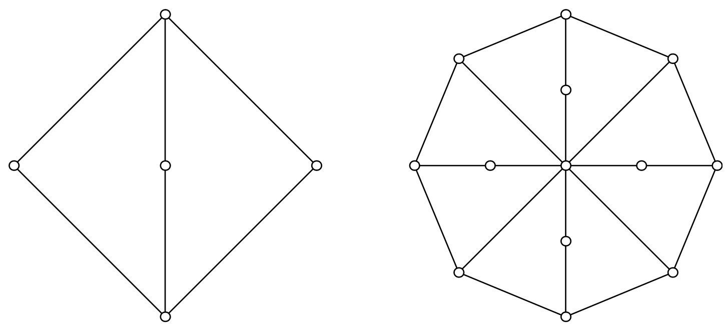

Figure 1: A series-parallel graph and an ISK4-free graph

If the second situation does not occur then it implies that for every vertex v of the tree, there exists a connected component of $T \setminus v$ that contains a vertex with label x and a vertex with label y. Thus for every vertex v, orient the corresponding edge (the one pointing towards this connected component) out from v. Because, $|E(T)| < |V(T)|$ some edge uv has to be oriented both ways, which means that the two components of $T \setminus uv$ contains vertices with both labels, which clearly implies the first situation. □

A *bad triple* is a triple (G, x, y) such that the following hold:

- G is a graph and x and y are vertices of G.
- G does not have the (x, y)-property.
- If $x \neq y$, then G has the x-property and G has the y-property.

It is clear that for every graph G and all vertices $x, y \in V(G)$ such that $x = y$ or $xy \in E(G)$, one of the following cases holds:

- G has the (x, y)-property;
- (G, x, x) is a bad triple;
- (G, y, y) is a bad triple;
- (G, x, y) is a bad triple.

Indeed, if G does not have the x-property, then (G, x, x) is a bad triple. Similarly, if G does not have the y-property, then (G, y, y) is a bad triple. So, we may assume that G has the x-property and the y-property. If G has the (x, y)-property, then we are done, and otherwise, all the requirements in the definition of bad triples are fulfilled. It follows that a structural description of bad triples really discribes all triples (G, x, y) such that G does not have the (x, y)-property. Such a description is given in the next lemma for triangle-free 2-connected series-parallel graphs with no clique cutset. Note that in the next lemma, we do not require the girth of the graph to be at least 5.

Lemma 6.2 *Let (G, x, y) be a bad triple, and suppose that G is triangle-free, 2-connected, series-parallel, has at least 5 vertices, and has no clique cutset. Then, G can be constructed as follows (and conversely, all graphs constructed as follows are triangle-free, 2-connected, series-parallel, have least 5 vertices, and have no clique cutset).*

- *If $x \neq y$, then build two non-empty trees T_x and T_y, not containing x, y, and consider the tree T obtained by gluing T_x and T_y along some vertex v (so $V(T_x) \cap V(T_y) = \{v\}$).*
- *If $x = y$ build a non-empty tree T, and set $T_x = T_y = T$.*
- *Add vertices of degree 2, each of them either adjacent to x and to some vertex in T_x, or to y and some vertex in T_y, in such a way that the following conditions are satisfied:*
  1. $|N(x) \setminus \{y\}|, |N(y) \setminus \{x\}| \geq 1$*;*
  2. *every vertex of T that has degree 2 in T has at least one neighbor in $(N[x] \cup N[y]) \setminus \{x, y\}$;*
  3. *every vertex of T that has degree 1 in T (so, a leaf of T) has at least two neighbors in $(N[x] \cup N[y]) \setminus \{x, y\}$;*
  4. *every vertex of T that has degree 0 in T (this happens only when* $|V(T)| = |V(T_x)| = |V(T_y)| = 1$*) has at least three neighbors in* $(N[x] \cup N[y]) \setminus \{x, y\}$.

PROOF — Let (G, x, y) be a bad triple, and suppose that G is triangle-free, 2-connected, series-parallel, has at least 5 vertices, and has no clique cutset.

(1) G is chordless.

Suppose that some cycle C of G has a chord ab. Thereofore, C is formed of two ab-paths R and R', both of length at least 2. Since $\{a, b\}$ is not a clique cutset of G, some path S of G is disjoint from $\{a, b\}$, has one end in R, the other one in R', and is internally disjoint from C. Therefore, C, ab and S

form a subdivision of K_4 (that is in G as a subgraph), and this contradicts G being series-parallel. This proves (1).

(2) Every cycle of G contains at least two vertices of degree 2.

If G is a cycle, our claim clearly holds. Otherwise, it follows directly from (1) and Theorem 4.2. This proves (2).

(3) All vertices in $N(x) \cup N(y)$ distinct from x and y have degree 2.

Suppose w.l.o.g. that a neighbor of x, $x' \neq y$ has degree at least 3. Let u and v be two neighbors of x' distinct from x. Because G is triangle-free, u and v are non-adjacent to x. Since G is 2-connected, there is a path $P = u \dots v$ in $G \setminus x'$. So, $C = x'uPvx'$ is a cycle of G. Note that C does not go through x, for otherwise xx' would be a chord a C, a contradiction to (1).

By (2), C contains two vertices a and b of degree 2. Vertices a and b must be adjacent to x or y because they have degree 2 and G does not have the (x, y)-property. If a and b are both adjacent to x (in particular, when $x = y$), then C must go through x (because a and b have degree 2), a contradiction.

Hence, at least one of a or b is adjacent to y, not to x, and in particular, $x \neq y$. If C does not go through y, then C, x and y form a subdivision of K_4, contradiciting G being series-parallel, so C goes through y. If x is adjacent to a or b, then again C, x and y form a subdivision of K_4. So, a and b are both adjacent to y, not to x. Since G has the y-property (from the definition of bad triples), there is a non-neighbor x'' of y that has degree 2, and since G does not have the (x, y)-property, x'' is a neighbor of x, distinct from x' because x' has degree 3. Since G is 2-connected, there exists a path $Q = x'' \dots c$ from x'' to C in $G \setminus x$. If $c \neq x', y$, then x, Q and C form a subdivision of K_4, a contradiction to G being series-parallel. Otherwise, $c \in \{x', c\}$ and xc is a chord of some cycle of G, a contradiction to (1). This proves (3).

(4) $N[x] \cup N[y] \subsetneq V(G)$.

Otherwise, $V(G) = N[x] \cup N[y]$ and all vertices of G are adjacent to x or y. If $x = y$, then $V(G) = N[x]$, a contradiction since G is triangle-free and 2-connected. So, $x \neq y$.

Let $x' \neq y$ be a neighbor of x. By (3), x' has degree 2. Its other neighbor y' is non-adjacent to x (because G is triangle-free), so it must be adjacent to y, and by (3), y' has degree 2. Since G contains at least five vertices, there must be other vertices, so w.l.o.g. another neighor x'' of x. Again, x'' has degree 2, a neighbor y'' (distinct from y' because y' has degree 2), and y'' is adjacent yo y. Now, $xx'y'yy''x''x$ is a cycle of G that has a chord (namely xy), a contradiction to (1). This proves (4).

(5) $(N[x] \cup N[y]) \setminus \{x, y\}$ *is a stable set.*

Otherwise, let u and v be two adjacent vertices in $(N(x) \cup N(y)) \setminus \{x, y\}$. By (3) they have degree 2, so $\{x, y\}$ is a clique cutset that separates them from $V(G) \setminus (N[x] \cup N[y])$ that is non-empty by (4). This proves (5).

(6) $G \setminus (N[x] \cup N[y])$ *is a tree* T*.*

Since (G, x, y) is a bad triple, $V(G) \setminus (N[x] \cup N[y])$ contains only vertices of degree at least 3 (in G), so by (2), it cannot contain a cycle. Also $G \setminus (N[x] \cup N[y])$ is connected, because otherwise let A and B be two components of $V(G) \setminus (N[x] \cup N[y])$. These two components must attach to disjoint sets of neighbors of x and y, because all neighbors of x and y (except x and y) have degree 2 by (3). It follows that $\{x, y\}$ is a clique cutset, a contradiction. This proves (6).

Let us now give label x (resp. y) to all vertices of T that have a neighbor adjacent to x (resp. y). Let us apply Lemma 6.1 to T.

If the first outcome holds (so in T, there exist two vertex-disjoint paths P and Q, and each of them is from a vertex with label x to a vertex with label y), then we reach a contradiction as follows. Let $P = p_x \dots p_y$ and $Q = q_x \dots q_y$ where p_x, q_x have label x and p_y, q_y have label y. So, let p'_x be a neighbor of p_x that is adjacent to x, and let p'_y, q'_x and q'_y be defined similarly. Observe that p'_x, p'_y, q'_x and q'_y are distinct, because all neighbors of x and y distinct from x and y have degree 2 by (3). Now the cycle $xp'_xp_xPp_yp'_yyq'_yq_yQq_xq'_xx$ has a chord (namely xy), a contradiction to (1).

Hence, the second outcome holds, so we keep the notation v, T_x and T_y from Lemma 6.1. Note that T can be obtained by gluing T_x and T_y along v. It follows that G can be constructed as we claim it should be. By (6) and (4), we really need to consider a non-empty tree. By (3), we have to add vertices of degree 2, and by (5), they all have one neighbor in $\{x, y\}$ and the other one in T. The last three conditions are here to ensure that the vertices of T really all have degree at least 3.

We do not prove the converse statement (every graph constructed as above is a bad triple, is triangle-free, 2-connected, series-parallel, has at least 5 vertices and has no clique cutset). It is easy to check and we do not need it in what follows. □

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