The structure of (theta, pyramid, 1-wheel, 3-wheel)-free graphs

Valerio Boncompagni *, Marko Radovanović † and Kristina Vušković ‡

August 4, 2018

Abstract

In this paper we study the class of graphs $\mathcal{C}$ defined by excluding the following structures as induced subgraphs: thetas, pyramids, 1-wheels and 3-wheels. We describe the structure of graphs in $\mathcal{C}$, and we give a polynomial-time recognition algorithm for this class. We also prove that K_4-free graphs in $\mathcal{C}$ are 4-colorable. We remark that $\mathcal{C}$ includes the class of chordal graphs, as well as the class of line graphs of triangle-free graphs.

Key words: structure, decomposition, clique cutsets, bisimplicial cutsets, 2-amalgams, recognition algorithm, vertex coloring

AMS Classification (2010): 05C75, 05C85

1 Introduction

Throughout the paper all graphs are finite and simple. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G, and it is F*-free* if it does not contain F. For a family of graphs $\mathcal{F}$ we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. A *hole* in a graph is a chordless cycle of length at least 4, and it is *even* or *odd* depending on the parity of its length.

In 1982 Truemper [18] gave a theorem that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities. The characterization states that this can be done for a graph G if an only if it can be done for all induced subgraphs of G that are of few specific types (depicted in Figure 1, note that in all figures solid lines denote an edge and a dashed line denotes a chordless path containing one or more edges), which we will call *Truemper configurations*, and will describe precisely later. We observe that in Figure 1 we have depicted a wheel in which the center vertex has 4 neighbors on the outer hole, but in general it can have any number of neighbors, greater than 2, on the outer hole. Truemper was originally motivated by the problem of obtaining a co-NP characterization of bipartite

*School of Computing, University of Leeds, Leeds LS2 9JT, UK. E-mail: scvb@leeds.ac.uk.

†Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia. E-mail: markor@matf.bg.ac.rs. Partially supported by Serbian Ministry of Education, Science and Technological Development grant 174033.

‡School of Computing, University of Leeds, Leeds LS2 9JT, UK and Faculty of Computer Science, Union University, Knez Mihailova 6/VI, 11000 Belgrade, Serbia. E-mail: k.vuskovic@leeds.ac.uk. Partially supported by Serbian Ministry of Education, Science and Technological Development grants III44006 and 174033, and EPSRC grants EP/K016423/1 and EP/N019660/1.

graphs that are signable to be balanced (i.e. bipartite graphs whose vertex-vertex adjacency matrices are balanceable matrices, a class of matrices that have important polyhedral properties).

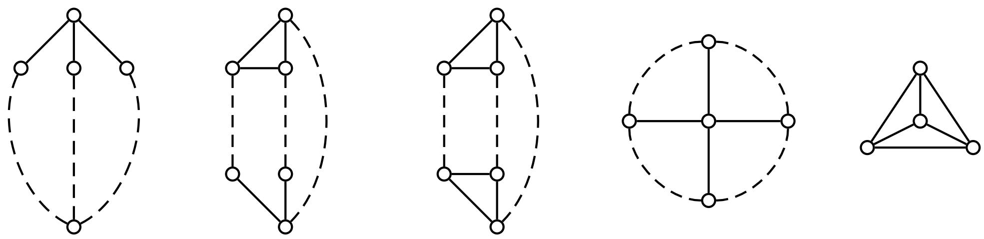

Figure 1: Truemper configurations and K_4

The configurations that Truemper identified in his theorem later played an important role in understanding the structure of several seemingly diverse classes of objects, such as regular matroids, balanceable matrices, perfect graphs, odd-hole-free and even-hole-free graphs (for a survey see [19]). All these classes were studied using the decomposition method. In these decomposition theorems Truemper configurations appear both as excluded structures that are convenient to work with, and as structures around which the actual decomposition takes place.

In this paper we study the class $\mathcal{C}$ of (theta, pyramid, 1-wheel, 3-wheel)-free graphs, which we formally define in Section 1.2. Observe that this is a hereditary class of graphs defined by excluding only cyclic structures. This class contains all chordal graphs and all line graphs of triangle-free graphs (or equivalently, (claw, diamond)-free graphs [10]). This class was first studied in [1] where it was shown that every graph in $\mathcal{C}$ has a vertex whose neighborhood is a disjoint union of two (possibly empty) cliques, and furthermore an ordering of such vertices can be found by LexBFS. A consequence of this is a linear-time algorithm for the maximum weight clique problem on $\mathcal{C}$, as well as a linear-time coloring algorithm that colors the graph with at most $2\omega(G) - 1$ colors, where $\omega(G)$ denotes the size of the largest clique in G. Coloring is NP-hard on line graphs of triangle-free graphs, and in fact it is NP-hard on $(K_4, \text{claw}, \text{diamond})$-free graphs [11]. The complexity of the stable set problem on $\mathcal{C}$ is open, and in fact it is open even for the subclass of K_4-free graphs in $\mathcal{C}$. On the other hand, stable set problem is polynomial-time solvable on line graphs – one just computes the root graph using the linear algorithm from [13] and then uses Edmond's algorithm from [8] to find a maximal matching in the root graph.

In this paper we describe the structure of graphs in $\mathcal{C}$, and as a consequence we obtain a series of decomposition theorems that use cutsets that combine star cutsets and 2-joins in the simplest possible ways. These theorems present a good setting for studying various problems, and in particular the stable set problem restricted to the class $\mathcal{C}$. Two much studied hereditary graph classes are even-hole-free graphs and perfect graphs (see for example surveys [19] and [17]). The complexity of the stable set problem on even-hole-free graphs is still not known, and also it is not known how to solve the stable set problem in polynomial time for perfect graphs by a purely graph theoretic algorithm (it is known that this problem can be solved in polynomial time for perfect graphs using the ellipsoid method [9]). The known decomposition theorems for these classes use star cutsets and 2-joins, as well as different generalizations of these. It is not clear how to make use of star cutsets for the stable set problem (and other problems), and therefore it would be of interest to understand how very structured star cutsets, such as the ones used in this paper, behave in algorithms.

The paper is organized as follows. In Sections 1.1 and 1.2 we introduce the terminology and notation that will be used throughout the paper. In Section 1.3 we give an overview of subclasses of $\mathcal{C}$ that were studied in literature. In Section 1.4 we give an overview of the complexity of recognizing different Truemper configurations, and in Section 2 we give two polynomial-time recognition algorithms for $\mathcal{C}$. In Section 1.5 we describe the structure of graphs in $\mathcal{C}$, which we prove in Sections 3 and 4. In Section 5, using the structure theorem for $\mathcal{C}$, we prove that K_4-free graphs in $\mathcal{C}$ are 4-colorable.

1.1 Terminology and notation

Let G be a graph. The vertex set of G is denoted by $V(G)$. Sometimes, when clear from context, for notational simplicity we will refer to $V(G)$ with just G. For $x \in V(G)$, $N(x)$ is the set of all neighbors of x in G, and $N[x] = N(x) \cup \{x\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, $G \setminus S = G[V(G) \setminus S]$, $N(S)$ denotes the set of vertices in $V(G) \setminus S$ with at least one neighbor in S, and $N[S] = N(S) \cup S$. Note that, if S is empty, then $N(S) = N[S] = \varnothing$.

Let A and B be two disjoint subsets of $V(G)$. A is *complete* to B if every vertex of A is adjacent to every vertex of B, and A is *anticomplete* to B if no vertex of A is adjacent to a vertex of B. Given a set $A \subset V(G)$ and a vertex $u \in V(G) \setminus A$, we will also say that u is complete (resp. anticomplete) to A if it is adjacent (resp. non-adjacent) to every vertex of A.

A *path* P is a sequence of distinct vertices $x_1, \dots, x_k$, $k \geq 1$, such that $x_i x_{i+1}$ is an edge for all $1 \leq i < k$. These are called the *edges* of P. Vertices x_1 and x_k are the *endpoints* of the path. The vertices of P that are not endpoints of P are called the *interior vertices* of P. Let x_i and x_j be two vertices of P such that $1 \leq i \leq j \leq k$. The path $x_i, x_{i+1}, \dots, x_j$ is called the *$x_i x_j$-subpath of* P, and is denoted by $P^{x_i x_j}$. For a $x_1 x_k$-path P and a subset S of the vertex set of P, we say that a vertex $u \in S$ is closest to x_1 if $V(P^{x_1 u}) \cap S = \{u\}$. A *cycle* C is a sequence of vertices $x_1, \dots, x_k, x_1$, $k \geq 3$, such that vertices $x_1, \dots, x_k$ form a path and $x_1 x_k$ is an edge. The edges of the path $x_1, \dots, x_k$, together with the edge $x_1 x_k$, are called the *edges* of C. Let Q be a path or a cycle. The vertex set of Q is denoted by $V(Q)$. The *length* of Q is the number of edges in Q.

Given a path or a cycle Q in a graph G, any edge of G between vertices of Q that is not an edge of Q is called a *chord* of Q. Q is *chordless* if no edge of G is a chord of Q. As mentioned earlier, a *hole* is a chordless cycle of length at least 4. It is called a *k-hole* if it has k edges. A k-hole is *even* if k is even, and it is *odd* otherwise.

In a graph G, a *clique* is a (possibly empty) subset of $V(G)$ consisting of pairwise adjacent vertices. The size of a largest clique in G is denoted by $\omega(G)$. A *complete graph* is a graph whose vertex set is a clique in that graph. A complete graph on n vertices is denoted by K_n, and a K_3 is also referred to as a *triangle*.

Given a graph G, a subset S of vertices and edges is a *cutset* if its removal results in a disconnected graph. A cutset S is a *clique cutset* if S is a clique. Note that a graph with no clique cutset is connected. A cutset S is a *star cutset* if, for some vertex $x \in S$, $S \subseteq N[x]$.

A *wheel* (H, x) is a graph induced by a hole H, called the *rim*, and a vertex x, called the *center*, that has at least three neighbors on H. A *sector* of a wheel is a subpath of the rim, of length at least 1, whose endpoints are adjacent to the center, but whose interior vertices are not. A sector is said to be *short* if it is of length 1, and *long* otherwise.

Throughout the paper, when we refer to a wheel (H, x), we will use the following associated terminology and notation. Let $x_1, \dots, x_n$ be the neighbors of x on H, appearing in this order when traversing H. For every $1 \leq i \leq n$, the sector of (H, x) with endpoints x_i and x_{i+1} (we assume that $x_{n+1} = x_1$) will be denoted by S_i (and throughout we will also assume that $S_{n+1} = S_1$). If S_i is a long sector, then we denote by x'_i (resp. x'_{i+1}) the neighbor of x_i (resp. x_{i+1}) in S_i. (We observe that the wheels in the class we will work with in this paper do not have consecutive long sectors, and hence x'_i and x'_{i+1} are well defined). Also, for a long sector S_i, the hole induced by $V(S_i) \cup \{x\}$ will be denoted by H_i.

A *k-coloring* of a graph G is a function $c : V(G) \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. A graph G is *k-colorable* if there exists a k-coloring of G. The chromatic number of G, denoted by $\chi(G)$, is the least k for which there exists a k-coloring of G.

1.2 Truemper configurations

The first three configurations in Figure 1 are referred to as *3-path-configurations* ($3PC$'s). They are structures induced by three paths P_1, P_2 and P_3, in such a way that, for every $i \neq j$, the vertices of P_i and P_j induce a hole. More specifically, a $3PC(x, y)$ is a structure induced by three paths that connect two non-adjacent vertices x and y; a $3PC(x_1x_2x_3, y)$, where $x_1x_2x_3$ is a triangle, is a structure induced by three paths having endpoints x_1, x_2 and x_3 respectively and a common endpoint y; a $3PC(x_1x_2x_3, y_1y_2y_3)$, where $x_1x_2x_3$ and $y_1y_2y_3$ are two vertex-disjoint triangles, is a structure induced by three paths P_1, P_2 and P_3 such that, for every $1 \leq i \leq 3$, path P_i has endpoints x_i and y_i. We say that a graph G contains a $3PC(\cdot, \cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$, a $3PC(\triangle, \cdot)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$, and a $3PC(\triangle, \triangle)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. Note that the condition that the vertices of P_i and P_j, for $i \neq j$, must induce a hole, implies that all paths of a $3PC(\cdot, \cdot)$ have length greater than one, and at most one path of a $3PC(\triangle, \cdot)$ has length one. In literature a $3PC(\cdot, \cdot)$ is also referred to as a *theta*, a $3PC(\triangle, \cdot)$ as a *pyramid*, and a $3PC(\triangle, \triangle)$ as a *prism*.

We refer to 3-path-configurations and wheels as *Truemper configurations*.

A wheel is a *1-wheel* if for some consecutive vertices x, y, z of the rim, the center is adjacent to y, but not to x and z. A wheel is a *2-wheel* if for some consecutive vertices x, y, z of the rim, the center is adjacent to x and y, but not to z. A wheel is a *3-wheel* if for some consecutive vertices x, y, z of the rim, the center is adjacent to x, y and z. Observe that a wheel can simultaneously be a 1-wheel, a 2-wheel and a 3-wheel, and that every wheel is a 1-wheel, a 2-wheel or a 3-wheel.

An *alternating wheel* is a wheel whose sectors alternate between short and long sectors. A *line wheel* is an alternating wheel with exactly two long sectors and two short sectors. A *long alternating wheel* is an alternating wheel that is not a line wheel.

From now on we will denote by $\mathcal{C}$ the class of (theta, pyramid, 1-wheel, 3-wheel)-free graphs. Note that the only Truemper configurations that these graphs may contain are prisms and alternating wheels.

1.3 Some subclasses of $\mathcal{C}$

The class $\mathcal{C}$ clearly contains all *chordal* graphs (i.e. hole-free graphs). We now describe some other subclasses of $\mathcal{C}$ that were studied in literature.

Let G be a graph and x and y two non-adjacent vertices of G. The *separability of x and y*, is the minimum cardinality of a set $S \subseteq V(G)$ such that x and y are in different components of $G \setminus S$. The *separability* *of* G is the maximum over all separabilities of pairs of non-adjacent vertices of G (unless G is complete, in which case it has separability 0). So the graphs of separability at most k are precisely the graphs in which every two non-adjacent vertices can be separated by removing a set of at most k other vertices. By Menger's Theorem, the separability of G is equal to the maximum number of internally vertex-disjoint paths connecting two non-adjacent vertices. Graphs of separability at most 2 were studied in [4] where the following characterization is obtained, and a number of other properties of this class. K_5^- is the graph obtained from a K_5 by removing a single edge.

Theorem 1.1 (Cicalese and Milanič [4]) *A graph G is of separability at most 2 if and only if it is $(K_5^-, \textit{theta}, \textit{pyramid}, \textit{prism}, \textit{wheel})$-free.*

Let γ be a $\{0,1\}$-vector whose entries are in one-to-one correspondence with the holes of a graph G. A graph G is *universally signable* if for all choices of vector γ, there exists a subset F of the edge set of G such that $|F \cap H| \equiv \gamma_H \pmod 2$, for all holes H of G. By the above mentioned theorem of Truemper [18], it is easy to obtain the following characterization of universally signable graphs in terms of forbidden induced subgraphs.

Theorem 1.2 (Conforti, Cornuéjols, Kapoor and Vušković [5]) *A graph is universally signable if and only if it is $(\textit{theta}, \textit{pyramid}, \textit{prism}, \textit{wheel})$-free.*

This characterization of universally signable graphs is then used to obtain the following decomposition theorem, which generalises the classical decomposition of chordal graphs with clique cutsets.

Theorem 1.3 (Conforti, Cornuéjols, Kapoor and Vušković [5]) *A connected universally signable graph is either a complete graph or a hole, or it admits a clique cutset.*

Clique cutsets have been studied extensively in literature and it is well understood how to use them in algorithms. So, in particular, Theorem 1.3 implies efficient algorithms for recognition of universally signable graphs, and for coloring, maximum clique and maximum stable set problems on this class.

As already observed, the only Truemper configurations that graphs in $\mathcal{C}$ may contain are prisms and alternating wheels. Graphs that may contain only prisms (and no other Truemper configuration) are studied in [6] where the following decomposition theorem is obtained. Given a graph G, its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge in G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. A graph is *chordless* if all of its cycles are chordless.

Theorem 1.4 (Diot, Radovanović, Trotignon and Vušković [6]) *If G is $(\textit{theta}, \textit{pyramid}, \textit{wheel})$-free, then G is the line graph of a triangle-free chordless graph or it admits a clique cutset.*

A *claw* is the complete bipartite graph with three vertices on one side of the bipartition and one vertex on the other. A *diamond* is the graph on four vertices that has exactly one pair of non-adjacent vertices. Note that the class of (claw, diamond)-free graphs is a subclass of $\mathcal{C}$.

Theorem 1.5 (Harary and Holzmann [10], see also Lemma 3.19 in [16]) *A graph is the line graph of a triangle-free graph if and only if it is $(\textit{claw}, \textit{diamond})$-free.*

By Theorem 1.5, the class of line graphs of triangle-free graphs is a subclass of $\mathcal{C}$. The main result in this paper is to show that graphs in $\mathcal{C}$ that are not line graphs of triangle-free graphs have a particular structure.

1.4 Recognizing Truemper configurations

A natural question to ask is whether Truemper configurations can be recognized in polynomial time. These questions in fact arose when studying how to recognize even-hole-free graphs and perfect graphs in polynomial time. Observe that if a graph contains a prism or a theta, then it must contain an even hole, and if it contains a pyramid, then it must contain an odd hole. In fact, the class of even-hole-free graphs is included in the class of (theta, prism, even wheel)-free graphs (where an *even wheel* is a wheel with an even number of sectors), and the class of odd-hole-free graphs, and hence perfect graphs, is included in the class of (pyramid, odd wheel)-free graphs (where an *odd wheel* is a wheel with an odd number of short sectors). We now briefly describe different general techniques that were developed when trying to recognize whether a graph contains a particular Truemper configuration.

In [2] it is shown that detecting whether a graph contains a pyramid can be done in $\mathcal{O}(n^9)$ time. This algorithm is based on the *shortest-paths detector* technique developed by Chudnovsky and Seymour. The idea of their algorithm is as follows. If G has a pyramid, then it has a pyramid Σ with fewest number of vertices. The algorithm "guesses" some vertices of Σ, and then finds shortest paths in G between the guessed vertices that are joined by a path in Σ. If the graph induced by the union of these paths is a pyramid, then clearly G contains a pyramid. If it is not, then it turns out that G is pyramid-free.

Chudnovsky and Seymour [3] show that detecting whether a graph contains a theta can be done in $\mathcal{O}(n^{11})$ time. For this detection problem, the shortest-paths detector technique does not work. The detection of thetas relies on being able to solve a more general problem called the *three-in-a-tree problem* defined as follows: given a graph G and three specified vertices a, b and c, the question is whether G contains a tree that passes through a, b and c. It is shown in [3] that this problem can be solved in $\mathcal{O}(n^4)$ time. What is interesting is that the algorithm for the three-in-a-tree problem is based on an explicit construction of the cases when the desired tree does not exist, and that this construction can be directly converted into an algorithm. The three-in-a-tree algorithm is quite general, and can be used to solve different detection problems, including the detection of a theta, and of a pyramid (the latter in $\mathcal{O}(n^{10})$ time).

Maffray and Trotignon show that detecting whether a graph contains a prism is NP-complete [14]. Also, detecting whether a graph contains a wheel is NP-complete, as shown by Diot, Tavenas and Trotignon [7]. In fact they prove that the problem remains NP-complete even when restricted to bipartite graphs. Since all wheels in bipartite graphs are 1-wheels, it follows that recognizing whether a graph is 1-wheel-free is NP-complete. A number of other detection problems related to graph classes defined by excluding some combination of Truemper configurations have been studied in literature. In Section 2 we will give two polynomial-time recognition algorithms for $\mathcal{C}$.

1.5 The structure of graphs in $\mathcal{C}$

We say that a connected graph G is *structured* if there exists a partition

$$\mathcal{S} = (\{x\}, X_1, X_2, X_3, Y_1, Y_2, Y_3, C_1, C_2, C_3, C_X, C_Y)$$

of $V(G)$ that satisfies the following:

(i) For $1 \leq i \leq 2$, X_i, Y_i and C_i are all non-empty. There exist $x_1 \in X_1$, $x_2 \in X_2$ such that x_1 is complete to $X_2 \cup X_3$ and x_2 is complete to $X_1 \cup X_3$, and $y_1 \in Y_1$, $y_2 \in Y_2$ such that y_1 is complete to $Y_2 \cup Y_3$ and y_2 is complete to $Y_1 \cup Y_3$.

(ii) Let $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$. Then x is complete to $X \cup Y$ and X is anticomplete to Y. Also, for every $1 \leq i, j \leq 3$, $i \neq j$, $X_i \cup Y_i$ is anticomplete to C_j, and for $1 \leq i \leq 3$ every vertex of $X_i \cup Y_i$ has a neighbor in C_i.

(iii) For every $1 \leq i \leq 3$, X_i and Y_i are both cliques, and X_3 (resp. Y_3) is complete to $X_1 \cup X_2$ (resp. $Y_1 \cup Y_2$).

(iv) C_1, C_2, C_3, C_X and C_Y are pairwise anticomplete to each other.

(v) $N(C_X) \subseteq X \cup \{x\}$ and $N(C_Y) \subseteq Y \cup \{x\}$.

If G is structured, we also say that $\mathcal{S}$ is a *structured partition* of G. We prove the following theorem.

Theorem 1.6 *If $G \in \mathcal{C}$ is not a line graph of a triangle-free graph and does not admit a clique cutset, then it is structured.*

The following decomposition theorems are immediate corollaries of Theorem 1.6.

A cutset S is a *bisimplicial cutset* if, for some vertex $x \in S$, $S \subseteq N(x) \cup \{x\}$ and $S \setminus \{x\}$ is the disjoint union of two cliques of size at least 2 that are anticomplete to each other.

A *2-amalgam* (K, V_1, V_2) of a connected graph G is a partition of $V(G)$ into subsets V_1, V_2 and K such that, for every $1 \leq i \leq 2$, W_i and Z_i are disjoint non-empty subsets of V_i and the following hold:

- $V_i \setminus (W_i \cup Z_i) \neq \varnothing$ for every $1 \leq i \leq 2$.
- W_1 (resp. Z_1) is complete to W_2 (resp. Z_2) and these are the only edges between V_1 and V_2.
- K is a clique that is complete to $W_1 \cup W_2 \cup Z_1 \cup Z_2$.

Note that the removal of K, together with the edges with one end in V_1 and one in V_2, disconnects G. A 2-amalgam is called *special* if K consists of a single vertex, W_i and Z_i are cliques for every $1 \leq i \leq 2$ and W_1 (resp. W_2) is anticomplete to Z_1 (resp. Z_2). A 2-amalgam is *small* if it is special and $|W_i| = |Z_i| = 1$ for every $1 \leq i \leq 2$. Note that if G admits a special 2-amalgam, then it has a bisimplicial cutset, that satisfies additional properties.

Theorem 1.7 *If $G \in \mathcal{C}$, then G is the line graph of a triangle-free graph or it admits a clique cutset or a bisimplicial cutset.*

Proof. If G is not the line graph of a triangle-free graph and does not admit a clique cutset then, by Theorem 1.6, it is structured. First observe that, for $1 \leq i \leq 2$, $\{x\} \cup X_i \cup Y_i$ is a cutset of G separating C_i from the rest of the graph. Now suppose that $\{x\} \cup X_1 \cup Y_1$ is not a bisimplicial cutset. Then w.l.o.g. $|X_1| = 1$. If $|Y_1| = 1$ or $|Y_2| = 1$ then $\{x\} \cup X_1 \cup X_2 \cup Y_1 \cup Y_2$ is a bisimplicial cutset. So we may assume that $|Y_1| \geq 2$ and $|Y_2| \geq 2$. But then $\{x\} \cup X_1 \cup X_2 \cup Y_1$ is a bisimplicial cutset. ■

Theorem 1.8 *If $G \in \mathcal{C}$ is a K_4-free graph, then G is the line graph of a triangle-free graph or it admits a clique cutset or a small 2-amalgam.*

Proof. Assume otherwise. By Theorem 1.6 we may assume that G is structured. If G is K_4-free, then $|X_1| = |X_2| = |Y_1| = |Y_2| = 1$ and $X_3 \cup Y_3 \cup C_3 = \varnothing$. Also, since X and Y are both cliques and G does not admit a clique cutset, $C_X = C_Y = \varnothing$. Therefore, if we define $K = \{x\}$, $W_1 = \{x_1\}$, $Z_1 = \{y_1\}$, $W_2 = \{x_2\}$, $Z_2 = \{y_2\}$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2$, then (K, V_1, V_2) is a small 2-amalgam of G, a contradiction. ■

Theorem 1.9 *If $G \in \mathcal{C}$ is a K_5^--free graph, then G is the line graph of a triangle-free graph or it admits a clique cutset or a special 2-amalgam.*

Proof. Assume not. By Theorem 1.6 G is structured. When G is K_5^--free, X and Y must both be cliques and therefore $C_X = C_Y = \varnothing$. So, if $K = \{x\}$, $W_1 = X_1$, $Z_1 = Y_1$, $W_2 = X_2 \cup X_3$, $Z_2 = Y_2 \cup Y_3$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2 \cup C_3$, then (K, V_1, V_2) is a special 2-amalgam of G, a contradiction. ■

As intermediate results, we also prove the following three theorems. Let (H, x) be a wheel of a graph $G \in \mathcal{C}$. Then we say that a chordless path $P = p_1, \ldots, p_k$, $k > 2$, in $G \setminus (V(H) \cup \{x\})$ is an *appendix* of (H, x) that attaches to S_i if, for some long sector S_i of (H, x), $N(p_1) \cap (V(H) \cup \{x\}) = \{x, x_i\}$, $N(p_k) \cap (V(H) \cup \{x\}) = \{x_i, x'_i\}$ and $N(p_j) \cap (V(H) \cup \{x\}) \subseteq \{x_i\}$ for every $1 < j < k$.

Theorem 1.10 *If $G \in \mathcal{C}$ does not contain a wheel with an appendix nor a long alternating wheel, then G is the line graph of a triangle-free graph or it admits a clique cutset.*

Theorem 1.11 *If $G \in \mathcal{C}$ does not contain a wheel with an appendix, then G is the line graph of a triangle-free graph, or it admits a clique cutset or a special 2-amalgam.*

Theorem 1.12 *If $G \in \mathcal{C}$ contains a wheel with an appendix or a long alternating wheel, then G admits a clique cutset or G is structured.*

Theorem 1.6 follows directly from Theorem 1.10 and Theorem 1.12. Theorem 1.10 is proved in Section 3, and Theorems 1.11 and 1.12 are proved in Section 4.

The following result is proved in Section 5.

Theorem 1.13 *If $G \in \mathcal{C}$ is a K_4-free graph, then G is 4-colorable.*

2 Recognizing graphs in $\mathcal{C}$

In this section we give two polynomial-time algorithms that decide whether an input graph G belongs to $\mathcal{C}$. The first algorithm is obtained by a direct search for certain Truemper configurations, so, although it is slower than the second one, we believe that its intermediate steps are of independent interest. The second algorithm has running time $\mathcal{O}(n^5)$ and is based on the description of the local structure of graphs in $\mathcal{C}$ that is obtained in [1]. (Throughout the section, for a graph G we let $n = |V(G)|$ and $m = |E(G)|$). Both of these algorithms do not use our main decomposition theorem for $\mathcal{C}$ (Theorem 1.7).

In [15], Maffray, Trotignon and Vušković give an $\mathcal{O}(n^7)$-time algorithm that decides whether a graph contains a theta or a pyramid. Recall that deciding whether a graph contains a 1-wheel is NP-complete [7]. In Lemma 2.1 we give an $\mathcal{O}(n^6)$-time algorithm that decides whether a graph contains a theta, a pyramid or a 1-wheel. In Lemma 2.2 we give an $\mathcal{O}(n^6)$-time algorithm that decides whether a graph contains a 3-wheel. Together these two algorithms give our first recognition algorithm for $\mathcal{C}$.

Lemma 2.1 *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *YES if G contains a theta, a pyramid or a 1-wheel, and NO otherwise.*

Running time: $\mathcal{O}(n^4m + n^5)$.

Proof. Consider the following algorithm.

Step 1: Let $\mathcal{L}$ be the set of all 4-element subsets of $V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return NO. Otherwise, take $S \in \mathcal{L}$ and remove S from $\mathcal{L}$.

Step 3: If S does not induce a claw, go to Step 2. Otherwise, let $S = \{u, a, b, c\}$ be such that u is complete to $\{a, b, c\}$.

Step 4: If there exists a connected component C of $G \setminus N[u]$ such that a, b and c all have a neighbor in C, then return YES. Otherwise, go to Step 2.

Since $\mathcal{L}$ has $\mathcal{O}(n^4)$ elements, and Step 4 takes $\mathcal{O}(n + m)$ time, the running time of this algorithm is $\mathcal{O}(n^4m + n^5)$.

Let us now prove its correctness. First suppose that, for some connected component C of $G \setminus N[u]$ (in Step 4), all a, b and c have a neighbor in C, and let C' be a minimal connected subgraph of C such that all a, b and c have a neighbor in C'. Let P be a chordless ac-path in the graph induced by $V(C') \cup \{a, c\}$. If b has a neighbor in P, then $V(P) \cup \{u, b\}$ induces a theta or a 1-wheel. Otherwise, let Q be a chordless bv-path of $G[V(C') \cup \{b\}]$ such that v has a neighbor in $P \setminus \{a, c\}$ and no vertex of $Q \setminus \{v\}$ has a neighbor in $P \setminus \{a, c\}$. By minimality of C', not both a and c can have a neighbor in Q, and v has one or two adjacent neighbors in P. So w.l.o.g. $N(c) \cap V(Q) = \varnothing$. Let H be the hole contained in $G[(V(P) \setminus \{a\}) \cup V(Q) \cup \{u\}]$ that contains Q, u and c. If a has at least three neighbors in H, then (H, a) is a 1-wheel. If a has exactly two neighbors in H, then $V(H) \cup \{a\}$ induces a theta. So we may assume that a has no neighbors in $H \setminus \{u\}$. But then the graph induced by $V(P) \cup V(Q) \cup \{u\}$ is a theta or a pyramid. It follows that the algorithm correctly returns YES in Step 4.

So, let us assume that the output is NO, but that G contains a theta, pyramid or a 1-wheel D. Let $\{u, a, b, c\}$ induce a claw contained in D and let u be complete to $\{a, b, c\}$. Additionally, in case D is a 1-wheel, then w.l.o.g. we assume that b is its center. So, clearly a connected component of $G \setminus N[u]$ has neighbors from all of a, b and c, and hence the algorithm returns YES in Step 4, a contradiction. ■

Lemma 2.2 *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *YES if G contains a 3-wheel, and NO otherwise.*

Running time: $\mathcal{O}(n^4m + n^5)$.

Proof. Consider the following algorithm.

Step 1: Let $\mathcal{L}$ be the set of all 4-element subsets of $V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return NO. Otherwise, take $S \in \mathcal{L}$ and remove S from $\mathcal{L}$.

Step 3: If S does not induce a diamond, go to Step 2. Otherwise, let $S = \{a, b, x, y\}$ be such that ab is not an edge.

Step 4: Let $N_x = N[x] \setminus \{a, b\}$ and $N_y = N[y] \setminus \{a, b\}$. If a and b are in the same connected component of $G \setminus N_x$ or in the same component of $G \setminus N_y$, then return YES. Otherwise, go to Step 2.

Since $\mathcal{L}$ has $\mathcal{O}(n^4)$ elements, and Step 4 takes $\mathcal{O}(n + m)$ time, the running time of this algorithm is $\mathcal{O}(n^4 m + n^5)$.

Let us now prove that the algorithm is correct. First, if the output is YES, then G contains a 3-wheel with center x or y. Indeed, if a and b are in the same connected component C of $G \setminus N_z$, for some $z \in \{x, y\}$, then a shortest path from a to b in C, together with $\{x, y\}$, induces a 3-wheel with center t, where $t \in \{x, y\} \setminus \{z\}$.

So, let us assume that the output is NO, but that G contains a 3-wheel. Let (H, x) be this 3-wheel, and let $a, b, y \in V(H) \cap N(x)$ be such that a and b are distinct neighbors of y. The vertex set $\{a, b, x, y\}$ induces a diamond and a and b are in the same connected component of $G \setminus N_y$. Therefore, the algorithm returns YES in Step 4, a contradiction. ■

To describe our second algorithm, we first recall some definitions from [1]. Let $\mathcal{F}$ be a set of graphs. A graph G is *locally $\mathcal{F}$-decomposable* if for every vertex v of G, every $F \in \mathcal{F}$ contained in $N(v)$ and every connected component C of $G \setminus N[v]$, there exists $y \in F$ such that y has a non-neighbor in F and no neighbors in C. A class of graphs $\mathcal{G}$ is *locally $\mathcal{F}$-decomposable* if every graph $G \in \mathcal{G}$ is locally $\mathcal{F}$-decomposable.

Let S_3 denote the stable graph on three vertices and P_3 the path on three vertices. The following theorem is the key for our second algorithm (we note that the class $\mathcal{C}$ is denoted by $\mathcal{C}_4$ in [1] – see also Table 1 from the same paper).

Theorem 2.3 ([1]) *The class $\mathcal{C}$ is exactly the class of locally $\{S_3, P_3\}$-decomposable graphs.*

Theorem 2.4 *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *YES if G is in $\mathcal{C}$, and NO otherwise.*

Running time: $\mathcal{O}(n^5)$.

Proof. Consider the following algorithm.

Step 1: Let $\mathcal{L} = V(G)$.

Step 2: If $\mathcal{L} = \varnothing$, then return YES. Otherwise, take $v \in \mathcal{L}$ and remove v from $\mathcal{L}$. Let $\mathcal{L}_v$ be the set of all 3-element subsets of $N(v)$ and $\mathcal{C}_v$ the set of all connected components of $G \setminus N[v]$.

Step 3: If $\mathcal{L}_v = \varnothing$, go to Step 2. Otherwise, take $S \in \mathcal{L}_v$ and remove S from $\mathcal{L}_v$.

Step 4: If S does not induce S_3 nor P_3, go to Step 3. Otherwise, for every $y \in S$ and every $C \in \mathcal{C}_v$ find $N(y) \cap C$ and $N(y) \cap S$. If the first set is empty and the second is not equal to $S \setminus \{y\}$ for some $y \in S$ and $C \in \mathcal{C}_v$, go to Step 3. Otherwise, return NO.

Let $v \in \mathcal{L}$. The set $\mathcal{L}_v$ has $\mathcal{O}(n^3)$ elements and $\mathcal{C}_v$ can be found in time $\mathcal{O}(n+m)$, so Step 2 takes $\mathcal{O}(n^3)$ time (for every $v \in \mathcal{L}$). Step 4 takes $\mathcal{O}(n)$ time (since $|\bigcup_{C \in \mathcal{C}_v} C| < n$ and $|S| = 3$), so the running time of the algorithm is $\mathcal{O}(n \cdot n^3 \cdot n) = \mathcal{O}(n^5)$.

The correctness of the algorithm follows directly from Theorem 2.3. ∎

3 Proof of Theorem 1.10

The following easy observation will be used throughout the paper.

Lemma 3.1 *Let $G \in \mathcal{C}$ and let H be a hole contained in G. If $x \in V(G) \setminus V(H)$ has at least two non-adjacent neighbors in H, then (H, x) is an alternating wheel.*

Proof. If x has exactly two neighbors in H, and they are not adjacent, then $G[V(H) \cup \{x\}]$ is a theta. So assume x has at least three neighbors in H. Then (H, x) is a wheel, and hence an alternating wheel. ∎

Theorem 1.10 immediately follows from Theorem 1.5 and from the two results below, whose proof is postponed to Section 3.1 and Section 3.2, respectively.

Theorem 3.2 *Assume that $G \in \mathcal{C}$ does not contain a wheel with an appendix. If G contains a diamond, then it admits a clique cutset.*

Theorem 3.3 *Assume that $G \in \mathcal{C}$ is a diamond-free graph that does not contain a long alternating wheel. If G contains a claw, then it admits a clique cutset.*

3.1 Proof of Theorem 3.2

In order to prove Theorem 3.2, it is convenient to work with *extended diamonds*. Given a graph G, let $K = \{v_1, \ldots, v_\ell\}$, $\ell \geq 2$, be a clique of G of size ℓ. An extended diamond $D = (K, x, y)$ of G is an induced subgraph of G with vertex set $V(D)$ given by the disjoint union of K and $\{x, y\}$, and such that x and y are distinct, non-adjacent and both complete to K. We say that an extended diamond D of G is *maximum* if G does not contain an extended diamond with more vertices. The above terminology and notation will be used throughout. Note that D is a diamond when $\ell = 2$.

Lemma 3.4 *Let $D = (K, x, y)$ be a maximum extended diamond of a graph $G \in \mathcal{C}$. Then for every vertex $u \in V(G) \setminus V(D)$, $N(u) \cap V(D)$ is a clique of size at most $\ell + 1$.*

Proof. Assume not. Then $x, y \in N(u) \cap V(D)$. Since, for every $1 \leq i, j \leq \ell$, $i \neq j$, $\{x, y, v_i, v_j, u\}$ cannot induce a 3-wheel, u is complete to $V(D)$. Let $K' = K \cup \{u\}$. Then the extended diamond induced by $K' \cup \{x, y\}$ contradicts the maximality of D. ∎

Proof of Theorem 3.2. Let $D = (K, x, y)$ be a maximum extended diamond of G. We prove that K is a clique cutset of G separating x from y. Assume not and let $Q = q_1, \dots, q_r$ be a shortest path in $G \setminus V(D)$ such that q_1 (resp. q_r) is adjacent to x (resp. y). By Lemma 3.4, $r \geq 2$. By minimality of Q, Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to x and no vertex of $Q \setminus \{q_r\}$ is adjacent to y, so that $N(q_i) \cap V(D) \subseteq K$ for every $1 < i < r$.

Since the graph induced by $V(Q) \cup V(D)$ cannot contain a 3-wheel, v_i has a neighbor in Q for every $1 \leq i \leq \ell$. Let q_j be the vertex of Q with lowest index that has a neighbor in K. W.l.o.g. $v_1 q_j \in E(G)$.

(1) q_j is not complete to K.

Proof of (1). Suppose it is. If $j > 1$, then $V(Q^{q_1 q_j}) \cup \{x, v_2\}$ induces a hole H and (H, v_1) is a 3-wheel, a contradiction. So, $j = 1$.

We now prove that q_2 is complete to K. Assume otherwise and w.l.o.g. suppose that $v_2 q_2$ is not an edge. Since $V(Q) \cup \{v_1, v_2, y\}$ cannot induce a 3-wheel, v_1 and v_2 both have a neighbor in $Q^{q_2 q_r}$. Let q_k (resp. q_h) be the vertex of $Q^{q_2 q_r}$ with lowest index that is adjacent to v_1 (resp. v_2). If $k = h$ then $k > 2$, and hence $V(Q^{q_1 q_k}) \cup \{v_1, v_2\}$ induces a 3-wheel, a contradiction. So w.l.o.g. $k < h$. Let H be the hole induced by $V(Q^{q_1 q_h}) \cup \{v_2\}$. Then, by Lemma 3.1, (H, v_1) is an alternating wheel and hence v_1 is not adjacent to q_h. Let q_m be the neighbor of v_1 in $Q^{q_1 q_h}$ with highest index. Note that $k < m < h$. Suppose that $N(v_1) \cap V(Q^{q_{h+1} q_r}) = \varnothing$. Since, by Lemma 3.1, $V(Q^{q_m q_r}) \cup \{y, v_1, v_2\}$ must induce an alternating wheel with center v_2, then $h < r - 1$ and v_2 is adjacent to q_{h+1}. It follows that the chordless path induced by $V(Q^{q_{h+1} q_r}) \cup \{y\}$ is an appendix of (H, v_1), a contradiction. So, let q_s be the neighbor of v_1 in $Q^{q_{h+1} q_r}$ with lowest index and let H' be the hole induced by $V(Q^{q_m q_s}) \cup \{v_1\}$. By Lemma 3.1, (H', v_2) is an alternating wheel. Also, $s > h + 2$ and q_{h+1} and q_s are both adjacent to v_2. But then $Q^{q_{h+1} q_s}$ is an appendix of (H, v_1), a contradiction. So, q_2 is complete to K.

Now let $K' = K \cup \{q_1\}$. Since q_1 is complete to K, K' is a clique. Also, q_2 is complete to K' and hence the extended diamond induced by $K' \cup \{x, q_2\}$ contradicts the maximality of D. □

By (1), w.l.o.g. $v_2 q_j \notin E(G)$. Let q_k be the neighbor of v_2 in $Q^{q_{j+1} q_r}$ with lowest index. Then $V(Q^{q_1 q_k}) \cup \{x, v_2\}$ induces a hole H' and, by Lemma 3.1, (H', v_1) is an alternating wheel. So, $j > 1$, $k > j + 1$ and v_1 is adjacent to q_{j+1} and not adjacent to q_k. For some $j + 1 \leq h < k$, let q_h be the neighbor of v_1 in Q with highest index.

First suppose that v_1 has no neighbors in $Q^{q_{k+1} q_r}$. By Lemma 3.1, $V(Q^{q_h q_r}) \cup \{y, v_1, v_2\}$ must induce an alternating wheel with center v_2, and hence $k < r - 1$ and v_2 is adjacent to q_{k+1}. But then the chordless path induced by $V(Q^{q_{k+1} q_r}) \cup \{y\}$ is an appendix of (H', v_1), a contradiction.

Therefore, let q_m be the neighbor of v_1 in $Q^{q_{k+1} q_r}$ with lowest index and let H'' be the hole induced by $V(Q^{q_h q_m}) \cup \{v_1\}$. By Lemma 3.1, (H'', v_2) is an alternating wheel. So, $m > k + 2$ and q_{k+1} and q_m are both adjacent to v_2. But then $Q^{q_{k+1} q_m}$ is an appendix of (H', v_1), a contradiction. This concludes the proof of Theorem 3.2. ■

3.2 Proof of Theorem 3.3

In order to prove Theorem 3.3, we first need some preliminary results. Throughout this subsection we assume that G is a diamond-free graph that belongs to $\mathcal{C}$ and does not contain a long alternating wheel.

Extended triangles.

An *extended triangle* $E = (K, S_1, S_2)$ of G is an induced subgraph of G defined as follows: $K = \{u, v_1, v_2\}$ is a clique of G of size 3, and $S_1 = x_1, \ldots, x_n$ and $S_2 = y_1, \ldots, y_m$ are vertex-disjoint chordless paths in $G \setminus K$ such that

- $N(x_1) \cap (V(S_2) \cup K) = \{u\}$, $N(x_n) \cap (V(S_2) \cup K) = \{v_1\}$ and $N(x_i) \cap (V(S_2) \cup K) = \varnothing$ for every $1 < i < n$.
- $N(y_1) \cap (V(S_1) \cup K) = \{u\}$, $N(y_m) \cap (V(S_1) \cup K) = \{v_2\}$ and $N(y_i) \cap (V(S_1) \cup K) = \varnothing$ for every $1 < i < m$.

Note that it follows that $m, n \geq 2$. For $1 \leq i \leq 2$, let H_i be the hole induced by $V(S_i) \cup \{u, v_i\}$. We say that an extended triangle E of G is *minimum* if G does not contain an extended triangle with a smaller number of vertices. This terminology and notation will be used in Lemma 3.5 and Lemma 3.6.

Lemma 3.5 *Let $E = (K, S_1, S_2)$ be a minimum extended triangle of G. For every vertex $z \in V(G) \backslash V(E)$, either z has at most one neighbor in K or z is complete to K. Also, one of the following holds:*

(i) $N(z) \cap V(E)$ is a clique of size at most 3.

(ii) $N(z) \cap V(E) = \{v_i, w_1, w_2\}$, where $1 \leq i \leq 2$ and w_1, w_2 are adjacent vertices of S_{3-i}.

Proof. Assume otherwise. Since G is diamond-free, either z has at most a single neighbor in K or z is complete to K. W.l.o.g. we may assume that z has a neighbor in S_1. Suppose that $N(z) \cap V(E) \subseteq V(H_1)$. Since (i) does not hold, by Lemma 3.1, (H_1, z) is an alternating wheel. But then $G[V(E) \cup \{z\}]$ contains an extended triangle with fewer vertices than E, a contradiction. It follows that z has a neighbor in $H_2 \setminus \{u\}$.

Suppose that $v_2 \in N(z) \cap V(E) \subseteq V(S_1) \cup K$. If z has a single neighbor in H_1, then such a neighbor belongs to S_1 and $V(H_1) \cup \{z, v_2\}$ induces a pyramid. So, since (ii) does not hold, by Lemma 3.1, (H_1, z) is an alternating wheel. Then z is complete to K, since otherwise the graph induced by $V(H_1) \cup \{z, v_2\}$ contains a $3PC(uv_1v_2, z)$. Let x_i be the neighbor of z in S_1 with lowest index. Then $(K \setminus \{v_1\}) \cup V(S_1^{x_1 x_i}) \cup V(S_2) \cup \{z\}$ induces an extended triangle that contradicts our choice of E.

Therefore z has a neighbor in both S_1 and S_2. Let x_i (resp. y_j) be the neighbor of z in S_1 (resp. S_2) with highest index. Assume that zu is an edge but z is not complete to K. Then $v_1, v_2 \notin N(z)$, and hence (by Lemma 3.1) $x_1, y_1 \in N(z)$, and so $\{u, z, x_1, y_1\}$ induces a diamond. If z is complete to K, then by Lemma 3.1, (H_1, z) and (H_2, z) are both alternating wheels and $(K \setminus \{u\}) \cup V(S_1^{x_i x_n}) \cup V(S_2^{y_j y_m}) \cup \{z\}$ induces an extended triangle with a smaller number of vertices. It follows that z is not adjacent to u. Let x_l (resp. y_k) be the neighbor of z in S_1 (resp. S_2) with lowest index and let $H = V(H_1) \cup V(S_2^{y_1 y_k}) \cup \{z\}$. If $V(H_1) \cup \{z\}$ induces an alternating wheel, then $G[H]$ contains a $3PC(u, z)$. If z has two neighbors in H_1 and they are adjacent, then H induces a pyramid. So, by symmetry, x_l and y_k are the only neighbors of z in E. If $x_l \neq x_1$, then H induces a $3PC(u, x_l)$. If $x_l = x_1$, then $V(H_1) \cup V(S_2^{y_k y_m}) \cup \{z, v_2\}$ induces a 1-wheel with center u, a contradiction. ■

Lemma 3.6 *If G contains an extended triangle, then it admits a clique cutset.*

PROOF. Let $E = (K, S_1, S_2)$ be a minimum extended triangle of G, and let W be the set of vertices of $G \setminus V(E)$ that are complete to K. We prove that $K \cup W$ is a clique cutset of G separating S_1 from S_2. First consider the following claim.

(1) $K \cup W$ is a clique of G.

Proof of (1). Assume not. Then there exist two vertices $w_1, w_2 \in W$, $w_1 \neq w_2$, such that w_1w_2 is not an edge. It follows that $\{v_1, v_2, w_1, w_2\}$ induces a diamond, a contradiction. □

By (1), we only need to show that $K \cup W$ is a cutset of G separating S_1 from S_2. Assume otherwise and let $Q = q_1, \ldots, q_r$ be a shortest path in $G \setminus (K \cup W)$ such that q_1 (resp. q_r) has a neighbor in S_1 (resp. S_2). By Lemma 3.5, $r \geq 2$. By minimality of Q, Q is chordless and no vertex of $Q \setminus \{q_1\}$ (resp. $Q \setminus \{q_r\}$) has a neighbor in S_1 (resp. S_2), and so $N(q_i) \cap V(E) \subset K$ for every $1 < i < r$. By Lemma 3.5, every vertex of Q has at most one neighbor in K.

(2) q_1 (resp. q_r) has a single neighbor in S_1 (resp. S_2).

Proof of (2). Suppose that q_1 has two adjacent neighbors in S_1 and no other neighbors in H_1. Let y_i be the neighbor of q_r in S_2 with lowest index. If u and v_1 do not have a neighbor in $Q \setminus \{q_1\}$, then $V(H_1) \cup V(S_2^{y_1y_i}) \cup V(Q)$ induces a pyramid. So, for some $1 < j \leq r$, let q_j be the vertex of Q with lowest index that is adjacent to a vertex of $\{u, v_1\}$. But then $V(H_1) \cup V(Q^{q_1q_j})$ induces a pyramid, a contradiction. By Lemma 3.5, it follows that q_1 has a single neighbor in S_1 and, by symmetry, q_r has a single neighbor in S_2. □

By Lemma 3.5 and (2), $N(q_1) \cap V(E) \subset V(H_1)$ and $N(q_r) \cap V(E) \subset V(H_2)$ are both cliques of size at most 2. Assume that K is not anticomplete to $V(Q)$ and let q_i (resp. q_j) be the vertex of Q with lowest (resp. highest) index that has a neighbor in K.

(3) q_i and q_j are adjacent to u.

Proof of (3). Suppose that q_i is not adjacent to u. If q_i is adjacent to v_2, then $i > 1$, and by (2), $V(H_1) \cup V(Q^{q_1q_i}) \cup \{v_2\}$ induces a pyramid, a contradiction. So, q_i is adjacent to v_1. If x_nq_1 is not an edge then, by (2), $V(H_1) \cup V(Q^{q_1q_i})$ induces a theta. So, q_1 is adjacent to x_n and has no other neighbors in S_1. Let R be the chordless uq_i-path contained in the graph induced by $V(S_2) \cup V(Q^{q_iq_r}) \cup \{u\}$. Then $V(H_1) \cup V(Q^{q_1q_i}) \cup V(R)$ induces a 1-wheel with center v_1, a contradiction. It follows that q_i is adjacent to u and, by symmetry, so is q_j. □

Note that $N(q_1) \cap V(S_1) = \{x_1\}$, since otherwise, by (2) and (3), $V(H_1) \cup V(Q^{q_1q_i})$ induces a theta. By symmetry, $N(q_r) \cap V(S_2) = \{y_1\}$. Also, $i \neq j$, since otherwise the vertex set $V(E) \cup V(Q)$ induces a 1-wheel with center u.

(4) $\{v_1, v_2\}$ is anticomplete to $V(Q)$.

Proof of (4). Assume not, w.l.o.g. suppose that v_1 has a neighbor in the interior of $Q^{q_iq_j}$ and, for some $i < k < j$, let q_k be the vertex of Q with lowest index that is adjacent to v_1. Then $V(H_1) \cup V(Q^{q_1q_k})$ induces a 1-wheel with center u, a contradiction. □

By (4), $V(E) \cup V(Q)$ induces a 1-wheel, a 3-wheel or a long alternating wheel with center u. It follows that K is anticomplete to $V(Q)$. By (2), q_1 (resp. q_r) has a single neighbor in E which belongs to S_1 (resp. S_2). If $N(q_1) \cap V(S_1) = \{x_1\}$ and $N(q_r) \cap V(S_2) = \{y_1\}$, then $V(S_1) \cup V(S_2) \cup V(Q) \cup \{v_1, v_2\}$

induces a hole H and (H, u) is a 1-wheel. So w.l.o.g. x_1 is not the neighbor of q_1 in S_1. But then the graph induced by $(V(E) \setminus \{v_2\}) \cup V(Q)$ contains a theta, a contradiction. ■

Unichord cycles.

A *unichord cycle* $U = (u, v, S_1, S_2)$ of G is an induced subgraph of G defined as follows: u and v are adjacent vertices of G and, for some $m, n \geq 2$, $S_1 = x_1, \dots, x_n$ and $S_2 = y_1, \dots, y_m$ are vertex-disjoint chordless paths in $G \setminus \{u, v\}$ such that

- $N(x_1) \cap (V(S_2) \cup \{u, v\}) = \{u\}$, $N(x_n) \cap (V(S_2) \cup \{u, v\}) = \{v\}$ and $N(x_i) \cap (V(S_2) \cup \{u, v\}) = \varnothing$ for every $1 < i < n$.
- $N(y_1) \cap (V(S_1) \cup \{u, v\}) = \{u\}$, $N(y_m) \cap (V(S_1) \cup \{u, v\}) = \{v\}$ and $N(y_i) \cap (V(S_1) \cup \{u, v\}) = \varnothing$ for every $1 < i < m$.

For $1 \leq i \leq 2$, let H_i be the hole induced by $V(S_i) \cup \{u, v\}$. We say that a unichord cycle U of G is *minimum* if G does not contain a unichord cycle with a smaller number of vertices. This terminology and notation will be used in Lemma 3.7 and Lemma 3.8.

Lemma 3.7 *Let $U = (u, v, S_1, S_2)$ be a minimum unichord cycle of G. For every vertex $z \in V(G) \setminus V(U)$, one of the following holds:*

(i) $N(z) \cap V(U)$ is a clique of size at most 2.

(ii) $|N(z) \cap V(U)| = 4$, $\{u, v\} \subset N(z)$ and (H_i, z) is a line wheel for some $1 \leq i \leq 2$.

Proof. Assume otherwise. W.l.o.g. we may assume that z has a neighbor in S_1. If z has non-adjacent neighbors in H_1 then, by Lemma 3.1, (H_1, z) is a line wheel. Suppose that $N(z) \cap V(U) \subseteq V(H_1)$. Since (i) and (ii) do not hold, $V(H_1) \cup \{z\}$ induces a line wheel and z is not complete to $\{u, v\}$. But then $G[V(U) \cup \{z\}]$ contains a unichord cycle that contradicts our choice of U.

It follows that z has a neighbor in both S_1 and S_2. Let x_i (resp. y_j) be the neighbor of z in S_1 (resp. S_2) with highest index. If z is complete to $\{u, v\}$, then (H_1, z) and (H_2, z) are both line wheels and $V(S_1^{x_i x_n}) \cup V(S_2^{y_j y_m}) \cup \{v, z\}$ induces a unichord cycle with a smaller number of vertices. Therefore, w.l.o.g. z is not adjacent to v. Let $H = V(H_1) \cup V(S_2^{y_j y_m}) \cup \{z\}$. Suppose that (H_1, z) is a line wheel. If $j > 1$ then $G[H]$ contains a $3PC(v, z)$. So $j = 1$, and hence either $G[H]$ contains a $3PC(uzy_1, v)$ (if zu is an edge) or $G[V(H_1) \cup \{y_1, z\}]$ contains a $3PC(u, z)$ (otherwise). Therefore, (H_1, z) is not a line wheel. If z has two neighbors in H_1, then by Lemma 3.1 they are adjacent, and hence $G[H]$ is a pyramid (if $j > 1$) or $G[V(H_1) \cup \{y_1, z\}]$ is a pyramid (if $j = 1$ and z is not adjacent to u), or $G[V(U) \cup \{z\}]$ is a 3-wheel with center u (otherwise). So by symmetry we may assume that $N(z) \cap V(U) = \{x_i, y_j\}$. If $i = j = 1$ (resp. $i = n$ and $j = m$) then $V(U) \cup \{z\}$ induces a 1-wheel with center u (resp. v). If $i < n$ then either $G[H]$ is a $3PC(v, x_i)$ (if $j > 1$) or $G[V(H_1) \cup \{y_1, z\}]$ is a $3PC(u, x_i)$ (otherwise). So $i = n$. But then $V(H_1) \cup V(S_2^{y_1 y_j}) \cup \{z\}$ induces a $3PC(u, x_n)$, a contradiction. ■

Lemma 3.8 *If G contains a unichord cycle, then it admits a clique cutset.*

Proof. Assume not and let $U = (u, v, S_1, S_2)$ be a minimum unichord cycle of G. By Lemma 3.6, G does not contain an extended triangle. Since $\{u, v\}$ is not a clique cutset of G separating S_1 from S_2, let $Q = q_1, \dots, q_r$ be a shortest path in $G \setminus \{u, v\}$ such that q_1 (resp. q_r) has a neighbor in S_1 (resp. S_2). By Lemma 3.7, $r \geq 2$. By minimality of Q, Q is chordless and no vertex of $Q \setminus \{q_1\}$ (resp. $Q \setminus \{q_r\}$) has a neighbor in S_1 (resp. S_2), so that $N(q_i) \cap V(U) \subseteq \{u, v\}$ for every $1 < i < r$.

(1) No vertex of Q is complete to $\{u, v\}$.

Proof of (1). Assume not and let q_i be such a vertex with lowest index. Suppose that a vertex of $Q^{q_1 q_{i-1}}$ has a neighbor in $\{u, v\}$, and let q_j be such a vertex with highest index. Then $V(H_2) \cup V(Q^{q_j q_i})$ induces an extended triangle, a contradiction. So, $V(Q^{q_1 q_{i-1}})$ is anticomplete to $\{u, v\}$. If $i = 1$ then by Lemma 3.7, $G[V(U) \cup \{q_1\}]$ contains an extended triangle. So $i > 1$, and by symmetry $i < r$. Let x_l be the neighbor of q_1 in S_1 with lowest index. If $l = n$ then $V(H_1) \cup V(Q^{q_1 q_i})$ induces a pyramid. So $l < n$, and hence, by Lemma 3.7, $V(H_2) \cup V(S_1^{x_1 x_l}) \cup V(Q^{q_1 q_i})$ induces an extended triangle, a contradiction. □

(2) $\{u, v\}$ is anticomplete to $V(Q)$.

Proof of (2). Assume not and let q_i be the vertex of Q with lowest index that has a neighbor in $\{u, v\}$. W.l.o.g. suppose that q_i is adjacent to u. By (1), vq_i is not an edge. If $N(q_1) \cap V(S_1) \neq \{x_1\}$ then, by Lemma 3.7, $V(H_1) \cup V(Q^{q_1 q_i})$ either induces a theta or a pyramid. So, q_1 is adjacent to x_1 and has no other neighbors in S_1. Let R be the chordless vq_i-path contained in the graph induced by $V(S_2) \cup V(Q^{q_i q_r}) \cup \{v\}$. Then by (1), $V(H_1) \cup V(Q^{q_1 q_i}) \cup V(R)$ induces a 1-wheel with center u, a contradiction. □

(3) q_1 (resp. q_r) has a single neighbor in S_1 (resp. S_2).

Proof of (3). Suppose that q_1 has at least two neighbors in S_1. Then, by Lemma 3.7 and (1), q_1 has two adjacent neighbors in S_1 and no other neighbors in U. Let y_i be the neighbor of q_r in S_2 with lowest index. By (2), u and v do not have a neighbor in Q. W.l.o.g. we may assume that $i < m$. But then $V(H_1) \cup V(S_2^{y_1 y_i}) \cup V(Q)$ induces a pyramid, a contradiction. So by symmetry, (3) holds. □

By (2) and (3), $N(q_1) \cap V(U) = \{x_i\}$ for some $1 \leq i \leq n$, $N(q_r) \cap V(U) = \{y_j\}$ for some $1 \leq j \leq m$ and no interior vertex of Q has a neighbor in U. If either $i = j = 1$ or $i = n$ and $j = m$, then the vertex set $V(U) \cup V(Q)$ induces a 1-wheel with center u or v. So w.l.o.g. $1 < i \leq n$ and $1 \leq j < m$. But then $V(H_1) \cup V(S_2^{y_1 y_j}) \cup V(Q)$ induces a $3PC(u, x_i)$, a contradiction. ■

Putting things together.

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let C be a claw contained in G, with vertex set $V(C) = \{u, v_1, v_2, v_3\}$ and edge set $E(C) = \{uv_1, uv_2, uv_3\}$, and assume that G does not admit a clique cutset. Since u is not a cut vertex of G, there exists a path $Q = q_1, \dots, q_r$ in $G \setminus \{u, v_1, v_2, v_3\}$ such that q_1 is adjacent to v_i for some $1 \leq i \leq 3$ and q_r has a neighbor in $\{v_1, v_2, v_3\} \setminus \{v_i\}$. Assume that the claw and the path are chosen so that Q is of shortest length, and w.l.o.g. suppose that q_1 is adjacent to v_1 and q_r is adjacent to v_2. It follows that Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to v_1, no vertex of $Q \setminus \{q_r\}$ is adjacent to v_2, and v_3 is anticomplete to $V(Q) \setminus \{q_1, q_r\}$. Also, G is diamond-free, so, by minimality of Q, u has no neighbors in Q, and hence $V(Q) \cup \{u, v_1, v_2\}$ induces a hole H. If v_3 is adjacent to q_1 or q_r, then $V(H) \cup \{v_3\}$ either induces a theta or a 1-wheel with center v_3. So, v_3 has no neighbors in Q.

Since $\{u\}$ is not a clique cutset of G, there exists a path $T = t_1, \dots, t_\ell$ in $G \setminus (V(H) \cup \{v_3\})$ such that t_1 is adjacent to v_3 and t_ℓ has a neighbor in $V(H) \setminus \{u\}$. In particular, let T be such a path of shortest length. Then T is chordless, no vertex of $T \setminus \{t_1\}$ is adjacent to v_3 and no vertex of $T \setminus \{t_\ell\}$ has a neighbor in $V(H) \setminus \{u\}$. If t_ℓ has non-adjacent neighbors in H then, by Lemma 3.1, (H, t_ℓ) is a line wheel.

First assume that t_ℓ is not adjacent to u and let T' be the chordless ut_ℓ-path contained in the graph induced by $V(T) \cup \{u, v_3\}$. Then t_ℓ must have a single neighbor in H, and this vertex must belong to $\{v_1, v_2\}$, since otherwise the graph induced by $V(H) \cup V(T')$ contains a theta or a pyramid. But then $V(H) \cup V(T')$ induces a unichord cycle, contradicting Lemma 3.8.

Therefore ut_ℓ is an edge. Then $N(t_\ell) \cap \{v_1, v_2\} \neq \varnothing$, since otherwise $V(H) \cup \{t_\ell\}$ either induces a theta or a 1-wheel with center t_ℓ. W.l.o.g. let t_ℓ be adjacent to v_1. Since G is diamond-free, it follows that v_2t_ℓ and v_3t_ℓ are not edges and hence $\ell \geq 2$. If (H, t_ℓ) is a line wheel, then the claw induced by $\{u, t_\ell, v_2, v_3\}$ and a proper subpath of Q contradict our choice of C and Q. So, $N(t_\ell) \cap V(H) = \{u, v_1\}$. Since G is diamond-free, u is not adjacent to $t_{\ell-1}$ and hence the graph induced by $V(H) \cup V(T)$ contains an extended triangle, contradicting Lemma 3.6. This concludes the proof of Theorem 3.3. ■

4 Proof of Theorem 1.12

Throughout this section we assume that $G \in \mathcal{C}$ contains a wheel with an appendix or a long alternating wheel, but does not admit a clique cutset. We want to show that G is structured. By our assumptions, w.l.o.g. G satisfies exactly one of the properties below, and we define a graph $H^\star$ depending on which property is satisfied.

Property 1: G contains a wheel with an appendix. Let (H, x) be a wheel with an appendix of G with shortest rim and let $P = p_1, \dots, p_k$ be its appendix with shortest length. Assume that (H, x) has short odd sectors and P is attached to S_2, and let $H^\star = G[V(H) \cup V(P) \cup \{x\}]$.

Property 2: G does not contain an alternating wheel with an appendix, but contains a long alternating wheel. Let (H, x) be a long alternating wheel of G with shortest rim, assume that (H, x) has short odd sectors and let $H^\star = G[V(H) \cup \{x\}]$.

Suppose that Property 1 holds and let W be the hole induced by the vertex set $(V(S_2) \setminus \{x_2\}) \cup V(P) \cup \{x\}$. Then we say that $y \in V(G) \setminus (V(H) \cup V(P) \cup \{x\})$ is a *special vertex* of G if it is complete to $\{x, x_1, x_2\}$, $N(y) \cap (V(H) \setminus \{x_1, x_2\}) = \varnothing$ and $\{p_1\} \subset N(y) \cap V(P)$ in such a way that $V(W) \cup \{y\}$ induces an alternating wheel.

We prove Theorem 1.12 by the following sequence of lemmas.

Lemma 4.1 *For every vertex $y \in V(G) \setminus V(H^\star)$, either $N(y) \cap V(H^\star)$ is a clique of size at most 3, or G satisfies Property 1 and y is special.*

We postpone the proof of Lemma 4.1 to Section 4.2.

Now let $M = V(H) \setminus (V(S_2) \cup \{x_1, x_4\})$ and

$$N = \begin{cases} (V(S_2) \setminus \{x_2, x_3\}) \cup V(P) & \text{if Property 1 holds,} \\ V(S_2) \setminus \{x_2, x_3\} & \text{if Property 2 holds.} \end{cases}$$

Also, we denote by A (resp. B) the set of vertices in $V(G) \setminus V(H^\star)$ that are complete to $\{x, x_1, x_2\}$ (resp. $\{x, x_3, x_4\}$). Note that, by Lemma 4.1, $A \cap B = \varnothing$. In particular, if $u \in A$, either u is a special vertex of G (when Property 1 holds) or $N(u) \cap V(H^\star) = \{x, x_1, x_2\}$. If $u \in B$, then $N(u) \cap V(H^\star) = \{x, x_3, x_4\}$.

Lemma 4.2 *$A \cup B \cup \{x, x_1, x_2, x_3, x_4\}$ is a cutset of G that separates N from M.*

Proof. Assume not. Then $G \setminus (V(H^\star) \cup A \cup B)$ contains a chordless path $T = t_1, \ldots, t_m$ such that no vertex in $T \setminus \{t_1, t_m\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$, t_1 has a neighbor in N and t_m has a neighbor in M. By Lemma 4.1, $m \geq 2$, $N(t_1) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$ and $N(t_m) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$. It suffices to consider the following two cases.

<u>Case 1</u>: (H, x) is a line wheel, and hence Property 1 holds.

We have $N(t_m) \cap V(H^\star) \subset V(S_4)$. Let u (resp. v) be the neighbor of t_m in S_4 that is closest to x_1 (resp. x_4). By Lemma 4.1, either $u = v$ (and if that is the case, $u \notin \{x_1, x_4\}$) or $uv \in E(G)$.

(1) At least one of the sets $\{x_1, x_2\}$, $\{x_3, x_4\}$ is anticomplete to $V(T) \setminus \{t_1, t_m\}$.

Proof of (1). Assume otherwise. Then there exists a minimal subpath $T^{t_i t_j}$ of $T \setminus \{t_1, t_m\}$ such that t_i is adjacent to a vertex of $\{x_1, x_2\}$ and t_j is adjacent to a vertex of $\{x_3, x_4\}$. Note that no interior vertex of $T^{t_i t_j}$ has a neighbor in H. Also, by Lemma 4.1, $i \neq j$. So, since $V(T^{t_i t_j}) \cup V(H)$ cannot induce a theta nor a pyramid, it follows that $N(t_i) \cap V(H) = \{x_1, x_2\}$ and $N(t_j) \cap V(H) = \{x_3, x_4\}$, and hence (by definition of T) neither t_i nor t_j is adjacent to x. But then $V(S_4) \cup V(T^{t_i t_j}) \cup \{x\}$ either induces a theta or a 1-wheel with center x, a contradiction. $\square$

(2) x has a neighbor in $T \setminus \{t_1, t_m\}$.

Proof of (2). Assume not and let R be the chordless $x_1 t_1$-path contained in the graph induced by the ux_1-subpath of S_4 together with $V(T)$. First suppose that t_1 is adjacent to x, so that, by Lemma 4.1, $N(t_1) \cap N = \{p_1\}$. If $x_2 t_1$ is not an edge, let R' be the chordless $x_2 t_1$-path contained in the graph induced by the ux_2-subpath of $H \setminus \{x_2'\}$ together with $V(T)$. Then $V(R') \cup \{p_1\}$ induces a hole H' and (H', x) is a 3-wheel. So, $N(t_1) \cap V(H^\star) = \{x, x_2, p_1\}$. But then $V(R) \cup \{x, x_2\}$ induces a 3-wheel with center x_2. It follows that t_1 is not adjacent to x. Let D be the chordless $t_1 p_1$-path contained in the graph induced by $N \cup \{t_1\}$. Then $V(R) \cup V(D) \cup \{x\}$ induces a hole H'' and (H'', x_2) is a 3-wheel, a contradiction. $\square$

By (2), let t_i be the neighbor of x in $T \setminus \{t_1, t_m\}$ with highest index.

(3) Either x_1 or x_4 is adjacent to t_i.

Proof of (3). Assume otherwise. If x_1 and x_4 have no neighbors in $T^{t_i t_{m-1}}$, then $V(S_4) \cup V(T^{t_i t_m}) \cup \{x\}$ either induces a theta or a pyramid. So, let t_j be the vertex of $T^{t_{i+1} t_{m-1}}$ with highest index that has a neighbor in $\{x_1, x_4\}$. W.l.o.g. let t_j be adjacent to x_1, and hence, by Lemma 4.1, not adjacent to x_4. Then it must be that either $u = x_1$ or $u = v = x_1'$, since otherwise $V(S_4) \cup V(T^{t_j t_m}) \cup \{x\}$ induces a theta or a pyramid. Also, by (1), $N(x_4) \cap V(T^{t_{i+1} t_{j-1}}) = \varnothing$. So, let H' be the hole induced by $(V(S_4) \setminus \{x_1\}) \cup V(T^{t_i t_m}) \cup \{x\}$. Then (H', x_1) is a 1-wheel, a contradiction. $\square$

(4) x_1 is adjacent to t_i.

Proof of (4). Assume not. Then, by (3), t_i is adjacent to x_4 and hence, since $t_i \notin B$, not adjacent to x_3. By (1), x_1 and x_2 have no neighbors in the interior of T. Let R be a chordless $x_2 t_1$-path contained in the graph induced by $N \cup \{x_2, t_1\}$. If $N(x_4) \cap V(T^{t_2 t_{i-1}}) \neq \varnothing$, let t_j be the neighbor of x_4 in $T^{t_2 t_{i-1}}$ with lowest index. Then t_j is adjacent to x, since otherwise $V(S_4) \cup V(R) \cup V(T^{t_1 t_j}) \cup \{x\}$ induces a 1-wheel with center x. So, $x_3 t_j$ is not an edge. Now let R' be the chordless $x_3 t_j$-path contained in the graph induced by $N \cup V(T^{t_1 t_j}) \cup \{x_3\}$. It follows that $V(R') \cup \{x, x_4\}$ induces a 3-wheel with center x, a contradiction. So, $N(x_4) \cap V(T^{t_2 t_{i-1}}) = \varnothing$. If we denote by D the chordless $x_3 t_i$-path contained in the graph induced by $N \cup V(T^{t_1 t_i}) \cup \{x_3\}$, then $V(D) \cup \{x_4\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction. $\square$

(5) $\{x_3, x_4\}$ is anticomplete to $V(T) \setminus \{t_1, t_m\}$.

Proof of (5). It follows from (1) and (4). $\square$

By (4), t_i is adjacent to x_1. Since $t_i \notin A$, t_i is not adjacent to x_2. Let R be the chordless $x_3 t_1$-path contained in the graph induced by $N \cup \{x_3, t_1\}$. If $N(x_1) \cap V(T^{t_2 t_{i-1}}) \neq \varnothing$, let t_j be the neighbor of x_1 in $T^{t_2 t_{i-1}}$ with lowest index. Then t_j is adjacent to x, since otherwise, by (5), $V(S_4) \cup V(R) \cup V(T^{t_1 t_j}) \cup \{x\}$ induces a 1-wheel with center x. So, $x_2 t_j$ is not an edge. Now let R' be a chordless $x_2 t_j$-path contained in the graph induced by $N \cup V(T^{t_1 t_j}) \cup \{x_2\}$. It follows that $V(R') \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. So, x_1 has no neighbors in $T^{t_2 t_{i-1}}$. If we denote by D a chordless $x_2 t_i$-path contained in the graph induced by $N \cup V(T^{t_1 t_i}) \cup \{x_2\}$, then $V(D) \cup \{x_1\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction.

<u>Case 2</u>: (H, x) is a long alternating wheel.

First assume that t_1 has a neighbor in $N \setminus V(P)$, and let u (resp. v) be the neighbor of t_1 in S_2 that is closest to x_2 (resp. x_3). By Lemma 4.1, t_1 is not adjacent to x.

(6) A vertex of $\{x_2, x_3\}$ has a neighbor in $T \setminus \{t_1, t_m\}$.

Proof of (6). Assume otherwise and let R be a chordless $x t_1$-path contained in the graph induced by $M \cup V(T) \cup \{x\}$. If $u = v$, then $u \notin \{x_2, x_3\}$ and hence $V(S_2) \cup V(R)$ induces a $3PC(x, u)$. So, by Lemma 4.1, uv is an edge. But then the same vertex set induces a $3PC(uvt_1, x)$, a contradiction. $\square$

By (6), let t_i be the vertex of $T \setminus \{t_1, t_m\}$ with highest index that has a neighbor in $\{x_2, x_3\}$. W.l.o.g. let t_i be adjacent to x_2. Then, by Lemma 4.1, t_i is anticomplete to $\{x_3, x_4\}$.

(7) t_i is adjacent to x.

Proof of (7). The graph induced by $(V(H) \setminus \{x_1\}) \cup V(T^{t_i t_m})$ contains a hole H' that contains x_4, $V(S_2)$ and t_i. By Lemma 3.1, (H', x) is an alternating wheel, and hence (7) holds. $\square$

By (7), Lemma 4.1 and definition of T, $N(t_i) \cap (V(H) \cup \{x\}) = \{x, x_2\}$. Let R be the chordless $x_1 t_i$-path contained in the graph induced by $M \cup V(T^{t_i t_m}) \cup \{x_1\}$. By our choice of t_i, $V(R) \cup \{x_2\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction.

So, $N(t_1) \cap N \subseteq V(P)$. Let p_j be the neighbor of t_1 in P with highest index. Instead of T, consider the chordless path induced by $\{p_k, \ldots, p_j, t_1, \ldots, t_m\}$, and the arguments above still apply. This concludes the proof of Lemma 4.2. ■

Attachments.

Let $u \in A \cup B$ and let $Q = q_1, \ldots, q_r$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that $N(u) \cap V(Q) = \{q_1\}$, no vertex in $Q \setminus \{q_r\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$ and q_r has a neighbor in M. Then we say that Q is an *attachment* of u to M. By Lemma 4.1, $N(q_r) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$. Also, let $X_1' \subseteq A$ (resp. $Y_1' \subseteq B$) be the set of vertices in A (resp. B) that have an attachment to M, and let $X_1 = X_1' \cup \{x_1\}$ (resp. $Y_1 = Y_1' \cup \{x_4\}$).

Lemma 4.3 *Let $u \in X_1'$ and $Q = q_1, \ldots, q_r$ be an attachment of u to M. Then the following hold:*

(i) x_2 and x_3 have no neighbors in Q.

(ii) x_4 has no neighbors in $Q \setminus \{q_r\}$.

Proof. Let v (resp. w) be the neighbor of q_r in $H \setminus V(S_2)$ that is closest to x_1 (resp. x_4). Since $N(q_r) \cap V(H^\star) \subset M \cup \{x, x_1, x_4\}$, the following hold.

(1) q_r is anticomplete to $\{x_2, x_3\}$.

(2) x_3 and x_4 have no neighbors in $Q \setminus \{q_r\}$. In particular, (ii) holds.

Proof of (2). Assume otherwise. Let q_i be the lowest indexed vertex of $Q \setminus \{q_r\}$ that has a neighbor in $\{x_3, x_4\}$ and let R be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_1q_i}) \cup \{x_2, u\}$. First suppose that q_i is adjacent to x_4. Then the graph induced by $(V(H) \setminus V(S_2)) \cup V(Q^{q_1q_i}) \cup \{u\}$ contains a hole H' that contains $V(H) \setminus V(S_2)$ and q_i. Also, by Lemma 3.1, (H', x) is an alternating wheel. So, q_i is adjacent to x. By definition of Q, q_i is not adjacent to x_3. But then $V(S_2) \cup V(R) \cup \{x, x_4\}$ induces a 3-wheel with center x. It follows that q_i is adjacent to x_3 and not adjacent to x_4. Then q_i is adjacent to x, since otherwise $V(S_2) \cup V(R) \cup \{x\}$ either induces a theta or a 1-wheel with center x. Furthermore, the graph induced by the vertex set $(V(H) \setminus V(S_2)) \cup V(Q^{q_1q_i}) \cup \{x_3, u\}$ contains a hole H'' that contains $V(H) \setminus V(S_2)$, x_3 and q_i. But then (H'', x) is a 3-wheel, a contradiction. □

(3) x_2 has no neighbors in $Q \setminus \{q_r\}$.

Proof of (3). Assume not and let q_i be the highest indexed vertex of $Q \setminus \{q_r\}$ that is adjacent to x_2. Then q_i must be adjacent to x, since otherwise, by (1) and (2), the x_2w-subpath of $H \setminus \{x_1\}$, together with $V(Q^{q_iq_r}) \cup \{x\}$, induces a 1-wheel with center x. Let R be the chordless x_1q_i-path contained in the graph induced by $V(Q^{q_iq_r})$ together with the x_1v-subpath of $H \setminus \{x_2\}$. Since $q_i \notin A$, q_i is not adjacent to x_1 and hence $V(R) \cup \{x_2\}$ induces a hole H' that contains x_1, x_2 and q_i, and (H', x) is a 3-wheel, a contradiction. □

By (1), (2) and (3), (i) holds. ■

Analogous arguments prove Lemma 4.4.

Lemma 4.4 *Let $u \in Y_1'$ and $Q = q_1, \ldots, q_r$ be an attachment of u to M. Then the following hold:*

(i) x_2 and x_3 have no neighbors in Q.

(ii) x_1 has no neighbors in $Q \setminus \{q_r\}$.

Now let $u \in A \cup B$ be a vertex that is not special, and let $Q = q_1, \ldots, q_r$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that $N(u) \cap V(Q) = \{q_1\}$, no vertex in $Q \setminus \{q_r\}$ has a neighbor in $H^\star \setminus$

$\{x, x_1, x_2, x_3, x_4\}$ and q_r has a neighbor in N. Then we say that Q is an *attachment* of u to N. By Lemma 4.1, $N(q_r) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$. Let $X_2' \subseteq A$ be the set of vertices in A that are either special or that have an attachment to N, let $Y_2' \subseteq B$ be the set of vertices in B that have an attachment to N, and let $X_2 = X_2' \cup \{x_2\}$ and $Y_2 = Y_2' \cup \{x_3\}$.

Lemma 4.5 *Let $u \in X_2'$ be a vertex that is not special and let $Q = q_1, \dots, q_r$ be an attachment of u to N. Then the following hold:*

(i) x_1 and x_4 have no neighbors in Q.

(ii) x_3 has no neighbors in $Q \setminus \{q_r\}$.

Proof. Since $N(q_r) \cap V(H^\star) \subset N \cup \{x, x_2, x_3\}$, the following holds.

(1) q_r is anticomplete to $\{x_1, x_4\}$.

(2) x_3 and x_4 have no neighbors in $Q \setminus \{q_r\}$. In particular, (ii) holds.

Proof of (2). Assume otherwise. Let q_i be the lowest indexed vertex of $Q \setminus \{q_r\}$ that has a neighbor in $\{x_3, x_4\}$ and let R be the chordless x_1q_i-path contained in the graph induced by $V(Q^{q_1q_i}) \cup \{x_1, u\}$. First suppose that q_i is not adjacent to x_4, so $x_3q_i \in E(G)$. The graph induced by $V(S_2) \cup V(Q^{q_1q_i}) \cup \{u\}$ contains a hole H' that contains $V(S_2)$ and q_i. By Lemma 3.1, (H', x) is an alternating wheel. So, q_i is adjacent to x and $(V(H) \setminus V(S_2)) \cup V(R) \cup \{x, x_3\}$ induces a 3-wheel with center x, a contradiction. So, x_4q_i is an edge. Then q_i is adjacent to x, since otherwise $(V(H) \setminus V(S_2)) \cup V(R) \cup \{x\}$ induces a 1-wheel with center x or a theta. Since $q_i \notin B$, q_i is not adjacent to x_3. Therefore the graph induced by $V(S_2) \cup V(Q^{q_1q_i}) \cup \{x_4, u\}$ contains a hole H'' that contains $V(S_2)$, x_4 and q_i. But then (H'', x) is a 3-wheel, a contradiction. □

(3) x_1 has no neighbors in $Q \setminus \{q_r\}$.

Proof of (3). Assume not and let q_i be the highest indexed vertex of $Q \setminus \{q_r\}$ that is adjacent to x_1. First suppose that q_r has a neighbor in the interior of S_2 and let v (resp. w) be the neighbor of q_r in S_2 that is closest to x_2 (resp. x_3). Then q_i must be adjacent to x, since otherwise, by (1) and (2), the x_1w-subpath of $H \setminus \{x_2\}$, together with $V(Q^{q_iq_r}) \cup \{x\}$, induces a 1-wheel with center x. Let R be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_iq_r})$ together with the x_2v-subpath of $H \setminus \{x_1\}$. Since $q_i \notin A$, q_i is not adjacent to x_2 and hence $V(R) \cup \{x_1\}$ induces a hole H' that contains x_1, x_2 and q_i, and (H', x) is a 3-wheel, a contradiction. It follows that q_r has no neighbors in the interior of S_2 and therefore $N(q_r) \cap V(P) \neq \varnothing$. Let p_j be the neighbor of q_r in P with highest index and, by (1) and (2), let H'' be the hole induced by the $x_2'x_1$-subpath of $H \setminus \{x_2\}$ together with $V(Q^{q_iq_r}) \cup V(P^{p_jp_k})$. Then $xq_i \in E(G)$, since otherwise (H'', x) is a 1-wheel. Since q_i cannot be complete to $\{x, x_1, x_2\}$, then q_i is not adjacent to x_2. Let R'' be the chordless x_2q_i-path contained in the graph induced by $V(Q^{q_iq_r}) \cup V(P^{p_jp_k}) \cup \{x_2\}$. Then $V(R'') \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. □

By (1), (2) and (3), (i) holds. ■

Analogous arguments prove Lemma 4.6.

Lemma 4.6 *Let $u \in Y_2'$ and $Q = q_1, \dots, q_r$ be an attachment of u to N. Then the following hold:*

(i) x_1 and x_4 have no neighbors in Q.

(ii) x_2 *has no neighbors in* $Q \setminus \{q_r\}$*.*

Note that, by Lemma 4.1, $X_i \cap Y_j = \varnothing$ for every $1 \leq i, j \leq 2$. We also have the following.

Lemma 4.7 $X_1 \cap X_2 = Y_1 \cap Y_2 = \varnothing$*.*

PROOF. Assume that $X_1 \cap X_2 \neq \varnothing$ and let $u \in A$ be a vertex that is not special and has an attachment $Q = q_1, \ldots, q_r$ to N and an attachment $T = t_1, \ldots, t_m$ to M. By Lemma 4.2, $V(Q) \cap V(T) = \varnothing$ and $V(Q)$ is anticomplete to $V(T)$. Let v be the neighbor of t_m in $H \setminus V(S_2)$ that is closest to x_4. Suppose that q_r has a neighbor in the interior of S_2 and let w be the neighbor of q_r in S_2 that is closest to x_3.

First assume that x is not adjacent to t_1. By Lemma 4.5, x_3 (resp. x_4) has no neighbors in $Q \setminus \{q_r\}$ (resp. Q), and, by Lemma 4.3, x_4 (resp. x_3) has no neighbors in $T \setminus \{t_m\}$ (resp. T). So the vw-subpath of $H \setminus V(S_1)$, together with $V(Q) \cup V(T) \cup \{u\}$, induces a hole H' that contains x_3, x_4 and u. By Lemma 3.1, (H', x) is an alternating wheel and hence x is adjacent to q_1. By Lemma 4.5, x_1 has no neighbors in Q, and therefore the x_1w-subpath of $H \setminus \{x_2\}$, together with $V(Q) \cup \{x, u\}$, induces a 3-wheel with center x, a contradiction. So, xt_1 is an edge. By Lemma 4.3, x_2 has no neighbors in T. It follows that the x_2v-subpath of $H \setminus \{x_1\}$, together with $V(T) \cup \{x, u\}$, induces a 3-wheel with center x, a contradiction.

It follows that q_r has no neighbors in the interior of S_2 but has a neighbor in P. Instead of Q, consider now the chordless p_kq_1-path contained in the graph induced by $V(Q^{q_1q_r}) \cup V(P)$, and the arguments above still apply. The same approach works when u is a special vertex. This proves that $X_1 \cap X_2 = \varnothing$. Analogously, it can be shown that $Y_1 \cap Y_2 = \varnothing$. ■

Lemma 4.8 X_1*,* X_2*,* Y_1 *and* Y_2 *are all cliques of* G*.*

PROOF. Suppose that $u, v \in X_1$, $u \neq v$ and uv is not an edge, and let Q (resp. T) be an attachment of u (resp. v) to M. Let R be a chordless uv-path contained in the graph induced by $M \cup V(Q) \cup V(T) \cup \{u, v\}$. By Lemma 4.3, $V(R) \cup \{x_2\}$ induces a hole H' and hence (H', x) is a 3-wheel, a contradiction. So, X_1 is a clique. Analogous arguments show that X_2, Y_1 and Y_2 are cliques too. ■

Ears.

Let $T = t_1, \ldots, t_m$ be a chordless path in $G \setminus (V(H^\star) \cup A \cup B)$ such that $N(t_i) \cap (V(H^\star) \setminus \{x, x_1, x_2, x_3, x_4\}) = \varnothing$, for every $1 \leq i \leq m$, and let $u \in A$ and $v \in B$ be such that $N(u) \cap V(T) = \{t_1\}$ and $N(v) \cap V(T) = \{t_m\}$. Then we say that T is an *ear* of $H^\star$, while u and v are said to be the *attachments* of T. Let X_3 (resp. Y_3) be the set of vertices of $A \setminus (X_1 \cup X_2)$ (resp. $B \setminus (Y_1 \cup Y_2)$) that are attachments of an ear of $H^\star$.

Lemma 4.9 *If* $T = t_1, \ldots, t_m$ *is an ear of* $H^\star$*, then* $N(t_i) \cap V(H) = \varnothing$ *for every* $1 \leq i \leq m$*.*

PROOF. By definition, $N(t_i) \cap V(H) \subseteq \{x_1, x_2, x_3, x_4\}$ for every $1 \leq i \leq m$. We now show that t_i is anticomplete to $\{x_1, x_2\}$ for every $1 \leq i \leq m$. Assume otherwise and let t_j be the vertex of T with highest index that is adjacent to a vertex of $\{x_1, x_2\}$, and let $u \in A$ and $v \in B$ be the attachments of T. By Lemma 4.1, t_j is anticomplete to $\{x_3, x_4\}$. Furthermore, let R (resp. R') be the chordless t_jx_4-path (resp. t_jx_3-path) contained in the graph induced by $V(T^{t_jt_m}) \cup \{x_4, v\}$ (resp. $V(T^{t_jt_m}) \cup \{x_3, v\}$). First suppose that t_jx_1 is an edge, and let H' be the hole induced by the x_1x_4-subpath of $H \setminus \{x_2\}$ together with R. Then by Lemma 3.1, (H', x) is an alternating wheel, and hence t_j is adjacent to x. Since $t_j \notin A$, it follows that t_j is not adjacent to x_2. Now let H'' be the hole induced by $V(S_2) \cup V(R') \cup \{x_1\}$. Then (H'', x) is a 3-wheel, a contradiction. Hence, t_j is adjacent to x_2 and not adjacent to x_1. Also, t_j is adjacent to x, since otherwise $V(S_2) \cup V(R') \cup \{x\}$ either induces a theta or a 1-wheel with center x. Therefore the x_2x_4-subpath of $H \setminus \{x_3\}$, together with $V(R) \cup \{x\}$, induces a 3-wheel with center x, a contradiction. So t_i is anticomplete to $\{x_1, x_2\}$ for every $1 \leq i \leq m$. Analogously it can be shown that t_i is also anticomplete to $\{x_3, x_4\}$ for every $1 \leq i \leq m$. ■

Lemma 4.10 *X_3 (resp. Y_3) is a clique of G that is complete to $X_1 \cup X_2$ (resp. $Y_1 \cup Y_2$).*

Proof. We only prove that X_3 is a clique of G and that it is complete to $X_1 \cup X_2$, since similar arguments show that Y_3 is a clique of G that is complete to $Y_1 \cup Y_2$.

(1) X_3 is a clique of G.

Proof of (1). Assume not and let $u, v \in X_3$, $u \neq v$, be such that $uv \notin E(G)$. Let $T = t_1, \ldots, t_m$ (resp. $D = d_1, \ldots, d_h$) be an ear of $H^\star$ with attachments $u \in A$ and $w \in B$ (resp. $v \in A$ and $z \in B$). Let R be a chordless uv-path contained in the graph induced by $V(T) \cup V(D) \cup \{x_3, u, v, w, z\}$. By Lemma 4.9, $V(R) \cup \{x_1\}$ induces a hole H', and (H', x) is a 3-wheel, a contradiction. □

(2) X_3 is complete to $X_1 \cup X_2$.

Proof of (2). Assume that X_3 is not complete to X_2. Let $u \in X_3$ and $v \in X_2$ (v not special), be such that $uv \notin E(G)$. Let Q be an attachment of v to N and let T be an ear of $H^\star$ with attachments $u \in A$ and $w \in B$. Let R be a chordless uv-path contained in the graph induced by $N \cup V(Q) \cup V(T) \cup \{u, v, w, x_3\}$. By Lemma 4.5 and Lemma 4.9, $V(R) \cup \{x_1\}$ induces a hole H'. But then (H', x) is a 3-wheel, a contradiction. A similar argument applies when v is special. So X_3 is complete to X_2. Analogously, it can be shown that X_3 is also complete to X_1. □

This proves the lemma. ■

Let $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$.

Lemma 4.11 *X and Y are disjoint and anticomplete.*

Proof. Sets X and Y are disjoint by Lemma 4.1. Suppose that $u \in X$ and $v \in Y$ are such that uv is an edge. Then $V(S_2) \cup \{x, u, v\}$ induces a 3-wheel with center x, a contradiction. So, X and Y are anticomplete to each other. ■

Putting things together.

We are ready to conclude the proof of Theorem 1.12. Let $S = \{x\} \cup X \cup Y$.

Lemma 4.12 *S is a cutset of G that separates N from M.*

Proof. Suppose that S is not a cutset of G that separates N from M. Then there exists a path $Q = q_1, \ldots, q_r$ in $G \setminus S$, such that q_1 (resp. q_r) has a neighbor in N (resp. M), and let Q be chosen such that it has the shortest length. It follows that Q is chordless and no vertex of $Q \setminus \{q_1, q_r\}$ has a neighbor in $H^\star \setminus \{x, x_1, x_2, x_3, x_4\}$. By Lemma 4.2, $V(Q) \cap (A \cup B) \neq \varnothing$. So, let q_i be the vertex of $V(Q) \cap (A \cup B)$ with the lowest index, and w.l.o.g. assume that $q_i \in A$. But then either $i = 1$ and q_1 is special, or $i > 1$ and $Q^{q_1 q_{i-1}}$ is an attachment of q_i to N, and hence $q_i \in X_2$, a contradiction. ■

Let $\mathcal{C}^*$ be the set of all connected components of $G \setminus S$. The sets C_1, C_2, C_3, C_X, C_Y are defined as follows:

- C_1 (resp. C_2) is the vertex set of the connected component from $\mathcal{C}^*$ that contains M (resp. N);
- C_X (resp. C_Y) is the union of vertex sets of all $C \in \mathcal{C}^*$, such that $N(C) \subseteq \{x\} \cup X$ (resp. $N(C) \subseteq \{x\} \cup Y$);
- C_3 is the union of vertex sets of all $C \in \mathcal{C}^*$ such that $V(C) \not\subseteq C_1 \cup C_2 \cup C_X \cup C_Y$.

Lemma 4.13 $C_X \cap C_Y = \varnothing$.

Proof. Assume otherwise. Then C_X and C_Y are both non-empty and such that $N(C_X) \subseteq \{x\}$ and $N(C_Y) \subseteq \{x\}$. But then G admits a clique cutset, a contradiction. ■

Lemma 4.14 *If $C \in \mathcal{C}^*$ is such that $V(C) \not\subseteq C_1 \cup C_X \cup C_Y$, then x_1 and x_4 have no neighbors in C.*

Proof. Consider first the following claim.

(1) x_1 and x_4 have no neighbors in C_2.

Proof of (1). Assume otherwise and let $Q = q_1, \dots, q_r$ be a shortest path in $G[C_2 \setminus N]$ such that q_1 (resp. q_r) has a neighbor in $\{x_1, x_4\}$ (resp. N). Note that (since $q_i \notin S$ for every $1 \leq i \leq r$) q_r is not special, and hence by Lemma 4.1, $r \geq 2$. By minimality of Q, Q is chordless, no vertex of $Q \setminus \{q_1\}$ is adjacent to x_1 or x_4, and no vertex of $Q \setminus \{q_r\}$ has a neighbor in N. Also, by Lemma 4.12, no vertex of Q has a neighbor in M.

W.l.o.g. suppose $x_1 q_1 \in E(G)$. Then, by our choice of Q and Lemma 4.1, x_4 is anticomplete to $V(Q)$. Let R be the chordless $x_3 q_1$-path contained in the graph induced by $V(Q) \cup N \cup \{x_3\}$. Then the $x_1 x_3$-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{x\}$, induces a 1-wheel with center x, unless x is adjacent to q_1. So, $x q_1 \in E(G)$. If $x_2 q_1$ is an edge, then $Q^{q_2 q_r}$ is an attachment of q_1 to N, and hence $q_1 \in X_2$, a contradiction. So, $x_2 q_1$ is not an edge. Now let R' be a chordless $x_2 q_1$-path contained in the graph induced by $V(Q) \cup N \cup \{x_2\}$. Then $V(R') \cup \{x_1\}$ induces a hole H' and (H', x) is a 3-wheel, a contradiction. □

By (1), w.l.o.g. we may assume that x_1 has a neighbor in $C' \in \mathcal{C}^*$ such that $V(C') \not\subseteq C_1 \cup C_2 \cup C_X \cup C_Y$. So, C' contains a chordless path $T = t_1, \dots, t_m$ such that t_1 is adjacent to x_1, t_m is adjacent to a vertex of Y and $N(t_i) \cap V(H^\star) \subseteq \{x, x_1, x_2, x_3, x_4\}$ for every $1 \leq i \leq m$. Pick such a path of minimum length. Then no vertex of $T \setminus \{t_1\}$ is adjacent to x_1 and no vertex of $T \setminus \{t_m\}$ has a neighbor in Y.

(2) t_m is anticomplete to $\{x_3, x_4\}$.

Proof of (2). Assume otherwise and let H' be the hole induced by $(V(H) \setminus V(S_2)) \cup V(T)$, together with x_3 if t_m is not adjacent to x_4. By Lemma 3.1, (H', x) is an alternating wheel. It follows that t_1 is adjacent to x. Suppose that $x_2 t_1 \notin E(G)$, and let R be the chordless $x_2 t_1$-path contained in the graph induced by $V(S_2) \cup V(T)$, together with x_4 if t_m is not adjacent to x_3. Then $V(R) \cup \{x_1\}$ induces a hole H'' and (H'', x) is a 3-wheel, a contradiction. So, $x_2t_1 \in E(G)$. Let R' be the chordless x_2t_m-path contained in $G[V(T) \cup \{x_2\}]$. First suppose that t_m is adjacent to x_3. Then xt_m is an edge, since otherwise $V(S_2) \cup V(R') \cup \{x\}$ induces a theta or a 1-wheel with center x. Also, if $x_4t_m \notin E(G)$, then (H', x) is a 3-wheel. It follows that $x_3t_m \in E(G)$, and hence $m > 2$, since otherwise (H', x) is a 3-wheel. But then $T^{t_2t_{m-1}}$ is an ear of $H^\star$ with attachments t_1 and t_m, implying that $t_1 \in X$ and $t_m \in Y$, a contradiction. So $x_3t_m \notin E(G)$ and $x_4t_m \in E(G)$. Since (H', x) is an alternating wheel, $xt_m \in E(G)$. But then $V(S_2) \cup V(R') \cup \{x, x_4\}$ induces a 3-wheel with center x, a contradiction. □

By (2), t_m has a neighbor $u \in Y \setminus \{x_3, x_4\}$. Note that xt_1 is an edge, since otherwise $(V(H) \setminus V(S_2)) \cup V(T) \cup \{x, u\}$ induces a 1-wheel with center x. If $m = 1$ then $(V(H) \setminus V(S_2)) \cup \{u, t_1\}$ induces a hole H' and (H', x) is a 3-wheel. So $m \geq 2$. If x_2 is adjacent to t_1, then $T^{t_2t_m}$ is an ear of $H^\star$, and hence $t_1 \in X$, a contradiction. So, t_1 is not adjacent to x_2. If R denotes the chordless x_2t_1-path contained in the graph induced by $V(S_2) \cup V(T) \cup \{u\}$, then $V(R) \cup \{x, x_1\}$ induces a 3-wheel with center x, a contradiction. ■

Analogous arguments prove the following lemma.

Lemma 4.15 *If $C \in \mathcal{C}^*$ is such that $V(C) \not\subseteq C_X \cup C_Y \cup C_2$, then x_2 and x_3 have no neighbors in C.*

Lemma 4.16 *$X \setminus X_i$ and $Y \setminus Y_i$ are anticomplete to C_i, for $1 \leq i \leq 2$.*

Proof. Suppose that $u \in X \setminus (X_1 \cup \{x_2\})$ has a neighbor in C_1. It follows that $G[C_1 \setminus M]$ contains a chordless path $Q = q_1, \dots, q_r$ such that q_1 is adjacent to u and q_r has a neighbor in M. By Lemma 4.12, no vertex of Q has a neighbor in N. Also, if we choose Q to be of shortest length, then no vertex of $Q \setminus \{q_1\}$ is adjacent to u and no vertex of $Q \setminus \{q_r\}$ has a neighbor in M. So, since $u \notin X_1$ and hence Q is not an attachment of u to M, $V(Q) \cap (A \cup B) \neq \varnothing$, contradicting Lemma 4.15. It follows that $X \setminus X_1$ is anticomplete to C_1. Similar arguments show that $Y \setminus Y_1$ is anticomplete to C_1, and that $X \setminus X_2$ and $Y \setminus Y_2$ are anticomplete to C_2. ■

Lemma 4.17 *X_i and Y_i are anticomplete to C_3, for $1 \leq i \leq 2$.*

Proof. By Lemma 4.14 and Lemma 4.15, $\{x_1, x_2, x_3, x_4\}$ is anticomplete to C_3. Suppose that $u \in X_1 \setminus \{x_1\}$ has a neighbor in C_3. Then C_3 contains a chordless path $T = t_1, \dots, t_m$, such that t_1 is adjacent to u, t_m is adjacent to a vertex $v \in Y \setminus \{x_3, x_4\}$ and $N(t_i) \cap V(H^\star) \subseteq \{x\}$, for every $1 \leq i \leq m$. Pick such a path of minimum length. Then no vertex of $T \setminus \{t_1\}$ is adjacent to x_1 and no vertex of $T \setminus \{t_m\}$ has a neighbor in $Y \setminus \{x_3, x_4\}$. Note that T is an ear of $H^\star$, with attachments u and v. By Lemma 4.11, uv is not an edge. Let $D = d_1, \dots, d_h$ be an attachment of u to M. Since T and D belong to different connected components of $G \setminus S$, $V(T) \cap V(D) = \varnothing$ and $V(T)$ is anticomplete to $V(D)$. Also, $V(D) \cap \{v\} = \varnothing$. Now let R be the chordless uv-path contained in the graph induced by $M \cup V(D) \cup \{u, v, x_4\}$. Then $V(R) \cup V(T)$ induces a hole H' and, by Lemma 3.1, (H', x) is an alternating wheel. If x is adjacent to d_1 (note that x_3u is not an edge by Lemma 4.1), let R' be the chordless x_3u-path contained in the graph induced by $M \cup V(D) \cup \{u, x_3, x_4\}$. Then, by Lemma 4.3, $V(R') \cup V(S_2)$ induces a hole H'', and (H'', x) is a 3-wheel, a contradiction. Hence, x is adjacent to t_1. But then $V(S_2) \cup V(T) \cup \{x, u, v\}$ induces a 3-wheel with center x, a contradiction. Therefore X_1 is anticomplete to C_3 and, similarly, so is Y_1. Analogous arguments show that $X_2 \cup Y_2$ is anticomplete to C_3. ■

Proof of Theorem 1.12. By definitions, Lemma 4.7, Lemma 4.11, Lemma 4.12 and Lemma 4.13, $\mathcal{S} = (\{x\}, X_1, X_2, X_3, Y_1, Y_2, Y_3, C_1, C_2, C_X, C_Y)$ is a partition of $V(G)$. Also, if we set $y_1 = x_4$ and

$y_2 = x_3$, $\mathcal{S}$ satisfies (i) by the definition of sets X_i, Y_i and C_i, for $1 \leq i \leq 2$, and X_3, Y_3. By Lemma 4.11, X is anticomplete to Y, by Lemma 4.16 and Lemma 4.17, $X_i \cup Y_i$ is anticomplete to C_j if $i \neq j$, and by definitions, x is complete to $X \cup Y$ and a vertex from $X_i \cup Y_i$ has a neighbor in C_i, for $1 \leq i \leq 3$. So, $\mathcal{S}$ satisfies (ii). By Lemma 4.8 and Lemma 4.10, $\mathcal{S}$ satisfies (iii). Finally, properties (iv) and (v) follow from definitions of sets C_1, C_2, C_3, C_X and C_Y, and Lemma 4.12. So G is structured. ■

4.1 Proof of Theorem 1.11

In this subsection we prove Theorem 1.11, so we further assume that $G \in \mathcal{C}$ does not contain a wheel with an appendix. By Theorem 1.10, it is enough to consider the case when G contains a long alternating wheel, i.e. we may assume that G satisfies Property 2. So we keep the same notation as before, and prove that the structured partition $\mathcal{S}$ of G, obtained in Theorem 1.12, has some additional properties (under the assumption that G does not admit a clique cutset).

Lemma 4.18 *Let $u \in X_2'$ and $Q = q_1, \dots, q_r$ be an attachment of u to N. Then:*

(1) x is anticomplete to $V(Q)$;

(2) if x_2 is not adjacent to q_r, then it is anticomplete to $V(Q)$.

Proof. (1) Assume not. Let q_i be the highest indexed vertex of Q that is adjacent to x and let v be the neighbor of q_r in S_2 that is closest to x_3. By Lemma 4.1 x is not adjacent to q_r, and hence $i < r$.

First suppose $x_2v \notin E(G)$. Then, by Lemma 4.1, x_2 is not adjacent to q_r, and x_2 must have a neighbor in $Q^{q_i q_{r-1}}$, since otherwise, by Lemma 4.1 and Lemma 4.5, $V(S_2) \cup V(Q^{q_i q_r}) \cup \{x\}$ induces a theta or a pyramid. If x_2 has a neighbor in $Q^{q_{i+1} q_{r-1}}$, then the graph induced by $V(S_2) \cup V(Q^{q_{i+1} q_r}) \cup \{x\}$ contains a theta or a pyramid. So, $N(x_2) \cap V(Q^{q_i q_r}) = \{q_i\}$. By Lemma 4.5, the vx_1-subpath of $H \setminus \{x_2\}$, together with $V(Q^{q_i q_r}) \cup \{x, x_2\}$, induces a 3-wheel with center x, a contradiction. It follows that $x_2v \in E(G)$. By Lemma 4.5, the vx_3-subpath of S_2, together with $V(Q^{q_i q_r}) \cup \{x\}$, induces a hole H', and, by Lemma 3.1, (H', x_2) is an alternating wheel. So, $i < r-1$ and x_2 is adjacent to q_i and q_r. But then, by Lemma 4.5, $Q^{q_i q_r}$ is an appendix of (H, x), a contradiction. □

(2) Assume not. Let q_i be the highest indexed vertex of Q that is adjacent to x_2 and let v be the neighbor of q_r in S_2 that is closest to x_3. By Lemma 4.5 and (1), the vx_3-subpath of S_2 together with $V(Q) \cup \{x, u\}$ induces a hole H', and therefore x_2v is not an edge, since otherwise (H', x_2) is a 1-wheel. Now, by Lemma 4.1, Lemma 4.5 and (1), $V(S_2) \cup V(Q^{q_i q_r}) \cup \{x\}$ induces a theta or a pyramid, a contradiction. ■

Lemma 4.19 *X_1 (resp. Y_1) is complete to X_2 (resp. Y_2).*

Proof. Let $u \in X_1$ and $v \in X_2$, and suppose that uv is not an edge. Let $Q = q_1, \dots, q_r$ (resp. $T = t_1, \dots, t_m$) be an attachment of v (resp. u) to N (resp. M). By Lemma 4.2, $V(Q) \cap V(T) = \varnothing$ and $V(Q)$ is anticomplete to $V(T)$.

Denote by w (resp. z) the neighbor of q_r (resp. t_m) in S_2 (resp. $V(H) \setminus V(S_2)$) that is closest to x_3 (resp. x_4). First suppose $x_2w \in E(G)$. It follows that $r > 1$ and q_r is adjacent to x_2 since otherwise, by Lemma 4.5 and Lemma 4.18, the vertex set $V(S_2) \cup V(Q) \cup \{x, v\}$ induces a 1-wheel or a 3-wheel with center x_2. By Lemma 4.3, the x_2z-subpath of $H \setminus \{x_1\}$, together with $V(T) \cup \{u\}$, induces a hole H' and, by Lemma 3.1, (H', x) is an alternating wheel. Furthermore, by Lemma 4.5 and Lemma 4.18, the chordless path induced by $V(Q) \cup \{v\}$ is an appendix of (H', x), a contradiction. Therefore x_2w is not an edge, and hence, by Lemma 4.1, x_2 is not adjacent to q_r. But then by Lemma 4.3, Lemma 4.5 and Lemma 4.18, the wz-subpath of $H \setminus \{x_1\}$, together with $V(Q) \cup V(T) \cup \{x_2, u, v\}$, induces a hole H'', and (H'', x) is a 3-wheel, a contradiction. So X_1 is complete to X_2, and by symmetry Y_1 is complete to Y_2. ■

Proof of Theorem 1.11. Let $K = \{x\}$, $W_1 = X_1$, $Z_1 = Y_1$, $W_2 = X_2 \cup X_3$, $Z_2 = Y_2 \cup Y_3$, $V_1 = W_1 \cup Z_1 \cup C_1$ and $V_2 = W_2 \cup Z_2 \cup C_2 \cup C_3$. We now show that, if G does not admit a clique cutset, then (K, V_1, V_2) is a special 2-amalgam of G. By Lemma 4.8, Lemma 4.10, and Lemma 4.19, the sets $X_1 \cup X_2 \cup X_3$ and $Y_1 \cup Y_2 \cup Y_3$ are cliques, and hence so are the sets $N(C_X)$ and $N(C_Y)$. So, by our assumptions, $C_X = C_Y = \varnothing$, and hence (K, V_1, V_2) is a special 2-amalgam of G. ■

4.2 Proof of Lemma 4.1

We prove Lemma 4.1 by considering Property 1 and Property 2 separately.

The following simple result will be used throughout.

Lemma 4.20 *Let (H, x) be an alternating wheel of a graph $G \in \mathcal{C}$, and let $y \in V(G) \setminus (V(H) \cup \{x\})$ be adjacent to x. If u and v are consecutive neighbors of y in H, then they cannot belong to the interior of two different long sectors of (H, x).*

Proof. Otherwise the uv-subpath of H that does not contain any other neighbor of y in H, together with $\{x, y\}$, induces a 1-wheel with center x. ■

Property 1 holds.

We first assume that G satisfies Property 1. The wheel (H, x), its appendix P and other associated notation are as in the beginning of Section 4. Let y be a vertex of $G \setminus (V(H) \cup V(P) \cup \{x\})$. We also use the following notation in this part: $N = V(S_2) \setminus \{x_2, x_3\}$, $M = V(H) \setminus (V(S_2) \cup \{x_1, x_4\})$ and $N' = N \cup V(P)$.

Lemma 4.21 *If y is adjacent to x and not adjacent to p_1, then $N(y) \cap V(P) = \varnothing$.*

Proof. Assume otherwise and let p_i be the neighbor of y in P with lowest index. If y is adjacent to x_2, then $V(P^{p_1p_i}) \cup \{x, x_2, y\}$ induces a 3-wheel with center x_2. So, x_2y is not an edge. By Lemma 3.1, (W, y) is an alternating wheel, and so y is adjacent to x_3. If $i = k$, then y is also adjacent to x_2' and hence $\{x, x_2, x_2', y, p_k\}$ induces a 3-wheel with center p_k. So, $i < k$. Let y' be the neighbor of y in S_2 that is closest to x_2 and let R be the x_2y'-subpath of S_2. First assume that $y' \neq x_3$. If $N(p_j) \cap \{x_2\} = \varnothing$ for every $1 < j \leq i$, $V(R) \cup V(P^{p_1p_i}) \cup \{x, y\}$ induces a $3PC(xx_2p_1, y)$. Otherwise let T be a chordless x_2y-path contained in the graph induced by $V(P^{p_2p_i}) \cup \{x_2, y\}$. It follows that $V(T) \cup V(R) \cup \{x\}$ induces a $3PC(x_2, y)$. So, $y' = x_3$. If there exists a chordless x_2y-path T contained in the graph induced by $V(P^{p_2p_i}) \cup \{x_2, y\}$, then $V(S_2) \cup V(T) \cup \{x\}$ induces a $3PC(xx_3y, x_2)$. So x_2 has no neighbors in $P^{p_2p_i}$. Let y'' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. Then the $y''x_1$-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_1p_i}) \cup \{x_2, y\}$, induces a hole H' and (H', x) is a 3-wheel, a contradiction. ■

Lemma 4.22 *If y is adjacent to p_1, then $N(y) \cap (V(H) \cup \{x\}) \subseteq \{x, x_1, x_2\}$.*

PROOF. Assume that y is adjacent to p_1, but $N(y) \cap (V(H) \cup \{x\}) \not\subseteq \{x, x_1, x_2\}$.

(1) y is adjacent to x_2.

Proof of (1). Assume otherwise and let y' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. If $y' \neq x'_2$, then the $y'x_1$-subpath of $H \setminus \{x_2\}$, together with $\{x, x_2, y, p_1\}$, induces a 3-wheel with center x. So, $y' = x'_2$. By Lemma 3.1, (W, y) is an alternating wheel, and hence y is adjacent to p_k. Let H' be the hole induced by $\{x_2, x'_2, y, p_1\}$. Then (H', p_k) is a 3-wheel, a contradiction. □

(2) y is adjacent to x.

Proof of (2). Assume not. By (1), $x_2y \in E(G)$. The graph induced by $(V(H) \setminus \{x_1, x_2\}) \cup \{x, y\}$ contains a chordless xy-path R, $V(R) \cup \{p_1\}$ induces a hole H' and (H', x_2) is a 3-wheel, a contradiction. □

(3) y is adjacent to x_1.

Proof of (3). Assume not. By (1) and (2), xy and x_2y are both edges. Let y' be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_1. First assume $y' \neq x'_2$, and let R be the $y'x_1$-subpath of $H \setminus \{x_2\}$. Then $V(R) \cup \{x_2, y\}$ induces a hole H' and (H', x) is a 3-wheel. So, $y' = x'_2$. Let H'' be the hole induced by $(V(S_2) \setminus \{x_2\}) \cup \{x, y\}$. Then (H'', x_2) is a 3-wheel, a contradiction. □

By (1), (2) and (3), $\{x, x_1, x_2\} \subseteq N(y) \cap (V(H) \cup \{x\})$.

(4) y is anticomplete to $\{x_3, \dots, x_n\}$.

Proof of (4). y is not adjacent to x_3 (resp. x_n), since otherwise (H_2, y) (resp. (H_n, y)) is a 3-wheel. Now assume that y is adjacent to x_i for some $3 < i < n$. In particular, let x_i be such a neighbor of y in H with lowest index. By Lemma 3.1, (H, y) is an alternating wheel. If i is even, then (H_i, y) is a 3-wheel. So, i is odd.

Let R be the x_2x_i-subpath of $H \setminus \{x_1\}$. If y does not have a neighbor in $R \setminus \{x_2, x_i\}$, then $V(R) \cup \{x, y\}$ induces a 3-wheel with center x. So, y has a neighbor in $R \setminus \{x_2, \dots, x_i\}$. Such a neighbor cannot belong to the interior of any long sector S_j for $2 < j < i - 1$, since otherwise y and H_j contradict Lemma 3.1. Also, by Lemma 4.20, y does not have a neighbor in the interior of both S_2 and S_{i-1}. W.l.o.g. assume that y has a neighbor in the interior of S_{i-1}. Then (H_{i-1}, y) is a wheel, and hence an alternating wheel, with appendix given by the x_2x_{i-2}-subpath of R. Since $|V(H_{i-1})| < |V(H)|$, our choice of (H, x) is contradicted. □

(5) y has no neighbors in the interior of any long sector of (H, x) that is not S_2.

Proof of (5). Assume otherwise. If y has a neighbor in the interior of a long sector S_i of (H, x), for some $2 < i < n$, then, by (4), y and H_i contradict Lemma 3.1. So y has a neighbor in the interior of S_n and (H_n, y) is a wheel, and hence an alternating wheel, with rim shorter than H. By Lemma 4.20 and (4), y is anticomplete to $V(S_2) \setminus \{x_2\}$, and hence $N(y) \cap V(H) \subseteq V(S_n) \cup \{x_2\}$. If $N(y) \cap V(P) = \{p_1\}$, then (H_n, y) has an appendix induced by the x'_2x_{n-1}-subpath of $H \setminus \{x_2\}$ together with $V(P)$, contradicting our choice of (H, x). So, y has a neighbor in $P \setminus \{p_1\}$ and let p_j be such a neighbor with highest index. Let y' be the neighbor of y in the interior of S_n that is closest to x_n on S_n. Then the x'_2y'-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_jp_k}) \cup \{x, y\}$, induces a 1-wheel with center x, a contradiction. □

By (4) and (5), $N(y) \cap V(H) \subseteq (V(S_2) \setminus \{x_3\}) \cup \{x_1\}$ and, by our initial assumption, y has a neighbor in the interior of S_2. It follows that (H_2, y) is a wheel and hence an alternating wheel. Let y' be the neighbor of y in S_2 that is closest to x_3 and let R be the $y'x_1$-subpath of $H \setminus \{x_2\}$. Then the vertex set $V(R) \cup \{y\}$ induces a hole H' and (H', x) is an alternating wheel with appendix induced by $(V(H) \setminus (V(R) \cup \{x_2\})) \cup V(P)$. Since $|V(H')| < |V(H)|$, our choice of (H, x) is contradicted. ■

Lemma 4.23 *y is anticomplete to at least one of N', M.*

Proof. Suppose that y has a neighbor in both N' and M. It suffices to consider the following two cases.

Case 1: y has a neighbor in M and a neighbor in N.

(1) (H, y) is an alternating wheel.

Proof of (1). It follows from our assumptions and Lemma 3.1. □

(2) y is not adjacent to x.

Proof of (2). Assume it is. By Lemmas 4.21 and 4.22, $N(y) \cap V(P) = \varnothing$. Let u (resp. v) be the neighbor of y in $H \setminus \{x_2\}$ that is closest to x_2' (resp. x_1). By our assumptions, $u \in N$ and $v \in M \cup \{x_1\}$. Let R be the x_2u-subpath of $H \setminus \{x_1\}$. If y is not adjacent to x_2, then the vx_1-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{y\}$, induces a hole H'. Also, (H', x) is a wheel, and hence an alternating wheel, with appendix P and such that $|V(H')| < |V(H)|$, a contradiction. It follows that x_2y is an edge. But then $V(P) \cup V(R) \cup \{x, y\}$ induces a 3-wheel with center x_2, a contradiction. □

(3) (H, x) is a long alternating wheel.

Proof of (3). Assume not. So, (H, x) is a line wheel. By (2), xy is not an edge. Let R be the chordless xy-path contained in the graph induced by $N' \cup \{x, y\}$. If y has a single neighbor in S_4, then this neighbor belongs to the interior of S_4, which contradicts (1). If y has two neighbors in S_4, and these neighbors are adjacent, then $V(S_4) \cup V(R)$ induces a pyramid. It follows that y has non-adjacent neighbors in S_4. But then the graph induced by $V(S_4) \cup V(R)$ contains a $3PC(x, y)$, a contradiction. □

By (2), y is not adjacent to x. By (3), the graph induced by $M \cup \{x, y\}$ contains a chordless xy-path R. If y has non-adjacent neighbors in S_2, then the graph induced by $V(S_2) \cup V(R)$ contains a $3PC(x, y)$. So, by (1), y has two neighbors in S_2 and these neighbors are adjacent. But then $V(S_2) \cup V(R)$ induces a pyramid, a contradiction.

Case 2: y has a neighbor in M, a neighbor in P and no neighbors in N.

(4) y is not adjacent to x.

Proof of (4). Otherwise, by Lemma 4.21, yp_1 is an edge, and so Lemma 4.22 is contradicted. □

By (4), xy is not an edge. Let p_i be the neighbor of y in P with lowest index and let R be the chordless xy-path induced by $V(P^{p_1p_i}) \cup \{x, y\}$.

(5) y has at least two neighbors in $H \setminus V(S_2)$.

Proof of (5). Suppose that y' is the unique neighbor of y in $H \setminus V(S_2)$. Then $y' \in M$ and hence, by Lemma 3.1, $N(y) \cap V(H) = \{y'\}$. If y' belongs to the interior of a long sector S_i of (H, x), for some $4 \leq i \leq n$, then $V(S_i) \cup V(R)$ induces a $3PC(x, y')$. So $y' = x_j$ for some $4 < j \leq n$. First assume that j is even. Let R' be the chordless x_2p_i-path contained in the graph induced by $V(P^{p_1p_i}) \cup \{x_2\}$. Then the x_jx_2-subpath of $H \setminus \{x_3\}$, together with $V(R') \cup \{y\}$, induces a hole H' and (H', x) is a 1-wheel. So, j is odd. Let p_r be the neighbor of y in P with highest index. Then the $x_2'x_j$-subpath of $H \setminus \{x_2\}$, together with $V(P^{p_r p_k}) \cup \{x, y\}$, induces a 1-wheel with center x, a contradiction. □

(6) y does not have non-adjacent neighbors in $H \setminus V(S_2)$.

Proof of (6). Otherwise the graph induced by the vertex set $(V(H) \setminus V(S_2)) \cup V(R)$ contains a $3PC(x, y)$, a contradiction. □

By (5) and (6), y has two neighbors in $H \setminus V(S_2)$, say y' and y'', and $y'y''$ is an edge. If they both belong to the same long sector S_i of (H, x), for some $4 \leq i \leq n$, then $V(S_i) \cup V(R)$ induces a $3PC(yy'y'', x)$. So, w.l.o.g. $y' = x_j$ and $y'' = x_{j+1}$ for some $4 < j < n$, j odd. By Lemma 3.1, y is not adjacent to x_2. Let R' be the chordless x_2p_i-path contained in the graph induced by $V(P^{p_1 p_i}) \cup \{x_2\}$. Then the $x_{j+1}x_2$-subpath of $H \setminus \{x_3\}$, together with $V(R') \cup \{y\}$, induces a hole H' and (H', x) is a 1-wheel, a contradiction. ■

Proof of Lemma 4.1 (under the assumption that Property 1 holds). Assume otherwise.

(1) y has no neighbors in M.

Proof of (1). Assume it does. By Lemma 4.23, y has no neighbors in N'. First suppose that y has non-adjacent neighbors in $H \setminus N$. Let y' (resp. y'') be the one that is closest to x_2 (resp. x_3). Then the $y'x_2$-subpath of $H \setminus N$, together with the x_3y''-subpath of $H \setminus N$ and $V(S_2) \cup \{y\}$, induces a hole H'. By Lemma 3.1, (H', x) is an alternating wheel with appendix P. By Lemma 3.1, (H, y) is an alternating wheel and hence $|V(H')| < |V(H)|$, so that (H', x) contradicts our choice of (H, x). So, y does not have non-adjacent neighbors in $H \setminus N$. It follows that y is adjacent to x and has a neighbor that belongs to the interior of a long sector S_i of (H, x), for some $4 \leq i \leq n$. But then y and H_i contradict Lemma 3.1. □

(2) y is not adjacent to x_1.

Proof of (2). Assume it is. By (1), y has no neighbors in M. Suppose that y has no neighbors in $V(H) \setminus \{x_1, x_2\}$. It follows that y has a neighbor in P and let p_i be such a neighbor with highest index. Then y is adjacent to x, since otherwise $(V(H) \setminus \{x_2\}) \cup V(P^{p_i p_k}) \cup \{x, y\}$ induces a 1-wheel with center x. So, by Lemma 4.21, yp_1 is an edge. Since $\{x, x_1, x_2, y, p_1\}$ cannot induce a 3-wheel with center x, y is adjacent to x_2. Therefore, y is complete to $\{x, x_1, x_2, p_1\}$. If $N(y) \cap V(P) = \{p_1\}$, then $(V(H) \setminus \{x_2\}) \cup V(P) \cup \{x, y\}$ induces a 3-wheel with center x. It follows that $\{p_1\} \subset N(y) \cap V(P)$. But then, by Lemma 3.1 applied to W and y, y is a special vertex of G, a contradiction.

So, y has a neighbor in $H \setminus (M \cup \{x_1, x_2\})$. Let y' be such a neighbor that is closest to x_4 and let H' be the hole induced by the $y'x_1$-subpath of $H \setminus \{x_2\}$ together with y. By Lemma 3.1, (H', x) is an alternating wheel, and hence $xy \in E(G)$ and $y' \notin \{x_3, x_4\}$. By Lemma 3.1, (H, y) is an alternating wheel, and so x_2y is an edge. By Lemma 3.1, (W, y) is an alternating wheel, and hence $yp_1 \in E(G)$, contradicting Lemma 4.22. □

(3) y is not adjacent to x_4.

Proof of (3). Assume it is. By (1) and (2), y has no neighbors in $M \cup \{x_1\}$. Suppose that y has a neighbor in P. Then, by Lemmas 4.21 and 4.22, y is not adjacent to x. Let R be a chordless x_2y-path contained in the graph induced by $V(P) \cup \{x_2, y\}$, and let H' be the hole induced by $(V(H) \setminus V(S_2)) \cup V(R)$. But then (since xy is not an edge) (H', x) is a 1-wheel. Therefore y has no neighbors in P. Since $N(y) \cap V(H)$ is not a clique, y must have a neighbor in $S_2 \setminus \{x_3\}$, and let y' be such a neighbor closest to x_2. Let H'' be the hole induced by y together with the x_4y'-subpath of $H \setminus \{x_3\}$. By Lemma 3.1, (H'', x) is an alternating wheel and hence y is adjacent to x and $y' \neq x_2$. Also, by Lemma 3.1, (H, y) is an alternating wheel, and so $|V(H'')| < |V(H)|$. Note that P is an appendix of (H'', x), and hence our choice of (H, x) is contradicted. $\square$

(4) y is not adjacent to x_2.

Proof of (4). Assume it is. By (1), (2) and (3), y has no neighbors in $H \setminus V(S_2)$. First suppose that y has no neighbors in N. Then y is not adjacent to x_3, since otherwise $V(S_2) \cup \{x, y\}$ induces a theta or a 3-wheel with center x. Now assume that xy is an edge. By our assumptions, y has a neighbor in $P \setminus \{p_1\}$. Then, by Lemma 4.21, y is adjacent to p_1. Let p_i be the neighbor of y in P with highest index, and note that $i > 2$ since, by Lemma 3.1, $V(W) \cup \{y\}$ must induce an alternating wheel. It follows that the chordless path induced by $V(P^{p_ip_k}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction.

So, y is not adjacent to x and has a neighbor in P. Let p_j (resp. p_r) be the neighbor of y in P with lowest (resp. highest) index. First suppose that $j \neq r$ and p_jp_r is not an edge. By Lemma 3.1, (W, y) is an alternating wheel, and so $r > j + 3$. Therefore the chordless path induced by $V(P^{p_1p_j}) \cup V(P^{p_rp_k}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction. Now assume that p_jp_r is an edge. Since $N(y) \cap (V(P) \cup \{x_2\})$ is not a clique of size 3, x_2 is not adjacent to at least one of p_j, p_r and hence the graph induced by $V(P) \cup \{x_2, y\}$ contains a 1-wheel or a 3-wheel with center y. It follows that $j = r$ and x_2p_j is not an edge. But then the graph induced by $V(P) \cup \{x_2, y\}$ contains a theta, a contradiction.

So, y has a neighbor in N, and let y' be the one that is closest to x_3 on S_2. First assume $y' = x_2'$. Then y is anticomplete to $\{x, x_3\}$, since otherwise $V(S_2) \cup \{x, y\}$ induces a 1-wheel or a 3-wheel with center y. So, by our assumptions, y has a neighbor in $P \setminus \{p_k\}$ and let p_ℓ be the one with lowest index. By Lemma 3.1, (W, y) is an alternating wheel, and so $\ell < k - 1$. By Lemma 4.22, $\ell > 1$, and hence the chordless path induced by $V(P^{p_1p_\ell}) \cup \{y\}$ is an appendix of (H, x) that is shorter than P, a contradiction.

So, $y' \neq x_2'$ and $V(S_2) \cup \{x, y\}$ induces an alternating wheel with center y. If y is adjacent to x, then the $y'x_2$-subpath of $H \setminus \{x_2'\}$, together with $\{x, y\}$, induces a 3-wheel with center x. So, y is adjacent to x_2' and not adjacent to x. Also, (W, y) is an alternating wheel and hence y is adjacent to p_k. Let p_s be the neighbor of y in P with lowest index. By Lemma 4.22, $s > 1$. In particular, $s > 2$ and x_2p_s is an edge, since otherwise the $y'x_3$-subpath of S_2, together with $V(P^{p_1p_s}) \cup \{x, x_2, y\}$, induces a 1-wheel or a 3-wheel with center x_2. Let H' be the hole induced by the $y'x_2$-subpath of $H \setminus \{x_2'\}$ together with y. Then (H', x) is an alternating wheel with appendix $P^{p_1p_s}$ and such that $|V(H')| < |V(H)|$, which contradicts our choice of (H, x). $\square$

(5) y has no neighbors in $V(S_2) \setminus \{x_2\}$.

Proof of (5). Assume it does and let y' be such a neighbor that is closest to x_3. If $y' = x_2'$, then y is not adjacent to x (else $V(S_2) \cup \{x, y\}$ induces a theta) and so, since $N(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, y has a neighbor in $P \setminus \{p_k\}$. This implies that the graph induced by $V(S_2) \cup (V(P) \setminus \{p_k\}) \cup \{x, y\}$ contains a $3PC(xx_2p_1, x_2')$. So, $y' \neq x_2'$. By (1), (2), (3) and (4), y is anticomplete to $M \cup \{x_1, x_2, x_4\}$. Let R be the $y'x_2$-subpath of $H \setminus \{x_2'\}$. First suppose that $N(y) \cap V(P) \neq \varnothing$. Let p_i (resp. p_j) be the neighbor of y in P with lowest (resp. highest) index. By Lemma 4.22, $i > 1$. Then, by Lemma 4.21, xy is not an edge. If $i \neq j$ and p_ip_j is not an edge, then the graph induced by $V(R) \cup V(P^{p_1p_i}) \cup V(P^{p_jp_k}) \cup \{y\}$ contains a $3PC(y, x_2)$. Now assume that p_ip_j is an edge. If x_2 is not adjacent to both p_i and p_j, then the graph induced by $V(R) \cup V(P) \cup \{y\}$ contains a $3PC(p_ip_jy, x_2)$. So, x_2 is adjacent to both p_i and p_j. Since (W, x_2) is an alternating wheel, $i > 2$ and p_{i-1} is not adjacent to x_2. It follows that the $y'x_3$-subpath of S_2, together with $V(P^{p_1p_i}) \cup \{x, x_2, y\}$, induces a 1-wheel with center x_2, a contradiction.

Therefore, $i = j$. By Lemma 3.1, (W, y) is an alternating wheel. So, $N(y) \cap V(P) = \{p_k\}$, y is adjacent to x'_2 and $x'_2 y'$ is not an edge. Then the $y'x_3$-subpath of S_2, together with $\{x, x_2, x'_2, y, p_k\}$, induces a 3-wheel with center p_k, a contradiction.

It follows that y has no neighbors in P. Let y'' be the neighbor of y in $V(S_2) \setminus \{x_2\}$ that is closest to x'_2. If y is adjacent to x then, by Lemma 3.1, (W, y) is an alternating wheel and hence $y' = x_3$, $y' \neq y''$ and $y'y''$ is not an edge. So, the $x'_2 y''$-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{x, y\}$, induces a 3-wheel with center x. Therefore, y is not adjacent to x. Since $N(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, $y' \neq y''$, $y'y''$ is not an edge and hence, by Lemma 3.1, (W, y) is an alternating wheel. It follows that the $x'_2 y''$-subpath of $H \setminus \{x_2\}$, together with $V(R) \cup \{y\}$, induces a hole H' that is shorter than H. Also, (H', x) is an alternating wheel with appendix P, a contradiction. □

By (1), (2), (3), (4) and (5), $N(y) \cap (V(H) \cup V(P) \cup \{x\}) \subseteq V(P) \cup \{x\}$ and, since $N(y) \cap (V(H) \cup V(P) \cup \{x\})$ is not a clique, by Lemma 3.1, (W, y) is an alternating wheel. If y is not adjacent to x, then the graph induced by $V(P) \cup \{y\}$ contains a chordless $p_1 p_k$-path that contains y and is an appendix of (H, x) that is shorter than P, a contradiction. So, y is adjacent to x and hence $\{p_1\} \subset N(y) \cap V(P)$. Let p_i be the neighbor of y in P with highest index. Then the graph induced by $V(P^{p_i p_k}) \cup \{x, x_2, y, p_1\}$ contains a 3-wheel with center p_1, a contradiction. ■

Property 2 holds.

We now assume that G satisfies Property 2. The wheel (H, x) and other associated notation are as in the beginning of Section 4.

Proof of Lemma 4.1 (under the assumption that Property 2 holds). Let $y \in V(G) \setminus (V(H) \cup \{x\})$ and assume that $N(y) \cap (V(H) \cup \{x\})$ is not a clique. We consider the following two cases.

Case 1: y is adjacent to x.

(1) For every long sector S_i of (H, x), either $N(y) \cap V(S_i) \subseteq \{x_j\}$ for $i \leq j \leq i+1$ or (H_i, y) is a line wheel.

Proof of (1). Note that y cannot be adjacent to both x_i and x_{i+1}, since otherwise (H_i, y) is a 3-wheel. If y has a neighbor in the interior of S_i then, by Lemma 3.1, (H_i, y) is an alternating wheel. In particular, by our choice of (H, x), (H_i, y) is a line wheel and hence (1) holds. □

(2) y is complete to a short sector of (H, x).

Proof of (2). Assume otherwise. W.l.o.g. y has a neighbor in a long sector S_i of (H, x). Therefore, by (1), y has a neighbor in $\{x_i, x_{i+1}\}$. So w.l.o.g. assume that y is adjacent to x_i. Let y' be the neighbor of y in $H \setminus \{x_i\}$ that is closest to x_{i-1} (it exists since $N(y) \cap (V(H) \cup \{x\})$ is not a clique). By (1), $y' \neq x'_i$. But then the $y'x_i$-subpath of H that contains x_{i-1}, together with $\{x, y\}$, induces a 3-wheel with center x, a contradiction. □

By (2), w.l.o.g. we may assume that y is complete to S_1. Also, by our assumptions, y has a neighbor in $H \setminus \{x_1, x_2\}$.

(3) y has a neighbor in $\{x_3, \ldots, x_n\}$.

Proof of (3). Assume otherwise. Then y has a neighbor in the interior of a long sector of (H, x), say S_i. Note that $i = 2$ or $i = n$, since otherwise (1) is contradicted. W.l.o.g. $i = 2$ and let y' be the neighbor of y in the interior of S_2 that is closest to x_3 on S_2. By (1), $y' \neq x'_2$. By Lemma 4.20, y has no neighbors in the interior of S_n. But then the $y'x_1$-subpath of $H \setminus \{x_2\}$, together with $\{x, y\}$, induces a long alternating wheel with center x and rim shorter than H, a contradiction. □

(4) y is anticomplete to $\{x_3, x_4, x_{n-1}, x_n\}$.

Proof of (4). Assume not and w.l.o.g. suppose that y has a neighbor in $\{x_3, x_4\}$. By (1), x_3y is not an edge. But then the graph induced by $V(S_2) \cup \{x, x_4, y\}$ contains a 3-wheel with center x, a contradiction. □

By (3) and (4), y is adjacent to x_i for some $4 < i < n - 1$. W.l.o.g. assume that y has no neighbors in $\{x_3, x_4, \dots, x_{i-1}\}$.

(5) i is odd, and x_2 and x_i are not consecutive neighbors of y in H.

Proof of (5). Let R be the x_2x_i-subpath of $H \setminus \{x_1\}$ and let y' be the neighbor of y in $R \setminus \{x_i\}$ that is closest to x_i on R. Let R' be the $y'x_i$-subpath of R. If i is even or $y' = x_2$, then $V(R') \cup \{x, y\}$ induces a 3-wheel with center x. Therefore, (5) holds. □

By (5), y has a neighbor in the interior of a long sector S_j of (H, x), for some $1 < j < i$. By (1), $j = 2$ or $j = i - 1$. So, w.l.o.g. assume $j = 2$ and, by (1), let y' and y'' be the adjacent neighbors of y in the interior of S_2, where y' is closer to x_2 on S_2. By Lemma 4.20, y'' and x_i are consecutive neighbors of y in H. Also, since (H_2, y) is a line wheel, y' is not adjacent to x_2. Let H' be the hole induced by the $y''x_i$-subpath of $H \setminus \{x_2\}$ together with y. Then (H', x) is an alternating wheel with appendix given by the x_2y'-subpath of S_2, a contradiction.

Case 2: y is not adjacent to x.

(6) (H, y) is an alternating wheel.

Proof of (6). Since $N(y) \cap V(H)$ is not a clique, y has at least two non-adjacent neighbors in H and so, by Lemma 3.1, (6) holds. □

W.l.o.g. assume that y has a neighbor is S_2.

(7) $N(y) \cap V(H) \subseteq V(S_2) \cup \{x_1, x_4\}$.

Proof of (7). Assume not. Then the graph induced by $(V(H) \setminus (V(S_2) \cup \{x_1, x_4\})) \cup \{x, y\}$ contains a chordless xy-path R. If y has non-adjacent neighbors in S_2, then the graph induced by $V(S_2) \cup V(R)$ contains a $3PC(x, y)$. If y has exactly two neighbors in S_2 then, by Lemma 3.1 applied to y and H_2, they are adjacent and hence $V(S_2) \cup V(R)$ induces a pyramid. Therefore y has a unique neighbor y' in S_2. If $y' \notin \{x_2, x_3\}$ then $V(S_2) \cup V(R)$ induces a $3PC(x, y')$. So w.l.o.g. $y' = x_2$. Let y'' be the neighbor of y in $H \setminus V(S_2)$ that is closest to x_4. Since $y'' \neq x_1$, the x_2y''-subpath of H that contains x_3, together with $\{x, y\}$, induces a 1-wheel with center x, a contradiction. □

Let y' (resp. y'') be the neighbor of y in the path induced by $V(S_2) \cup \{x_1, x_4\}$ that is closest to x_1 (resp. x_4). By (6) and (7), the $y'y''$-subpath of H that contains x_5, together with y, induces a hole H' that is shorter than H. By Lemma 3.1, (H', x) is an alternating wheel, thus contradicting our choice of (H, x). ■

5 Proof of Theorem 1.13

Proof of Theorem 1.13. Our proof is by induction on $|V(G)|$. By Theorem 1.8, G is the line graph of a triangle-free graph or it admits a clique cutset or a small 2-amalgam. Let us consider these three cases.

Case 1: G is the line graph of a triangle-free graph.

By Vizing's theorem, $\chi(G) \leq \omega(G) + 1 \leq 4$, which completes the proof.

Case 2: G admits a clique cutset.

Let K be a clique cutset of G, let $C_1, \dots, C_k$, $k \geq 2$, be the connected components of $G \setminus K$, and, for every $1 \leq i \leq k$, let $G_i = G\big[V(C_i) \cup K\big]$. By induction, for every $1 \leq i \leq k$, G_i is 4-colorable. For $1 \leq i \leq k$, let c_i be a 4-coloring of G_i. Since K is a clique, vertices of K must have different colors in all of these colorings. So, we can permute the colors of c_i's so that they all agree on the colors of the vertices of K, and by putting together such colorings we get a 4-coloring of G.

Case 3: G admits a small 2-amalgam.

Let $(\{x\}, V_1, V_2)$ be a small 2-amalgam of G, and let $W_1 = \{x_1\}$, $W_2 = \{x_2\}$, $Z_1 = \{x_4\}$ and $Z_2 = \{x_3\}$. Let $G_1 = G[V_1 \cup \{x, x_2, x_3\}]$ and $G_2 = G[V_2 \cup \{x, x_1, x_4\}]$. Since G is K_4-free, $\omega(G) = \omega(G_1) = \omega(G_2) = 3$. Let c_1 (resp. c_2) be a 4-coloring of G_1 (resp. G_2). W.l.o.g. assume that c_1 and c_2 agree on $\{x, x_1, x_2\}$. If they also agree on $\{x_3, x_4\}$, then we are done. So, consider the case where they do not agree. W.l.o.g. suppose $c_1(x) = c_2(x) = 1$, $c_1(x_1) = c_2(x_1) = 2$ and $c_1(x_2) = c_2(x_2) = 3$.

If $c_1(x_4) \neq c_2(x_3)$, then we can obtain a 4-coloring of G by coloring every vertex from $V(G_1) \setminus \{x_3\}$ with the same color as in the coloring c_1, and by coloring every vertex from $V(G_2) \setminus \{x_4\}$ with the same color as in the coloring c_2.

So, assume that $c_1(x_4) = c_2(x_3)$. First suppose that $c_1(x_4) \in \{2, 3\}$. Then w.l.o.g. we may assume that $c_1(x_4) = 2$. To obtain a 4-coloring c of G we first color every vertex from $V(G_1) \setminus \{x_3\}$ with the same color as in the coloring c_1. Then, for a vertex $v \in V_2$ we define $c(v)$ in the following way: $c(v) = c_2(v)$ if $c_2(v) \in \{1, 3\}$, $c(v) = 2$ if $c_2(v) = 4$, and $c(v) = 4$ if $c_2(v) = 2$.

Finally, let $c_1(x_4) = c_2(x_3) = 4$. To obtain a 4-coloring c of G we first color every vertex from $V(G_1) \setminus \{x_3\}$ with the same color as in the coloring c_1. Then, for a vertex $v \in V_2$ we define $c(v)$ in the following way: $c(v) = c_2(v)$ if $c_2(v) \in \{1, 3\}$, $c(v) = 2$ if $c_2(v) = 4$, and $c(v) = 4$ if $c_2(v) = 2$. ■

References

[1] P. Aboulker, P. Charbit, N. Trotignon and K. Vušković, Vertex elimination orderings for hereditary graph classes, Discrete Math. 338 (2015), 825–834.

[2] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, Recognizing Berge graphs, Combinatorica 25 (2005), 143–186.

[3] M. Chudnovsky and P. Seymour, The three-in-a-tree problem, Combinatorica 30 (4) (2010), 387–417.

[4] F. Cicalese and M. Milanič, Graphs of separability at most 2, Discrete Appl. Math. 160 (2012), 685–696.

[5] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Universally signable graphs, Combinatorica 17 (1) (1997), 67–77.

[6] E. Diot, M. Radovanović, N. Trotignon and K. Vušković, The (theta, wheel)-free graphs Part I: Only-prism and only pyramid graphs, to appear in J. Comb. Theory B, https://doi.org/10.1016/j.jctb.2017.12.004

[7] E. Diot, S. Tavenas and N. Trotignon, Detecting wheels, Appl. Anal. Discr. Math. 8 (1) (2014), 111–122.

[8] J. Edmonds, Paths, trees and flowers, Canad. J. Math. 17 (1965), 449–467.

[9] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169–197.

[10] F. Harary and C. Holtzmann, Line graphs of bipartite graphs, Revista de la Sociedad Matematica de Chile 1 (1974), 67–77.

[11] D. Král', J. Kratochvíl, Z. Tuza and G. J. Woeginger, Complexity of coloring graphs with forbidden induced subgraphs, Proceedings of the 27^{th} International Workshop on Graph-Theoretic Concepts in Computer Science, WG 01, LNCS 2204 (2001), 254–262.

[12] D. Kühn and D. Osthus, Induced subdivisions in $K_{s,s}$-free graphs of large average degree, Combinatorica 24 (2) (2004), 287–304.

[13] P.G.H. Lehot, An optimal algorithm to detect a line graph and output its root graph, J. ACM 21 (4) (1974), 569–575.

[14] F. Maffray and N. Trotignon, Algorithms for perfectly contractile graphs, SIAM J. Discrete Math. 19 (3) (2005), 553–574.

[15] F. Maffray, N. Trotignon and K. Vušković, Algorithms for square-$3PC(\cdot,\cdot)$-free Berge graphs, SIAM J. Discrete Math. 22 (1) (2008), 51-71.

[16] N. Trotignon, Graphes parfaits: structure et algorithmes, PhD thesis, Université Grenoble I (2004).

[17] N. Trotignon, Perfect graphs: a survey, (2015), arXiv:1301.5149v7.

[18] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids, J. Comb. Theory B 32 (1982), 112-139.

[19] K. Vušković, The world of hereditary graph classes viewed through Truemper configurations, Surveys in Combinatorics 2013, London Mathematical Society Lecture Note Series 409, Cambridge University Press (2013), 265–325.