On Roussel–Rubio-type lemmas and their consequences

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Abstract

Roussel and Rubio proved a lemma which is essential in the proof of the Strong Perfect Graph Theorem. We give a new short proof of the main case of this lemma. In this note, we also give a short proof of Hayward's decomposition theorem for weakly chordal graphs, relying on a Roussel–Rubio-type lemma. We recall how Roussel–Rubio-type lemmas yield very short proofs of the existence of even pairs in weakly chordal graphs and Meyniel graphs.

1 Introduction

A *hole* in a graph is an induced cycle of length at least 4. An *antihole* is the complement of a hole. A graph is a *Berge* graph if it contains no odd hole and no odd antihole, where *odd* refers to the length of the hole. By $\chi(G)$ we denote the chromatic number of G, and by $\omega(G)$ we denote the maximum size of a clique in G. A graph G is *perfect* if for any induced subgraph H of G, $\chi(H) = \omega(H)$. Berge conjectured that every Berge graph is perfect [1]. This was known as the *Strong Perfect Graph Conjecture*, was the object of

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much research and was finally proved by Chudnovsky, Robertson, Seymour, and Thomas [3].

A lemma due to Roussel and Rubio [16] is used at many steps in [3]. In fact, the authors of [3] rediscovered it (in joint work with Thomassen) and initially named it the *wonderful lemma* because of its many applications. The first aim of this note is to give a previously unpublished short proof of the Roussel–Rubio Lemma, see Section 2. In the same section, we recall variants of this lemma for classes of perfect graphs.

A hole or an antihole is said to be *long* if it contains at least 5 vertices. A graph is *weakly chordal*, also called *weakly triangulated* if it contains no long hole and no long antihole. Weakly chordal graphs were investigated by Chvátal [6] and Hayward et al. [9, 10, 8, 12]. Hayward proved a decomposition theorem for weakly chordal graphs that implies their perfectness. The second aim of this note is to give a short proof of this decomposition theorem, relying on a version of the Roussel–Rubio Lemma, see Section 3.

An *even pair* in a graph is a pair of vertices such that all induced paths linking them have even length. In Section 4, we recall known proofs of the existence of even pairs in weakly chordal graphs and also Meyniel graphs, all relying on Roussel–Rubio-type lemmas.

We say that a graph G *contains* a graph H if G has an induced subgraph isomorphic to H. We say that G is H*-free* if G does not contain H.

2 Roussel–Rubio-type lemmas

A set of vertices in a graph is *anticonnected* if it induces a graph whose complement is connected. A vertex is *complete* to a set S if it is adjacent to all vertices in S. The Roussel–Rubio Lemma states that, in a sense, any anticonnected set of vertices of a Berge graph behaves like a single vertex. How does a vertex v "behave" in a Berge graph? If a chordless path of odd length (at least 3) has both ends adjacent to v, then v must have other neighbors in the path, for otherwise there is an odd hole. The lemma states roughly that an anticonnected set T of vertices behaves similarly: if a chordless path of odd length (at least 3) has both ends complete to T, then at least one internal vertex of the path is also complete to T. In fact, there are two situations where this statement fails, so the lemma is slightly more complicated.

Proofs of the Roussel–Rubio Lemma can be found in the original paper [16], and also in [5]. A simpler proof due to Maffray and Trotignon can be found in [13]. The proof given here, although previously unpublished, is due to Kapoor, Vušković, and Zambelli. Originally, the argument can be found in an unpublished work on *cleaning*, by Kapoor and Vušković, and Zambelli noticed that this argument yields an elegant proof of the Roussel–Rubio Lemma (cleaning is an essential component of the recognition algorithm for Berge graphs, see [2]). The proof is essentially the same as the other proofs (in particular the proof in [13]), except in the case when T is a stable set. So, we present here only this case.

When T is a set of vertices of a graph G, a set $S \subseteq V(G) \setminus T$ is T-*complete* if each vertex of S is adjacent to each vertex of T. An *antipath* is the complement of a chordless path. If $P = xx' \dots y'y$ is a chordless path of length at least 3 in a graph G, we say that a pair of non-adjacent vertices $\{u, v\}$, disjoint from P, is a *leap* for P if $N(u) \cap V(P) = \{x, x', y\}$ and $N(v) \cap V(P) = \{x, y', y\}$. Note that the following statement is a reformulation from [3] of the original lemma.

Lemma 2.1 (Roussel and Rubio [16]) *Let G be an odd-hole-free graph and let T be an anticonnected set of $V(G)$. Let P be a chordless path in $G \setminus T$, of odd length, at least 3. If the ends of P are T-complete, then at least one internal vertex of P is T-complete, or T contains a leap for P, or P has length 3, say $P = xx'y'y$, and G contains an antipath of length at least 3, from x' to y' whose interior is in T.*

PROOF — We prove that if T is a stable set, then at least one internal vertex of P is T-complete or T contains a leap for P (for the case when T is not a stable set, see the proof in [13]). In fact, we prove slightly more by induction on $|T|$: either there is a leap, or P has an odd number of T-complete edges.

Let $P = xx' \dots y'y$. Mark the vertices of P that have at least one neighbor in T. Call an *interval* any subpath of P, of length at least 1, whose ends are marked and whose internal vertices are not. Since x and y are marked, the edges of P are partitioned by the intervals of P.

If $|T| = 1$, then every interval P' of odd length has length 1, for otherwise $T \cup P'$ induces an odd hole. Since the length of P is odd, it has an odd number of intervals of odd length, and hence an odd number of T-complete edges. Now suppose $|T| > 1$ and there is no leap for P contained in T.

We claim that every interval of P either has even length or has length 1. Indeed, suppose there is an interval of odd length, at least 3, say $P' = x'' \dots y''$, named so that x, x'', y'', y appear in this order along P. Let u and v be neighbors of x'' and y'' in T, respectively. If x'' and y'' have a common neighbor t in T then $P' \cup \{t\}$ induces an odd hole. Hence $u \neq v$, $x'' \neq x$, $y'' \neq y$, v is not adjacent to x'', and u is not adjacent to y''. If $x'' \neq x'$, then $P' \cup \{u, x, v\}$ induces an odd hole. So, $x'' = x'$ and similarly, $y'' = y'$. Hence, $\{u, v\}$ is a leap, a contradiction. This proves our claim.

Hence, there is an odd number of intervals of length 1 in P. Moreover, we claim that for every interval of length 1, there is a vertex in T adjacent to both its ends. Indeed, suppose that there is an interval $x''y''$ such that x'' and y'' do not have a common neighbor in T. Let u be a neighbor of x'' in T, and let v be a neighbor of y'' with $u \neq v$, $uy'' \notin E(G)$, and $vx'' \notin E(G)$. Note that $x \neq x''$ and $y \neq y''$. If $x'' \neq x'$, then $\{u, x, v, x'', y''\}$ induces an odd hole. So, $x'' = x'$ and similarly $y'' = y'$. Now $\{u, v\}$ is a leap, a contradiction.

For every $v \in T$, denote by $f(v)$ the set of all $\{v\}$-complete edges of P. Let $v_1, \ldots, v_n$ be the elements of T. We know that $|f(v_1) \cup \cdots \cup f(v_n)|$ is odd, since, from the previous paragraph, it is equal to the number of the intervals of length 1. Moreover, by the sieve formula, also known as the inclusion-exclusion formula, we have:

$$\begin{aligned}
|f(v_1) \cup \cdots \cup f(v_n)| \quad = \quad & \sum_i |f(v_i)| \\
& - \sum_{i \neq j} |f(v_i) \cap f(v_j)| \\
& \vdots \\
& + (-1)^{(k+1)} \sum_{I \subset \{1,\ldots,n\}, |I| = k} | \cap_{i \in I} f(v_i)| \\
& \vdots \\
& + (-1)^{(n+1)} |f(v_1) \cap \cdots \cap f(v_n)|
\end{aligned}$$

By the induction hypothesis, we know that if $S \subsetneq T$, then P has an odd number of S-complete edges (note that a leap in S is a leap in T, and we are assuming that T has no leap). Hence if $I \subsetneq \{1, \ldots, n\}$, then $|\cap_{i \in I} f(v_i)|$ is odd. Thus, we can rewrite the above equality modulo 2 as:

$$|f(v_1) \cup \cdots \cup f(v_n)| = (2^n - 2) + (-1)^{(n+1)} |f(v_1) \cap \cdots \cap f(v_n)|$$

Since $|f(v_1) \cup \cdots \cup f(v_n)|$ is odd, it follows that $|f(v_1) \cap \cdots \cap f(v_n)|$ is odd, meaning that P has an odd number of T-complete edges. □

We now present several Roussel–Rubio-type lemmas. It seems that the first Roussel–Rubio-type lemma ever proved is the following simple lemma about Meyniel graphs, originally pointed out by Meyniel (and needed in Section 4). A graph is a *Meyniel* graph if every odd cycle of length at least 5 has at least two chords. We include the original proof for completeness.

Lemma 2.2 (Meyniel [14]) *Let G be a Meyniel graph, and $v \in V(G)$. Let P be a path of $G \setminus v$ of odd length, at least 3. If the ends of P are adjacent to v, then all vertices of P are adjacent to v.*

PROOF — Let an *interval* be any subpath of P, of length at least 1, whose ends are adjacent to v and whose internal vertices are not. The length of any interval is 1 or is even, because otherwise, together with v, it induces an odd hole. So, if v is not complete to P, then P does contain intervals of even length and also intervals of lengths 1 since P has odd length. So, there is an interval of length 1 and an interval of even length that are consecutive. This yields an odd cycle with a unique chord, a contradiction. □

Another Roussel–Rubio-type lemma is used in [8] to prove that every vertex in a minimal imperfect graph is in a long hole or in a long antihole, see Lemma 2 in [8]. The following can be seen as a Roussel–Rubio-type lemma for weakly chordal graphs; that is used in Sections 3 and 4. We include the proof from [13] for completeness.

Lemma 2.3 (Maffray and Trotignon [13]) *Let G be a weakly chordal graph, and let $T \subseteq V(G)$ be an anticonnected set. Let $P = x \dots y$ be a chordless path of $G \setminus T$ of length at least 3. If the ends of P are T-complete, then P has an internal vertex that is T-complete.*

PROOF — Note that no vertex $t \in T$ can be non-adjacent to two consecutive vertices of P, for otherwise $V(P) \cup \{t\}$ contains a long hole. Let z be an internal vertex of P adjacent to a maximum number of vertices of T. Suppose for a contradiction that there exists a vertex $u \in T \setminus N(z)$. Let x' and y' be the neighbors of z along P, so that x, x', z, y', y appear in this order along P. Then, from the first sentence of this proof, and subject to the conditions established so far, $ux', uy' \in E(G)$. Without loss of generality, we may assume $x' \neq x$. From the choice of z, since $ux' \in E(G)$ and $uz \notin E(G)$, there exists a vertex $v \in T$ such that $vz \in E(G)$ and $vx' \notin E(G)$. Since $G[T]$ is anticonnected, there exists an antipath Q of $G[T]$ from u to v. Suppose that u, v are chosen subject to the minimality of this antipath. From the first sentence and subject to the conditions established so far, internal vertices of Q are all adjacent to x' or z and from the minimality of Q, internal vertices of Q are all adjacent to x' and z. If $x'x \notin E(G)$ then $V(Q) \cup \{x, x', z\}$

induces a long antihole. So $x'x \in E(G)$. If $zy \notin E(G)$ then $V(Q) \cup \{z, x', y\}$ induces a long antihole. So $zy \in E(G)$ and $y = y'$. Now, $V(Q) \cup \{x, x', z, y\}$ induces a long antihole, a contradiction. □

Let $C(T)$ denote the set of all T-complete vertices. The following is implicit in [13].

Lemma 2.4 *Let G be a weakly chordal graph, and $T \subseteq V(G)$ a set of vertices such that $G[T]$ is anticonnected and $C(T)$ contains at least two non-adjacent vertices. If T is inclusion-wise maximal with respect to these properties, then any chordless path of $G \setminus T$ whose ends are in $C(T)$ has all its vertices in $C(T)$.*

PROOF — Let P be a chordless path in $G \setminus T$ whose ends are in $C(T)$. If some vertex of P is not in $C(T)$, then P contains a subpath P' of length at least 2 whose ends are in $C(T)$ and whose interior is disjoint from $C(T)$. If P' is of length 2, say $P' = atb$, then $T \cup \{t\}$ is a set that contradicts the maximality of T. If P' is of length greater than 2, then it contradicts Lemma 2.3. □

3 Weakly chordal graphs

Here we give a new simple proof of Hayward's decomposition theorem for weakly chordal graphs. A *cutset* in a graph is a set S of vertices such that $G \setminus S$ is disconnected. A *star cutset* of a graph is a set of vertices S that contains a vertex c such that $S \subseteq \{c\} \cup N(c))$ and that is a cutset.

Theorem 3.1 (Hayward [9, 12]) *If G is a weakly chordal graph, then one of the following holds:*

- *G is a complete graph;*
- *G is the complement of a perfect matching;*
- *G admits a star cutset.*

PROOF — We proceed by induction on $|V(G)|$. If G is a disjoint union of complete graphs (in particular when $|V(G)| = 1$), then the theorem holds: if there is more than one component, then some vertex is a star cutset, unless G has exactly 2 vertices, that are furthermore not adjacent, in which case it is the complement of a perfect matching. Otherwise, we may assume that

G contains a chordless path P on 3 vertices. Hence, there exists a set T of vertices such that $G[T]$ is anticonnected and $C(T)$ contains at least two non-adjacent vertices, because the center of P forms such a set T. Let us assume that T is maximal, as in Lemma 2.4. Since $C(T)$ is not a clique, by the induction hypothesis, we have two cases to consider: the graph induced by $C(T)$ has a star cutset S, or the graph induced by $C(T)$ is the complement of a perfect matching. In the first case, by Lemma 2.4, $T \cup S$ is a star cutset of G. So, we may assume that we are in the second case.

Suppose first that $V(G) = T \cup C(T)$. Then by the induction hypothesis, either T is a clique (in which case $|T| = 1$ since T is anticonnected), or T induces the complement of a perfect matching or T has a star cutset S. If $|T| = 1$, then $T \cup C(T) \setminus \{x, y\}$, where x, y are two non-adjacent vertices in $C(T)$, is a star cutset of G. In the second case, G itself is the complement of a perfect matching. In the third case, $S \cup C(T)$ is a star cutset of G.

So, we may assume that there exists a vertex $x \in V(G) \setminus (T \cup C(T))$. We choose x with a neighbor y in $C(T)$; this is possible otherwise T together with any vertex of $C(T)$ forms a star cutset of G.

Recall that $C(T)$ is the complement of a perfect matching. Let y' be the non-neighbor of y in $C(T)$. We claim that $S = T \cup C(T) \setminus \{y'\}$ is a star cutset of G separating x from y'. First, observe that $S \subseteq \{y\} \cup N(y)$. Also, if there is a path in $G \setminus S$ from x to y', then there is a chordless path, and by appending y to that chordless path we see that $G \setminus T$ contains a chordless path from y to y' that is not included in $C(T)$, which contradicts Lemma 2.4. This proves our claim. □

An easy corollary of Theorem 3.1 is another theorem of Hayward [10] stating that if G is weakly chordal on at least 3 vertices then G or $\overline{G}$ admits a star cutset. This implies the perfectness of weakly chordal graphs because Chvátal [6] proved that a minimally imperfect graph has no star cutset.

4 Even pairs

In this section, neither the results nor their proofs are new, but we include them because we think that it is interesting to see how they all rely on Roussel–Rubio-type lemmas.

Even pairs are a tool to prove perfectness of graphs and to give polynomial time coloring algorithms (see [7]). Roussel–Rubio-type lemmas provide a good tool to prove the existence of an even pair. Even pairs are used in [13] to give a polynomial time coloring algorithm for a class of graphs that generalizes Meyniel graphs, weakly chordal graphs, and perfectly orderable graphs, the class of so-called *Artemis graphs*. They are used in [4] to significantly shorten the proof of the Strong Perfect Graph Theorem. Here we give two very simple examples of this technique. The first one is due to Meyniel.

Theorem 4.1 (Meyniel [14]) *If G is a Meyniel graph, then either G is a complete graph or G has an even pair.*

PROOF — We proceed by induction on $|V(G)|$. If G is a disjoint union of complete graphs (in particular when $|V(G)| = 1$), then the theorem holds: if there is more than one component, then an even pair is obtained by taking two vertices in different components. Otherwise, G contains a chordless path P on 3 vertices. Let v be the internal vertex of P. By the induction hypothesis, $N(v)$ contains an even pair for $G[N(v)]$. By Lemma 2.2, this is an even pair for G. □

Since Meyniel [14] proved that a minimally imperfect graph does not contain an even pair, the theorem above implies that Meyniel graphs are perfect. The following is originally due to Hayward, Hoàng, and Maffray [11], but the proof given here is implicitly given in [13]. A *2-pair* of vertices is a pair a, b such that all chordless paths linking a to b have length 2.

Theorem 4.2 (Hayward, Hoàng, and Maffray [11]) *If G is a weakly chordal graph then either G is a clique or G has 2-pair.*

PROOF — We proceed by induction on $|V(G)|$. If G is a disjoint union of complete graphs (in particular when $|V(G)| = 1$), then the theorem holds trivially. We may therefore assume that G contains a chordless path P on 3 vertices. Hence there exists a set T as in Lemma 2.4 (start with the center of P to build T). Since $C(T)$ is not a clique, by the induction hypothesis, we know that $C(T)$ admits a 2-pair of $G[C(T)]$. By Lemma 2.4, it is a 2-pair of G. □

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