Universally Signable Graphs

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Abstract

In a graph, a chordless cycle of length greater than three is called a hole. Let γ be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the holes of a graph G. We characterize graphs for which, for all choices of the vector γ, we can pick a subset F of the edge set of G such that $|F \cap H| \equiv \gamma_H \pmod 2$, for all holes H of G and $|F \cap T| \equiv 1$ for all triangles T of G. We call these graphs *universally signable*. The subset F of edges is said to be labelled *odd*. All other edges are said to be labelled *even*. Clearly graphs with no holes (*triangulated graphs*) are universally signable with a labelling of odd on all edges, for all choices of the vector γ. We give a decomposition theorem which leads to a good characterization of graphs that are universally signable. This is a generalization of a theorem due to Hajnal and Suranyi [3] for triangulated graphs.

1 Introduction

In a graph a chordless cycle of length greater than three is called a *hole*. We consider the problem of assigning labels *odd* and *even* to the edges of a graph $G(V,E)$ so that, given any $\{0,1\}$ vector γ with entries in one-to-one correspondence with the holes of G, the parity of the number of odd edges in a hole H of G is γ_H and the parity of the number of odd edges in every triangle is odd. An assignment of odd and even labels to the edges of G is called a *signing* of G. Graphs for which there exists a signing in accordance with γ, for all choices of the vector γ, are called *universally signable*. Graphs that contain no holes (*triangulated graphs*), are seen to be universally signable by signing all edges odd.

A subset of the edge set E is *odd* (resp. *even*) if it contains an odd (resp. even) number of odd edges. A graph is *even signable* if there exists a labelling of its edges such that every triangle is odd and every hole even. A graph is *odd signable* if there exists a labelling of its edges such that every triangle is odd and every hole is also odd. Even signable (resp. odd signable) graphs generalize graphs with no odd holes (resp. even holes). Our interest in universally signable graphs is motivated by the easy observation that they are both odd and even signable.

We obtain a co-NP characterization of universally signable graphs as a corollary to a theorem of Truemper [5]. Using this result, we obtain a decomposition theorem for universally signable graphs. The decomposition theorem is a generalization of a theorem due to Hajnal and Suranyi [3] for triangulated graphs. The decomposition result leads to a polynomial time recognition algorithm for universally signable graphs.

2 Universally Signable Graphs

The following theorem of Truemper [5] is used to obtain a co-NP characterization of universally signable graphs.

Theorem 2.1 *Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C(\text{mod}2)$ for all chordless cycles C of G, if and only if for every induced subgraph G' of G of type H_0, H_1, H_2 or H_3, there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C(\text{mod}2)$, for all chordless cycles C of G'.*

The graphs H_0, H_1, H_2 and H_3 are shown in Figure 1. A dotted line indicates a chordless path containing one or more edges.

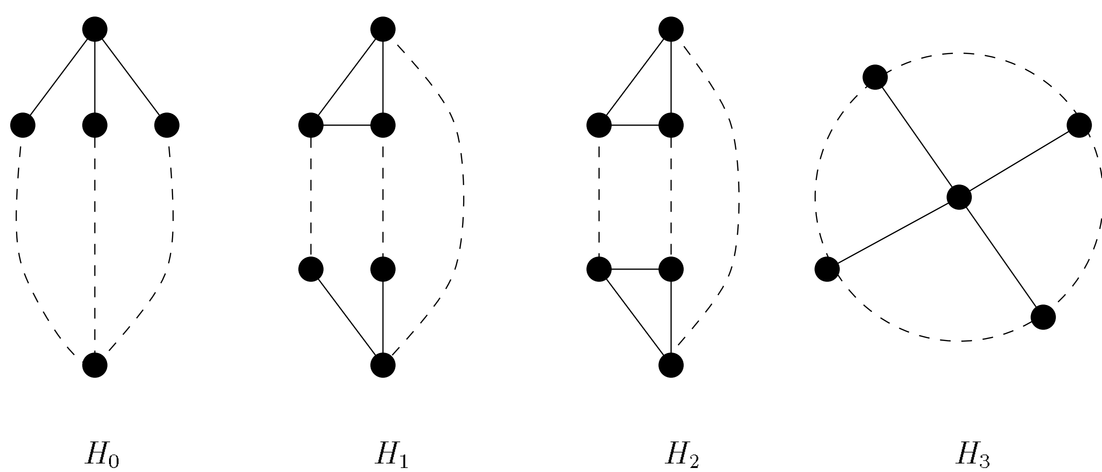

Figure 1: 3-path configurations and wheel

Subgraphs of type H_0, H_1 or H_2 are referred to as *3-path configurations* ($3PC$'s). A subgraph of type H_0 is called a $3PC(x, y)$ where node x and node y are connected by three paths P_1, P_2 and P_3. A subgraph of type H_1 is called a $3PC(xyz, u)$, where xyz is a triangle and P_1, P_2 and P_3 are three paths with endnodes x, y and z respectively and a common endnode u. A subgraph of type H_2 is called a $3PC(xyz, uvw)$, consists of two node disjoint triangles xyz and uvw and paths P_1, P_2 and P_3 with endnodes x and u, y and v and z and w respectively. Furthermore in all three cases the nodes of $P_i \cup P_j$ $i \neq j$ must induce a hole. This implies that all paths P_1, P_2, P_3 of H_0 have length greater than one, and at most one path of H_1 has length one.

Subgraphs of type H_3 are *wheels*. These consist of a chordless cycle called the *rim* together with a node called the *center* that has at least three neighbors on the rim. The edges of the wheel that do not belong to the rim are called *spokes*.

Since all cuts and cycles of G have even intersections, switching the labels on all edges of a cut does not change the parity of a cycle. In particular, given a labelled graph, switching the label on all edges incident to a node will not change the parity of any cycle. The operation of switching the labels on all the edges incident to a node is called *scaling*. Since every edge of a spanning forest T belongs to a cut of G which does not contain any other edge of T, it follows that, if G can be signed in accordance with β, then there exists such a signing with the edges of T labelled arbitrarily.

This implies that if a graph G can be signed in accordance with the vector β, one can produce such a signing as follows. Order the edges of G $e_1, \ldots, e_n$, so that the edges of T are the first in the sequence and all other edges e_j have the property that e_j closes a chordless cycle H_j of G together with edges having smaller indices. Sign the edges of T arbitrarily and label the remaining edges e_j so that H_j is signed in accordance with β.

The following theorem gives a co-NP characterization of universally signable graphs.

Theorem 2.2 *A graph is universally signable if and only if it contains no 3PC and no wheel whose rim is a hole.*

Proof: By Theorem 2.1 we only need to characterize the subgraphs of type H_0, H_1, H_2 and H_3 that are universally signable. Subgraphs of type H_0 are not odd signable. To see this, let $3PC(x, y)$ be a subgraph of type H_0. Pick the spanning tree that contains all the edges of the $3PC$ except two of the edges incident at node x. Label all edges of the spanning tree even. Now both the remaining edges close holes with the edges of the spanning tree and must be labelled odd. But now the hole of the $3PC(x, y)$ that contains both these edges is even. Similarly, it is easy to show that subgraphs of type H_2 are not odd signable and subgraphs of type H_1 are not even signable. Consequently, universally signable graphs contains no 3PC.

If the rim of a wheel (H, x) is a hole, then by choosing γ_H to be opposite in parity to the number of spokes of the wheel, and all other components of γ to be one, we show that the wheel is not universally signable. If the rim is not a hole, the wheel is a triangulated graph and so it is universally signable. $\Box$

3 Decomposition

Unless otherwise stated, we assume in the remainder that G is a universally signable graph. For triangulated graphs, Hajnal and Suranyi showed the following:

Theorem 3.1 *[3] A minimal node cutset C in a triangulated graph G is a clique.*

The following is the main result of this paper:

Theorem 3.2 *A minimal node cutset C in a universally signable graph G is either a clique or a stable set of size two. Furthermore in the latter case $G(V \setminus C)$ has exactly two connected components.*

Let $F \subset V$ and $u \in V \setminus F$. We say that node u is *adjacent* to F if it has a neighbor in F. Node u is *strongly adjacent* to F if it has more than one neighbor in F. We also loosely say that a node u is adjacent (strongly adjacent) to a subgraph H of G to mean that node u is adjacent (strongly adjacent) to the node set $V(H)$. In other instances as well, where our intent is clear from the context, we use H to mean $V(H)$.

Lemma 3.3 *A node x strongly adjacent to a hole H in G has exactly two neighbors in H which are furthermore adjacent.*

Proof: If node x has more than two neighbors in H, graph G contains a wheel whose rim is a hole. If node x is adjacent to exactly two nodes u and v in H, which are not adjacent, then G contains a $3PC(u, v)$. □

Proof of Theorem 3.2: Let C be a minimal node cutset. We first show that C is either a clique or a stable set of size two. Let two of the connected components of $G(V \setminus C)$ be the subgraphs G_1 and G_2 with node sets V_1 and V_2 respectively. By the minimality of C, every node $u \in C$ is adjacent to at least one node in each of the connected components of $G(V \setminus C)$. Suppose that C is not a clique. Let x, y be two nonadjacent nodes in C. Let P_i, $i = 1, 2$, be paths in G_i with one endnode adjacent to x, the other to y and with the property that the node set $V(P_i) \cup \{x, y\}$ induces a chordless path. Now the node set $V(P_1) \cup V(P_2) \cup \{x, y\}$ induces a hole H of G. Suppose that $|C| > 2$ then we consider three possible cases

1. there exists a node $z \in C$ adjacent to both x and y,
2. there exists a node $z \in C$ not adjacent to x or y,
3. there exists a node $z \in C$ adjacent to exactly one of x or y.

Case 1: There exists a node $z \in C$ adjacent to both nodes x and y.

Node z is strongly adjacent to H and contradicts Lemma 3.3.

Case 2: There exists a node $z \in C$ adjacent to neither x nor y.

Let Q_i for $i \in \{1, 2\}$ be a chordless path in $G(V_i \cup \{z\})$, with one endnode z and the other adjacent to a node in P_i. Furthermore no intermediate node of Q_i is adjacent to P_i. Note that by the minimality of C such paths must exist.

Claim 1: *We can pick paths P_i and Q_i, $i = 1, 2$, such that the path Q_i contains no neighbor of neither node x nor y except possibly the endnode adjacent to a node in P_i.*

Proof of Claim 1: Assume such a choice of P_i and Q_i is not possible. Assume w.l.o.g. that $i = 1$. Let z_1 be the node closest to z on Q_1 adjacent to either node x or y, say x. Let z_0 and z_2 be the neighbors of z_1 in Q_1, with z_0 closer to z than z_1. In the graph induced by the nodes $V(Q_1) \cup V(P_1) \cup \{y\}$ let P_1' be the shortest path from z_1 to y. Let P be the shortest path from node z to node y in the graph induced by $V(P_2) \cup V(Q_2) \cup \{y\}$. Since the nodes of P, P_1' and the $z, \ldots, z_1$ subpath of Q_1, together form a hole, by

Lemma 3.3 node x either has exactly one neighbor in this hole, namely node z_1, or possibly nodes z_1 and z_2. In the first case, the choice of P_1' and the $z, \ldots, z_0$ instead of P_1 and Q_1 contradicts the assumption. In the other case, the choice of $z_2, \ldots, y$ subpath of P_1' and the $z, \ldots, z_1$ subpath of Q_1 contradicts the assumption. By symmetry we can choose P_2 and Q_2 appropriately as well. This completes the proof of the claim.

Let P_i and Q_i, $i = 1, 2$ be chosen in accordance with Claim 1. Let the endnode of Q_1 adjacent to a node in P_1 be node u and the endnode of Q_2 adjacent to a node in P_2 be node v. Note that by Lemma 3.3 node z is not adjacent to both P_1 and P_2 and so is different from either node u or node v. By Lemma 3.3 node u (resp. v) is either adjacent to exactly one node of H or to two adjacent nodes of H. If u and v have a common neighbor in H, say x, then the node set $V(H) \cup V(Q_1) \cup V(Q_2)$ induces a wheel with center x and whose rim is a hole. Otherwise the node set $V(H) \cup V(Q_1) \cup V(Q_2)$ induces a $3PC$.

Case 3: There exists a node $z \in C$ adjacent to exactly one of the nodes x or y.

Assume w.l.o.g. that z is adjacent to x.

Let Q_i for $i \in \{1, 2\}$ be a chordless path in $G(V_i \cup \{z\})$, with one endnode z and the other adjacent to a node in P_i. Furthermore no intermediate node of Q_i is adjacent to P_i. Note that by the minimality of C such paths must exist.

Claim 2: *We can pick paths P_i and Q_i, $i = 1, 2$ such that the path Q_i contains no neighbor of node y except possibly the endnode of Q_i distinct from z and no neighbors of node x except node z.*

Proof of Claim 2: Assume such a choice of P_i and Q_i is not possible. Assume w.l.o.g. that $i = 1$. Let z_1 be the node of Q_1, distinct from and closest to node z, adjacent to either node x or y. If z_1 is adjacent to both nodes x and y then node z_1 and hole H contradict Lemma 3.3.

Assume first that node z_1 is adjacent to node y. By our assumption on P_1 and Q_1 node z_1 is not the endnode of Q_1 distinct from node z. If node z is not adjacent to a node of P_2 graph G contains a $3PC(x, y)$ with paths $x, z, \ldots, z_1, y$; x, P_2, y and x, P_1, y. If node z is adjacent to P_2 then by Lemma 3.3 applied to node z and hole H, z is adjacent to the neighbor of x in P_2. Let this node be z_2. But now G contains a $3PC(zxz_2, y)$.

So, we assume that node z_1 is adjacent to node x. Let P_1' be the shortest path from z_1 to y in $G(V(P_1) \cup V(Q_1) \cup \{y\})$. Let P_2' be the shortest path from z to y in $G(V(P_2) \cup V(Q_2) \cup \{y\})$. The nodes in P_1', P_2' together with the nodes of the $z, \dots, z_1$ subpath of Q_1 induce a hole H'. By Lemma 3.3 applied to H' and node x, node z_1 is adjacent to z and node x is adjacent to no node of H' other than z and z_1. Let the endnode of Q_1 adjacent to a node in P_1 be node u and the endnode of Q_2 adjacent to a node in P_2 be node v. Note that node z_1 is distinct from u since otherwise by Lemma 3.3 node u is adjacent to the neighbor of x in P_1 and consequently node x has three neighbors in H'. We first consider the case when no node of $Q_1 \setminus \{u\}$ is adjacent to node y. Now $Q_1 \setminus \{z\}$ is a subpath of P_1' and so has no neighbor of x other than z_1. If node u is adjacent to exactly one node u_1, of P_1, then G contains a $3PC(z_1zx, u_1)$. Otherwise u is adjacent to adjacent nodes u_1 and u_2 of $P_1 \cup \{y\}$. Note that since x is not adjacent to node u, it is distinct from both u_1 and u_2. Now G contains a $3PC(z_1zx, uu_1u_2)$. We now consider the case where a node of $Q_1 \setminus \{u\}$ is adjacent to node y. Note that in this case P_1' is a subpath of Q_1 together with node y. Let z_2 be the node of $Q_1 \setminus \{u\}$ closest to z adjacent to y. If v has a unique neighbor v_1 in $P_2 \cup \{y\}$, such that v_1x is an edge, then there is a wheel with center x whose rim is a hole. Otherwise there is a $3PC(zz_1x, y)$ with paths $zP_2'y$; $z_1, \dots, z_2, y$ and xP_1y. This completes the proof of the claim.

Let P_i and Q_i, $i = 1, 2$ be chosen in accordance with Claim 2. Let the endnode of Q_1 adjacent to a node in P_1 be node u and the endnode of Q_2 adjacent to a node in P_2 be node v. By Lemma 3.3 applied to node z and hole H we know that node z is different from either node u or v. We first consider the case when it is distinct from both. Suppose node u has a unique neighbor u_1 in H. If v has a neighbor in P_2 not adjacent to node x, then G contains a $3PC(u_1, z)$. If the only neighbor of node v in P_2, say v_1, is adjacent to node x then the graph either contains a $3PC(v_1, z)$ or a wheel whose rim is a hole, with center x, having spokes xv_1, xz and xu_1. So node u has adjacent neighbors u_1 and u_2 in H. If node x does not coincide with u_1 or u_2, graph G contains a $3PC(uu_1u_2, x)$. Otherwise, assume w.l.o.g. that x coincides with u_2, G contains a wheel whose rim is a hole with center x and at least the three spokes xu_1, xu and xz. So we assume w.l.o.g. that node z coincides with node v. Then by Lemma 3.3 node z is adjacent to the neighbor of x in P_2 say node v_1. If node u is adjacent to a unique node

u_1 of $P_1 \cup \{y\}$ the graph G contains a $3PC(zxv_1, u_1)$, otherwise node u has adjacent neighbors u_1 and u_2 in H. If x does not coincide with neither u_1 nor u_2, then G contains $3PC(uu_1u_2, zxv_1)$. Otherwise G contains a wheel, whose rim is a hole, with center x and spokes xv_1, xz, xu and xu_1. This shows that Case 3 cannot occur.

Therefore every minimal cutset C of G is either a clique or a stable set of size two. If $C = \{x, y\}$ is a stable set of size two and $G(V \setminus C)$ has more than two connected components then the graph G contains a $3PC(x, y)$ picking one path each in three of the components. $\Box$

4 Algorithm

Lemma 4.1 *Let H be a hole in G and let $G_1, \ldots, G_k$ be the connected components of $G \setminus V(H)$. Then for every $i \in \{1, \ldots k\}$, H contains an edge uv such that $N(G_i) \cap V(H) \subseteq \{u, v\}$.*

Proof: Let $i \in \{1, \ldots, k\}$. First assume that G_i contains a node z that is strongly adjacent to H. By Lemma 3.3 the neighbors of z in H are two adjacent nodes of H, say u and v. Suppose that $N(G_i) \cap V(H) \not\subseteq \{u, v\}$. Then G_i contains a chordless path P whose one endnode is z, the other, say y, is adjacent to a node in $H \setminus \{u, v\}$ and no intermediate node of P is adjacent to a node in $H \setminus \{u, v\}$. Let z' be the node of P which is closest to y and is adjacent to both u and v. Note that by Lemma 3.3, z' is distinct from y. Let P' be the $y, \ldots, z'$ subpath of P. Let C be a minimal node cutset separating z' from $H \setminus \{u, v\}$. C must contain the edge uv and so by Theorem 3.2 it must be a clique cutset. But by the choice of z', no node of $V(P') \setminus \{z'\}$ is adjacent to both u and v, contradicting our assumption that C is a cutset separating z from $H \setminus \{u, v\}$.

Now assume that no node of G_i is strongly adjacent to H. Suppose that $N(G_i) \cap V(H)$ contains two nonadjacent nodes u and w. Let P be a chordless path in G_i whose one endnode is adjacent to u and the other to w. Assume w.l.o.g. that no proper subpath of P has endnodes that are adjacent to two nonadjacent nodes of H. In particular, no intermediate node of P is adjacent to u or w. If no node of $V(H) \setminus \{u, w\}$ is adjacent to a node of P then the node set $V(P) \cup V(H)$ induces a $3PC(u, w)$. So assume that node $v \in V(H) \setminus \{u, w\}$ is adjacent to a node of P. Then by our assumption on P, both uv and vw are edges and no node of $V(H) \setminus \{u, v, w\}$ is adjacent to a node of P. But now the node set $V(P) \cup V(H)$ induces a wheel with center v, whose rim is a hole. □

Corollary 4.2 *A connected universally signable graph G with no clique cutset is either a clique or a hole.*

Proof: Suppose that G does not contain a clique cutset and that it is not a clique. Then, by Theorem 3.1, G is not triangulated. Let H be a hole of G. If G is not H itself, then by Lemma 4.1 G contains a clique cutset, contradicting our assumption. □

Definition 4.3 *A node of G is* simplicial *if its neighborhood set forms a clique.*

Definition 4.4 *A hole H of G is* simplicial *if $G \neq H$ and H contains an edge uv such that $N(G \setminus H) \cap V(H) \subseteq \{u, v\}$.*

Dirac [2] showed the following result for triangulated graphs.

Theorem 4.5 *[2] Every triangulated graph that is not a clique contains at least two nonadjacent simplicial nodes.*

We prove the following generalization:

Theorem 4.6 *Every connected universally signable graph G that is not a clique nor a hole, contains either a simplicial hole or two nonadjacent simplicial nodes.*

Proof: Let G be a connected universally signable graph that is not a clique nor a hole. Assume that G does not contain a simplicial hole. If G is triangulated then by Theorem 4.5 we are done. So let H be a hole of G. Let $G'_1, \ldots, G'_k$ be the connected components of $G \setminus H$. By Lemma 4.1, for $i \in \{1, \ldots, k\}$, let $u_i v_i$ be an edge of H so that $N(G'_i) \cap V(H) \subseteq \{u_i, v_i\}$, and let G_i be the graph induced by the node set $V(G'_i) \cup \{u_i, v_i\}$. We now show that every G_i contains a simplicial vertex that is also a simplicial vertex of G, i.e. it is distinct from both u_i and v_i. We consider the following two cases.

Case 1: G_i is triangulated.

If G_i is a clique then every vertex of G_i is simplicial in G_i, and every vertex in $V(G_i) \setminus \{u_i, v_i\}$ is simplicial also in G. If G_i is not a clique then by Theorem 4.5, it contains two nonadjacent simplicial vertices. One of these two vertices must be distinct from both u_i and v_i, and hence is also a simplicial vertex of G.

Case 2: G_i is not triangulated.

Let H_i be a hole of G_i such that the connected component of $G \setminus H_i$, which contains the nodes of $H \setminus \{u_i, v_i\}$, is maximal. Let C_1 be the connected component of $G \setminus H_i$ which contains the nodes of $H \setminus \{u_i, v_i\}$. Since H_i is not a simplicial hole of G, there exists a component C_2 of $G \setminus H_i$ such that $N(C_2) \cap V(H_i) \nsubseteq N(C_1) \cap V(H_i)$ and $N(C_1) \cap V(H_i) \nsubseteq N(C_2) \cap V(H_i)$. Let G_{C_1} be the graph induced by the node set $V(C_1) \cup (N(C_1) \cap V(H_i))$ and let G_{C_2} be the graph induced by the node set $V(C_2) \cup (N(C_2) \cap V(H_i))$. If G_{C_2} is triangulated, then by Case 1 we are done. So suppose that it is not and let H' be a hole in G_{C_2}. Let C_1' be a connected component of $G \setminus H'$ which contains $H \setminus \{u_i, v_i\}$. Let $G_{C_1'}$ be the graph induced by the node set $V(C_1') \cup (N(C_1') \cap V(H'))$. Then $G_{C_1'}$ contains both H_i and G_{C_1}, contradicting our choice of H_i.

Hence every G_i contains a simplicial vertex that is also a simplicial vertex of G. Now since H is not a simplicial hole of G, $k \geq 2$. So both G_1 and G_2 contain a simplicial vertex of G, and by definition of G_1 and G_2, these two vertices are not adjacent. $\square$

Lemma 4.7 *If a node u of G is simplicial then G is universally signable if and only if $G \setminus \{u\}$ is universally signable.*

Proof: Since $G \setminus \{u\}$ is an induced subgraph of G, if G is universally signable then so is $G \setminus \{u\}$. Now suppose that $G \setminus \{u\}$ is universally signable. Then by Theorem 2.2, $G \setminus \{u\}$ does not contain a $3PC$ nor a wheel whose rim is a hole. Since the neighborhood set of u in G induces a clique, u is not contained in any hole of G. Since every node of a $3PC$ is contained in at least one hole, u cannot be contained in any $3PC$. Since u can have at most two neighbors on a hole, u cannot be contained in any wheel whose rim is a hole. Hence graph G does not contain a $3PC$ nor a wheel whose rim is a hole, and so by Theorem 2.2 it is universally signable. $\square$

Lemma 4.8 *Let H be a simplicial hole of G and uv an edge of H such that $N(G \setminus H) \cap V(H) \subseteq \{u, v\}$. Then G is universally signable if and only if $G \setminus (V(H) \setminus \{u, v\})$ is universally signable.*

Proof: Let $G' = G \setminus (V(H) \setminus \{u, v\})$. Since G' is an induced subgraph of G, if G is universally signable then so is G'. Now assume that G' is universally signable. By Theorem 2.2 it is sufficient to show that G does not contain a $3PC$ nor a wheel whose rim is a hole, which contains a node of $V(H) \setminus \{u, v\}$. Since every node of a $3PC$ is contained in at least two holes, no node of $V(H) \setminus \{u, v\}$ can be contained in a $3PC$. If a node of $V(H) \setminus \{u, v\}$ is contained in a wheel then H is the rim of that wheel. But then there is a node of G with at least three neighbors in H, contradicting the assumption that H is simplicial. Hence no node of $V(H) \setminus \{u, v\}$ can be contained in a wheel either. $\square$

Recognition Algorithm

Input: A connected graph G.

Output: YES if G is universally signable, NO otherwise.

Step 1: If G is a clique or a hole return YES.

Step 2: If G contains a simplicial node, remove it and go to Step 1.

Step 3: If G contains a simplicial hole H, remove from G the node set $V(H) \setminus \{u, v\}$, where uv is an edge of H such that $N(G \setminus H) \cap V(H) \subseteq \{u, v\}$, and go to Step 1.

Step 4: Return NO.

The validity of the algorithm follows from Theorem 4.6, Lemma 4.7 and Lemma 4.8. Steps 1 and 2 can obviously be performed in polynomial time. Step 3 can be performed in polynomial time as follows: find two adjacent nodes u and v whose removal disconnects the graph in such a way that one of the connected components together with u and v induces a hole. In each iteration of the algorithm at least one node is removed, so there are at most $|V|$ iterations. Therefore the algorithm can be implemented to run in polynomial time.

5 Optimization

A graph G is *h-perfect* if its stable set polytope, $STAB(G)$ (the convex hull of the incidence vectors of all stable sets of G), is described by the clique inequalities, the odd hole inequalities and non-negativity of the variables. A graph is *perfect* if the clique inequalities and the non-negativity of the variables suffice. The following theorem of V. Chvátal appears in [1].

Theorem 5.1 *Let G be a graph that contains a clique cutset C. Let the graph induced by $V(G) \setminus V(C)$ induce components $G'_1, \ldots, G'_m$ and let G_i be the graph induced by $V(G'_i) \cup V(C)$. Let $STAB(G_i)$ be given by the linear system $A^i x^i \leq b^i, x^i \geq 0$, where x^i is a vector indexed by the nodes of $V(G_i)$. Then $STAB(G)$ is given by the linear system $A^i x^i \leq b^i, x^i \geq 0, i \in \{1, \ldots, m\}, x^i_v = x^j_v, \forall v \in C, i, j \in \{1, \ldots, m\}$.*

Holes and cliques are easily verified to be h-perfect. Now by Theorem 4.6 and Theorem 5.1 we have the following theorem.

Theorem 5.2 *Universally signable graphs are h-perfect.*

Corollary 5.3 *A universally signable graph is perfect if and only it contains no odd hole.*

Markossian, Gasparian and Reed [4] have introduced the notion of β-perfection. For a graph G let $\beta(G) = \max\{\delta(F) + 1 | F$ *an induced subgraph of* $G\}$, where $\delta(F)$ is the minimum degree in F. A graph is β-perfect if for all its induced subgraphs H, the chromatic number $\chi(H) = \beta(H)$. An even hole is not β-perfect since $\chi(G) = 2$ while $\beta(G) = 3$. In [4] the following lemma is proved:

Lemma 5.4 *A graph with no even holes is β-perfect if every induced subgraph of G contains a simplicial node or a vertex of degree two.*

By Theorem 4.6 and the above lemma we have the following theorem.

Theorem 5.5 *A universally signable graph is β-perfect if and only it contains no even hole.*

Finally we remark that it is an easy exercise to modify the procedure to find a maximum weight stable set or clique in a triangulated graph to obtain an algorithm to solve the corresponding problems in universally signable graphs.

References

[1] On certain polytopes associated with graphs, *Journal of Combinatorial Theory B 13* (1975), 138-154.

[2] G. A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961), 71-76.

[3] A. Hajnal, T. Suryani, Uber die Auflosung von Graphen vollstandiger Teilgraphen, Ann. Univ. Sc. Budapest, *Eotvos Sect. Math., 1 (1958)*.

[4] S.E. Markossian, G.S. Gasparian and B.A. Reed, β-perfect graphs, to appear in *Journal of Combinatorial Theory B*.

[5] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids, *Journal of Combinatorial Theory B 32* (1982) 112-139.