ELSEVIER Discrete Mathematics 199 (1999) 27–33

DISCRETE MATHEMATICS

Balanced cycles and holes in bipartite graphs [1]

Michele Conforti [a], Gérard Cornuéjols [b,*], Kristina Vušković [c]

[a] *Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy*

[b] *Carnegie Mellon University, Schenley Park, Pittsburgh, PA 15213, USA*

[c] *University of Kentucky, Department of Mathematics, Lexington, KY 40506, USA*

Received 31 March 1997; revised 1 July 1998; accepted 22 June 1998

Abstract

Bruce Reed asks the following question:

Can we determine whether a bipartite graph contains a chordless cycle whose length is a multiple of 4? We show that the two following more general questions are equivalent and we provide an answer. Given a bipartite graph G where each edge is assigned a weight $+1$ or -1,

- determine whether G contains a cycle whose weight is a multiple of 4,
- determine whether G contains a chordless cycle whose weight is a multiple of 4.

Given a bipartite graph, we can also decide whether it is possible to assign weights $+1$ or -1 to its edges so that the above two properties hold.

Keywords: Balanced; Hole; Signed graph; Series-parallel graph; 3-path configuration

1. Introduction

A *signing* of a graph is an assignment of a weight $+1$ or -1 to each of its edges. A cycle C of a signed bipartite graph G is *balanced* if the sum of the weights of the edges in C is congruent to $0 \bmod 4$. Otherwise C is *unbalanced*. (In this case, the weight is obviously congruent to $2 \bmod 4$).

Given a signed bipartite graph G, we provide an answer to the following questions:

(1) determine whether G contains a balanced cycle,

(2) determine whether G contains a balanced chordless cycle (hole).

When all the edges of G have weight $+1$, the answer to Question (2) provides also an answer to the following question of Bruce Reed:

* Corresponding author. E-mail: gc0v@andrew.cmu.edu.

[1] This work was supported in part by NSF grants DMI-9802773, DMS-9509581 and ONR grant N00014-9710196.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00283-0

Determine whether G contains a chordless cycle whose length is congruent to $0 \bmod 4$.

Given a bipartite graph, we can also decide whether it is possible to sign it so that the above two properties hold.

Two related questions can be asked for a signed bipartite graph G:

(3) determine if G contains an unbalanced cycle,

(4) determine if G contains an unbalanced hole.

Both of these questions have been answered. An answer to Question (3) yields a characterization of a subclass of totally unimodular matrices [9]. Question (4) is more complicated. It amounts to characterizing balanced matrices. This is done in [2]. An outline of the proof can be found in [1].

2. Balanced cycles and balanced holes

The following observation shows that Questions (1) and (2) are equivalent.

Remark 2.1. A signed bipartite graph G has a balanced cycle if and only if G has a balanced hole.

For, let C be a balanced cycle of G with smallest number of nodes. If C is not a hole, pick a chord uv of C and let P_1 and P_2 be the two paths connecting u and v in C and C_1, C_2 be the two cycles closed by edge uv with P_1 and P_2. Both C_1 and C_2 have fewer nodes than C and an easy counting argument shows that one of them is balanced, contradicting our choice of C. So C is a hole.

So the following properties are equivalent for a signed bipartite graph G and will be denoted by Properties 1 and 2.

Property 1. All cycles of G are unbalanced.

Property 2. All holes of G are unbalanced.

A signed bipartite graph G satisfying Properties 1 and 2 is said to be *totally unbalanced.*

3. Signing graphs and forbidden subgraphs

In this section we consider the following problem:

When can we sign the edges of a bipartite graph G so that it becomes totally unbalanced?

When this is possible, we say that G is *totally unbalanceable* and we call such a signing a *total unbalancing* of G.

Since minimal edge cuts and cycles in a graph G have even intersections, it follows that if a signed bipartite graph G is totally unbalanced, then the signed bipartite graph G', obtained from G by switching signs on the edges of a cut, is also totally unbalanced. For every edge uv of a spanning tree T of G, there is an edge cut containing uv and no other edge of T. So, if G is a totally unbalanceable bipartite graph, then there exists a total unbalancing of G with edges of T signed arbitrarily, and hence the following signing algorithm will totally unbalance G.

Pick a spanning tree T of G and sign its edges arbitrarily. Sign every edge e not in T so that the unique cycle of $T \cup \{e\}$ is unbalanced.

Let u, v be two nodes of a bipartite graph G. A *3-path configuration* connecting u and v, denoted by $3PC(u,v)$, is defined by three distinct internally node disjoint paths P_1, P_2, P_3 (i.e. no common internal nodes) connecting u and v. If paths P_1, P_2, P_3 are chordless (hence u and v are nonadjacent) and no edge connects internal nodes in distinct paths, the three path configuration is *induced.*

A $3PC(u,v)$ is *homogeneous* if u and v belong to the same side of the bipartition.

If G contains a homogeneous $3PC(u,v)$ then G cannot be signed to satisfy Properties 1 and 2, for since P_1, P_2, P_3 each has an even number of edges, any signing will make the weights of these paths conguent to $0 \bmod 4$ or $2 \bmod 4$ and two paths whose weights are congruent $\bmod 4$ induce a balanced cycle.

A *wheel* (C,x) is defined by a hole C and a node x having at least three neighbors in C.

Note that a wheel (C,x) is a homogeneous $3PC(x_i,x_j)$, where x_i and x_j are two neighbors of x in C. So, if G contains a wheel, then G cannot be signed to satisfy Properties 1 and 2.

In fact, a theorem of Truemper [6] (see also [7]) on alpha-balanced graphs implies the following result.

Theorem 3.1. *A bipartite graph G is totally unbalanceable if and only if G does not contain an induced homogeneous 3-path configuration or a wheel as an induced subgraph.*

An equivalent form of this theorem follows from the next lemma.

Lemma 3.2. *A bipartite graph G contains a homogeneous $3PC(u,v)$ if and only if G contains a wheel or an induced homogeneous $3PC(u,v)$.*

Proof. The ‘if’ direction was already observed above. We now prove the ‘only if’ direction. Let G^* be a homogeneous 3-path configuration with smallest number of nodes. Let $G^* = 3PC(u,v)$, where paths P_1, P_2 and P_3 connect u and v. Since G^* is smallest, paths P_1, P_2 P_3 are chordless. Assume there exists an edge xy where x is an internal node in, say, P_1 and y is an internal node in, say, P_2. Since one of the nodes x,y belongs to the same side of the bipartition as u and v, we can assume w.l.o.g. that nodes u, v and x belong to the same side of the bipartition. Then there is a homogeneous $3PC(x,v)$ which is smaller than G^* unless y is adjacent to u. The same argument shows that y is also adjacent to v. Now if, other than the edges of P_1 and P_3, there is no edge G^* having both endnodes distinct from y, then G^* is a wheel. Else there is a homogeneous 3-path configuration, connecting x and u or x and v which is smaller than G^*. □

Theorem 3.3. *A bipartite graph G is totally unbalanceable if and only if G does not contain a homogeneous 3-path configuration.*

4. Recognition algorithm

A graph is *series-parallel* if it can be obtained from a single edge by a sequence of the following operations: (i) inserting a node of degree 2 on an edge, or (ii) duplicating an edge. Biconnected totally unbalanceable bipartite graphs are series-parallel. This follows from the following characterization of series-parallel graphs.

Theorem 4.1 (Dirac [3]). *A biconnected graph is a series-parallel graph if and only if it has no K_4 minor.*

Theorem 4.2. *A biconnected totally unbalanceable bipartite graph is series-parallel.*

Proof. Let G be a biconnected totally unbalanceable bipartite graph. By Theorem 4.1 it is sufficient to show that G has no K_4 minor. Suppose it does. Then G contains three internally node disjoint paths P_1, P_2, P_3 from, say, node u to node v and a fourth path P_4 internally node disjoint from P_1, P_2, P_3, whose endnodes x and y are w.l.o.g. contained in P_1 and P_3, respectively. If u and v are on the same side of the bipartition, then the node set $V(P_1) \cup V(P_2) \cup V(P_3)$ induces a homogeneous 3-path configuration, contradicting Theorem 3.3. Otherwise, w.l.o.g. x and u belong to the same side of the bipartition, and hence there is a homogeneous $3PC(x,u)$, again contradicting Theorem 3.3. □

Note that a biconnected bipartite graph that is series-parallel is not necessarily totally unbalanceable. For example, an induced homogeneous 3-path configuration is series-parallel, but not totally unbalanceable.

Series-parallel graphs can be recognized in linear time, see [4]. In [8] Wagner observes that the linear time algorithm of Hopcroft and Tarjan [5] for computing the decomposition of a graph into triconnected components can be used to determine whether a graph is series-parallel. We now adapt the algorithm in [5] to an algorithm that determines whether a signed bipartite graph is totally unbalanced. This algorithm together with the signing algorithm yields a recognition of totally unbalanceable bipartite graphs.

Lemma 4.3. *Let G be a biconnected totally unbalanceable bipartite graph. Then G is either a hole or it contains a two node cutset $\{x, y\}$, such that x and y are on opposite sides of the bipartition and have degrees greater than two.*

Proof. Let G be a biconnected totally unbalanceable bipartite graph. Then G contains a hole H. Suppose that $G \neq H$. Then $G \backslash V(H)$ is nonempty. Let C be a connected component of $G \backslash V(H)$. Since G is biconnected, C has at least two neighbors in H. We now show that C has exactly two neighbors in H, and these two neighbors are furthermore on the opposite sides of the bipartition. Suppose not. Then C has distinct neighbors x and y in H that are on the same side of the bipartition. Let P be an xy-path whose intermediate nodes are in C. Then $V(P) \cup V(H)$ induces an homogeneous $3PC(x, y)$, contradicting the assumption that G is totally unbalanceable. So C has exactly two neighbors x and y in H, and they are on opposite sides of the bipartition. Hence $\{x, y\}$ is the desired cutset since in $G \backslash \{x, y\}$ the nodes of C and H are in distinct connected components. □

Given a one node cutset $\{x\}$ of a signed bipartite graph G, let $C_1, \ldots, C_n$ be the connected components of $G \backslash \{x\}$. The *blocks* of the decomposition of G by $\{x\}$ are graphs $G_1, \ldots, G_n$, where each G_i is a subgraph of G induced by the node set $V(C_i) \cup \{x\}$.

Lemma 4.4. *Let $G_1, \ldots, G_n$ be the blocks of a one node cutset decomposition of a signed bipartite graph G. Then G is totally unbalanced if and only if $G_1, \ldots, G_n$ are all totally unbalanced.*

Proof. The result follows trivially from the fact that every cycle of G is contained in one of the blocks $G_1, \ldots, G_n$. □

Given a two node cutset $\{x, y\}$ of a biconnected signed bipartite graph G, let $C_1, \ldots, C_n$ be the connected components of $G \backslash \{x, y\}$. The two *blocks* of the decomposition of G are obtained by partitioning $C_1, \ldots, C_n$ in two sets S_1 and S_2, adding to S_1 nodes x and y and an edge, connecting x and y, whose weight is congruent $\bmod 4$ to the weight of a chordless path connecting x and y in the graph induced by $V(S_2) \cup \{x, y\}$. The block corresponding to S_2 is defined analogously.

Remark 4.5. If G_1 and G_2 are blocks of a two node cutset decomposition of a biconnected bipartite graph G, then neither G_1 nor G_2 is isomorphic to G. Furthermore both G_1 and G_2 are biconnected.

Lemma 4.6. *Let G be a biconnected signed bipartite graph. Let $\{x, y\}$ be a two node cutset, where x and y are on opposite sides of the bipartition, and G_1 and G_2 be the two blocks of the decomposition of G. Then G is totally unbalanced if and only if both G_1 and G_2 are totally unbalanced.*

Proof. If G is totally unbalanced then clearly both of the blocks are totally unbalanced. Now suppose that G_1 and G_2 are totally unbalanced, but G is not. Let H be a balanced hole of G. If the nodes of H are contained in one of the blocks, say G_1, then H is a balanced cycle of G_1, contradicting the assumption that G_1 is totally unbalanced. Hence H passes through nodes x and y, and contains nodes in both $G_1 \backslash \{x, y\}$ and $G_2 \backslash \{x, y\}$. Let Q_1 and Q_2 be the two paths of H connecting x and y, such that Q_1 is contained in G_1. Since x and y are on the opposite sides of the bipartition and H is balanced, we may assume without loss of generality that the weight of Q_1 is congruent to $1 \bmod 4$ and the weight of Q_2 is congruent to $3 \bmod 4$. The nodes of Q_1 induce a hole in G_1, and since G_1 is totally unbalanced, the weight of the edge xy in G_1 is congruent to $1 \bmod 4$. Similarly the weight of the edge xy in G_2 is congruent to $3 \bmod 4$. By construction of the blocks, there exists a chordless path P_1, connecting x and y in the graph obtained from G_1 by removing the edge xy, of weight congruent to $3 \bmod 4$. But now the nodes of P_1 induce a balanced hole in G_1, contradicting the assumption that G_1 is totally unbalanced. □

Consider now the following algorithm:

INPUT: A signed bipartite graph.

OUTPUT: Checks if G is totally unbalanced.

Using cutnodes, decompose G into biconnected components. Recursively perform the following step:

Until a biconnected component is a hole, check if it contains a two node cutset $\{x, y\}$ *where x and y are in distinct sides of the bipartition of G. If no such nodes exist, stop: G is not totally unbalanced. Else decompose the component into two blocks.*

When all the biconnected components are holes check if they are totally unbalanced and stop: G is totally unbalanced if and only if every hole is totally unbalanced.

Theorem 4.7. *The above algorithm correctly identifies if G is totally unbalanced.*

Proof. By Lemma 4.4, G is totally unbalanced if and only if all its biconnected components are totally unbalanced. If a biconnected component is not a hole and does not contain a two node cutset, with the two nodes in the opposite sides of the bipartition, by Lemma 4.3 the algorithm correctly identifies G as not totally unbalanced. Now by Lemma 4.6 the validity of the algorithm is established. □

Remark 4.8. Due to the result of Hopcroft and Tarjan, see [5], the above algorithm can be implemented to run in linear time.

Remark 4.9. Together with the signing algorithm, the above algorithm detects if a bipartite graph is totally unbalanceable.

References

[1] M. Conforti, G. Cornuéjols, A. Kapoor, M.R. Rao, K. Vušković, Balanced matrices: a survey, Proc. 15th Int. Symp. on Mathematical Programming, Ann Arbor, Michigan 1994, pp. 1–33.
[2] M. Conforti, G. Cornuéjols, A. Kapoor, K. Vušković, Balanced $0, \pm 1$ matrices, parts I and II, preprint, Carnegie Mellon University, revised 1998, J. Combin. Theory B, submitted.
[3] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[4] R.C. Gelmacher, P.C. Liu, An $\mathcal{O}(\max(m,n))$ algorithm for finding a subgraph homeomorphic to K_4, Proc. 11th Southeastern Conf. on Combinatorics, Graph Theory and Computing, 1980, pp. 597–609.
[5] J.E. Hopcroft, R.E. Tarjan, Dividing a graph into triconnected components, SIAM J. Comput. 2 (3) (1973) 135–158.
[6] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids, J. Combin. Theory B 32 (1982) 112–139.
[7] K. Truemper, Matroid Decomposition, Academic Press, Boston, 1992.
[8] D.K. Wagner, Forbidden subgraphs and graph decompositions, Networks 17 (1987) 105–110.
[9] M.Yannakakis, On a class of totally unimodular matrices, Math. Oper. Res. 10 (1985) 280–304.