Triangle-Free Graphs that are Signable Without Even Holes

Michele Conforti[1], Gérard Cornuéjols[2], Ajai Kapoor[2], and Kristina Vušković[3]

[1]DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA
UNIVERSITÀ DI PADOVA
VIA BELZONI 7, 35131 PADOVA, ITALY

[2]GSIA CARNEGIE MELLON UNIVERSITY
SCHENLEY PARK
PITTSBURGH. PA 15213

[3]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KENTUCKY
LEXINGTON, KY 40506

Received July 22, 1996; revised December, 1998

Abstract: We characterize triangle-free graphs for which there exists a subset of edges that intersects every chordless cycle in an odd number of edges (TF odd-signable graphs). These graphs arise as building blocks of a decomposition theorem (for cap-free odd-signable graphs) obtained by the same authors. We give a polytime algorithm to test membership in this class. This algorithm is itself based on a decomposition theorem. © 2000 John Wiley & Sons, Inc. J Graph Theory 34: 204–220, 2000

Keywords: *graph decompositions, β-perfect graphs*

Contract grant sponsors: NSF, ONR, Gruppo Nazionale Delle Richerche
Contract grant number: DMI 9802773, DMS 9509581, N00014-97-1-0196.

1. INTRODUCTION

A *hole* in a graph is a chordless cycle containing at least 4 edges. A convenient setting for the study of graphs with no holes of given parity is that of signed graphs: a graph G is said to be *signed* if each edge of G is given an *odd* or *even* label. Let $E(G)$ denote the edge set of G and $V(G)$ its node set. In a signed graph G, a subset of $E(G)$ is *odd* (resp. *even*) if it contains an odd (resp. even) number of odd-labeled edges. A graph is *odd-signable* if it can be signed so that the edge set of every chordless cycle is odd. A signed graph is *odd-signed* if the edge set of every chordless cycle is odd. We say that a graph G *contains* a graph H if H is an induced subgraph of G. Note that G contains no hole of even cardinality if and only if G is odd-signable with all edges odd. The importance of these graphs in the study of β-perfection is discussed in [7].

In this article we study triangle-free (TF, for short) odd-signable graphs. These graphs arise as building blocks of cap-free odd-signable graphs [3] but their structure was not studied before. Cap-free odd-signable graphs are, in turn, building blocks for graphs with no even holes [4].

We give a decomposition theorem for this class of graphs. We then exploit this theorem to obtain a polytime algorithm for testing whether a TF graph is odd-signable. We also give an algorithm for testing whether a signed TF graph is odd-signed. As a special case, this yields a polytime algorithm to detect whether a TF graph has a hole of even cardinality. It is interesting to note that Bienstock [1] has shown that it is NP-complete to test whether a TF graph has a hole of even cardinality that contains a specific node. In the last section, we give a construction that generates all TF odd-signable graphs.

A *wheel* (H, v) in a graph consists of a hole H together with a node v, called the *center*, that has at least three neighbors on H. When the center has an even number of neighbors on the hole, the wheel is called an *even wheel*. Let $v_1, \ldots, v_n$ be the neighbors of v in H, appearing in this order when traversing H. A *sector* S_i is a subpath of H with endnodes v_i and v_{i+1} not containing any other neighbor of v (throughout the paper, we take indices mod n when appropriate. For example, here $v_{n+1} = v_1$).

A *three-path configuration* $3PC(x, y)$ consists of two nodes x and y connected by three paths P_1, P_2, and P_3 such that the nodes of $V(P_i) \cup V(P_j), i \neq j$, induce a hole. Therefore, all paths of a $3PC(x, y)$ are chordless and have length greater than one.

The following fact is a consequence of a fundamental theorem of Truemper [8] (see Theorem 2.3 in [3]) and gives a co-NP characterization of TF odd-signable graphs.

Theorem 1.1. *A TF graph is odd-signable if and only if it contains neither an even wheel nor a three-path configuration.*

If G is a TF graph, the only possible clique cutsets are the cliques K_1 and K_2 of cardinality one and two, respectively. The *blocks* of a decomposition of G by a clique cutset K_l are the induced subgraphs of G obtained from each connected components of $G\backslash K_l$ by adding back the nodes of K_l.

Corollary 1.2. *If G has a clique cutset, then G is odd-signable if and only if all the blocks of the clique cutset decomposition are odd-signable.*

In a graph G, we denote by $N(v)$ the set of neighbors of node v. We say that a path P of G is an xy-path if it has endnodes x and y.

2. DECOMPOSITION

The main result of this section is Theorem 2.11, a decomposition result for TF odd-signable graphs. We also prove various properties of these graphs that will be used in subsequent sections.

Remark 2.1. *Let H be a hole in a TF odd-signable graph G. Let* $P = x_1, \ldots, x_n$, $n \geq 3$, *be a path such that* x_1 *and* x_n *belong to H and the only adjacencies between the nodes* $x_2, \ldots, x_{n-1}$ *and the nodes of H are the two edges* x_1x_2 *and* $x_{n-1}x_n$. *Then* x_1 *and* x_n *are adjacent.*

Proof. Suppose not. Then the two x_1x_n-subpaths of H, together with a chordless subpath of P induce a $3PC(x_1, x_n)$. ■

Let G' be an induced subgraph of G. A node v is *strongly adjacent* to G' if $v \in V(G)\backslash V(G')$ and v has at least two neighbors in G'.

Remark 2.2. *Let* v *be a node that is strongly adjacent to a hole H in a TF odd-signable graph G. Then* v *has an odd number of neighbors in H.*

Proof. Since no wheel of G is even by Theorem 1.1, it suffices to show that v cannot have exactly two neighbors in H. Remark 2.1, applied to the subgraph of G induced by $V(H) \cup \{v\}$, shows that v together with its two neighbors in H should induce a triangle, a contradiction. ■

Definition 2.3. *The complete bipartite graph* $K_{4,4}$ *with a perfect matching removed is called cube. This graph is indeed the skeleton of a 3-dimensional cube. So a cube is a hole* $H = u_1, v_2, u_3, v_1, u_2, v_3$ *of length 6, together with two nonadjacent nodes, say* u_4 *and* v_4, *where* u_4 *is adjacent to* v_1, v_2 *and* v_3, *and* v_4 *is adjacent to* u_1, u_2 *and* u_3.

Note that a cube does not contain an even wheel nor a three-path configuration and hence, by Theorem 1.1, it is an odd-signable graph.

Theorem 2.4. *Let G be a connected TF odd-signable graph containing no* K_1 *or* K_2 *cutset. If G contains a cube M, then* $G = M$.

Proof. Assume G contains a cube M induced by the nodes $u_1, \ldots, u_4$, $v_1, \ldots, v_4$, where u_i is adjacent to v_j whenever $i \neq j$ and no other adjacencies exist.

Claim. *No node of G is strongly adjacent to M.*

Proof of Claim. Assume a node w, strongly adjacent to M, has neighbors in both sides of the bipartition of M. Since G is TF, we can assume w.l.o.g. that w is adjacent to u_1, v_1 and no other node of M. Now w, v_3, and v_4 are intermediate nodes in distinct paths of a $3PC(u_1, u_2)$.

So all the neighbors of w belong to one side of the bipartition of M. Assume w.l.o.g. that w is adjacent to u_1 and u_2 and possibly u_3 and u_4. Again, w, v_3, and v_4 are intermediate nodes in distinct paths of a $3PC(u_1, u_2)$. This completes the proof of the claim.

Assume $G \neq M$ and let C be a connected component of $G \backslash M$. Nodes of C that have a neighbor in M, have a unique neighbor in M (by the claim). Since G has no K_1 or K_2 cutset, nodes of C must have two nonadjacent neighbors in M. Therefore, C contains a chordless path $P = x_1, \ldots, x_n, n \geq 2$, such that the neighbors of x_1 and x_n in M are two nonadjacent nodes of M. Among all such paths in C, assume that P has the shortest length. Therefore, at most one node of M is adjacent to an intermediate node of P, and if this node exists, then it is adjacent to both neighbors of x_1 and x_n in M.

Case 1: *No node of M is adjacent to a node* $x_i, 2 \leq i \leq n-1$.

By symmetry, we can consider two possibilities: Either x_1 is adjacent to u_1 and x_n is adjacent to v_1 or x_1 is adjacent to u_1 and x_n is adjacent to u_2. The same argument used for the claim shows the existence of a $3PC(u_1, u_2)$.

Case 2: *One node of M is adjacent to a node* $x_i, 2 \leq i \leq n-1$.

Assume w.l.o.g. that x_1 is adjacent to u_1, x_n is adjacent to u_2 and v_3 is adjacent to a node $x_i, 2 \leq i \leq n-1$. Let $P' = u_1, v_2, u_4, v_1, u_2$ and H' be the hole made up by P and P'. Let H'' be the hole closed by v_4 with P. Then either (H', v_3) or (H'', v_3) is an even wheel. ■

Theorem 2.5. *Let u and v be two nodes, strongly adjacent to a hole H in a TF odd-signable graph G. Then u and v are nonadjacent and either* $V(H) \cup \{u, v\}$ *induces a cube or a sector of* (H, v) *contains all the neighbors of u.*

Proof. Assume first that u and v are adjacent. Then they have no common neighbor in H as G is TF. Also every sector of (H, v) has an even number of neighbors of u (by Remark 2.2 applied to the hole closed by v and a sector of (H, v)). But then (H, u) is an even wheel. Hence, u and v are not adjacent.

Next, assume that no node of H is adjacent to both u and v. Assume that u has at least one neighbor in sector S_1 of (H, v), with endnodes v_1 and v_2, but S_1 does not contain all the neighbors of u in H. Let h_1, h_2 be the neighbors of v_1, v_2 in H but not in S_1. Then all the neighbors of u in H are contained in $V(S_1) \cup \{h_1, h_2\}$, because if u has a unique neighbor u_1 in S_1, there is a $3PC(u_1, v)$, and if u has several neighbors in S_1, there is a $3PC(u, v)$. So u is adjacent to h_1 or h_2, say h_1. Assume u is not adjacent to h_2. Then u has several neighbors in S_1 and Remark 2.2, applied to the hole closed by v with S_1, shows that u has an odd number of neighbors in S_1. Therefore, (H, u) is an even wheel, a contradiction. So u is adjacent to both h_1 and h_2. Let S be the h_1h_2-subpath of H that does not contain S_1. If S is not of length 2, then there is a $3PC(h_1, v)$ or a $3PC(h_2, v)$ contained in the graph induced by the node set $V(S) \cup \{u, v, v_1, v_2\}$. Hence S is of length 2 and v is adjacent to the intermediate node of S. By symmetry, S_1 is also of length 2 and hence the node set $V(H) \cup \{u, v\}$ induces a cube.

Finally assume that u and v have a common neighbor u^* in H.

Claim 1. *If u has a unique neighbor u_i in a sector S_i of (H, v), the node u_i is an endnode of the sector S_i.*

Proof of Claim 1. Assume u has a unique neighbor u_1 in S_1 and u_1 is not an endnode of S_1. Let v_1, v_2 be the endnodes of S_1. Then there is a $3PC(u_1, v)$, where v_1, v_2 and u^* are intermediate nodes in the three distinct paths. This completes the proof of the claim.

Claim 2. *Both $(N(v)\backslash N(u)) \cap V(H)$ and $(N(u)\backslash N(v)) \cap V(H)$ are nonempty.*

Proof of Claim 2. Assume $N(u) \cap V(H) \subseteq N(v) \cap V(H)$. By Remark 2.2, $|N(u) \cap V(H)| \geq 3$, and no two nodes in $N(u) \cap V(H)$ are adjacent, else there is a triangle. So G contains a $3PC(u, v)$. This completes the proof of the claim.

By Claim 2, we can assume that u is adjacent to at least one intermediate node of a sector, say node u_1 of sector S_1 of (H, v) with v_1, v_2 as endnodes. Claim 1 shows that u_1 is not the unique neighbor of u in S_1. Let h_1, h_2 be the neighbors of v_1, v_2 in H but not in S_1. Then all the neighbors of u in H are contained in $V(S_1) \cup \{h_1, h_2\}$, else there is a $3PC(u, v)$. Assume u is adjacent to h_1. Since G is TF, u and v_1 are nonadjacent, and hence h_1 is the unique neighbor of u in the sector of (H, v) that contains h_1, which contradicts Claim 1. So all the neighbors of u in H belong to S_1.

Definition 2.6. *A chordless xz-path P is an ear of the hole H if the intermediate nodes of P belong to $V(G)\backslash V(H)$, nodes $x, z \in V(H)$ have a common neighbor y in H, and $(V(H)\backslash\{y\}) \cup V(P)$ induces a hole H'. We say that x and z are the* attachments *of P in H and that H' is obtained by* augmenting *H with P.*

Note that, in a TF odd-signable graph, Remark 2.2 shows that y has an odd number of neighbors in H'.

Lemma 2.7. *Let $H = u_1, \ldots, u_m$ be a hole in a TF odd-signable graph G, and let $P = u_i, x_1, \ldots, x_n, u_j$ be a chordless path such that u_i is not adjacent to u_j and no intermediate node of P belongs to H nor is strongly adjacent to H. Then P is an ear of H.*

Proof. Let H and P be chosen so that they contradict the lemma and P is shortest possible. By Remark 2.1, a node in $V(H)\backslash\{u_i, u_j\}$ is adjacent to a node in $V(P)\backslash\{u_i, u_j\}$. Let $x_r, r \geq 2$, be the node of lowest index adjacent to a node in H and let u_l be its unique neighbor in H. By Remark 2.1, u_l is adjacent to u_i. Let $x_s, s \geq r$, be the node of lowest index adjacent to a node in

$V(H)\backslash\{u_l\}$, say u_t. Again by Remark 2.1, u_t is adjacent to u_l and $u_t \neq u_i$, since P is chordless.

Claim. $t = j$.

Proof of Claim. Do not assume. Let H' be the hole induced by $(V(H)\backslash \{u_l\}) \cup \{x_1, \dots, x_s\}$. Now, since G is TF, the chordless path $P' = x_s, \dots, x_n, u_j$ and the hole H' satisfy the conditions of the lemma and P' is strictly shorter than P, so P' is an ear of H'. Let H'' be the hole obtained by augmenting H' with P'. Nodes u_t and u_l are strongly adjacent to H'' and are adjacent, contradicting Theorem 2.5. This completes the proof of the claim.

By the claim, x_s is adjacent to u_j, so $x_s = x_n$ and P is an ear of H. ∎

We do not know of any polytime algorithm to check whether a graph contains a wheel. By using the fact that a hole can be found in polytime, however, the following consequence of Remark 2.2 and Lemma 2.7 yields a polytime algorithm that either detects a wheel in G or else shows that G is not TF odd-signable.

Corollary 2.8. *Let G be a connected TF odd-signable graph containing no K_1 or K_2 cutset and let H be a hole in G. Then either $G = H$ or G contains a wheel (H, v) or H has an ear P and G contains a wheel (H', y) where H' is obtained by augmenting H with P and $\{y\} = V(H)\backslash V(H')$.*

Lemma 2.9. *Let (H, v) be a wheel of a TF odd-signable graph and let P be an ear of H, with attachments x,z. Then v has no neighbors in the interior of P and nodes x, z belong to the same sector of (H, v).*

Proof. Let H' be the hole obtained by augmenting H by P and let $\{y\} = V(H)\backslash V(H')$. Nodes y and v are both strongly adjacent to H'. By Theorem 2.5, y and v are nonadjacent. But then x,y and z all belong to the same sector of (H, v). Also, $V(H') \cup \{v, y\}$ does not induce a cube, so by Theorem 2.5, all neighbors of v are contained in one sector of (H', y). Thus v has no neighbors in the interior of P. ∎

Definition 2.10. *Let G be a connected TF graph that contains a wheel (H, v) and let $v_1, \dots, v_n$ be the neighbors of v in H, appearing in this order when traversing H. Then G can be decomposed with wheel (H, v) if the following holds:*

(a) $G\backslash\{v, v_1, \dots, v_n\}$ contains exactly n connected components $Q'_1, \dots, Q'_n$.

(b) The intermediate nodes of the sector with endnodes v_i and v_{i+1} belong to Q'_i and no node of Q'_i is adjacent to $v_j, j \neq i, i+1$.

Note that, given a wheel (H, v) in a TF graph G, one can check in polytime whether G can be decomposed with (H, v). The *blocks* $Q_i, 1 \leq i \leq n$ of the decomposition of G with (H, v) are the subgraphs of G induced by $V(Q'_i) \cup \{v_i, v, v_{i+1}\}$. A subgraph G' of G is *separated* in a decomposition if no block contains all of G'.

Theorem 2.11. *Let G be a connected TF odd-signable graph with at least three nodes and no K_1 or K_2 cutset. Furthermore, assume G is neither a cube nor a hole. Then G contains a wheel and it can be decomposed with any arbitrarily chosen wheel.*

Proof. G contains a cycle since it contains at least three nodes, is connected and contains no K_1 cutset. Therefore, G contains a hole since G is TF. By Corollary 2.8, G contains a wheel, say (H, v). Let $v_1, \ldots, v_n$ be the neighbors of v in H, appearing in this order when traversing H. Let S_l be the sector of (H, v) with endnodes v_l, v_{l+1}.

Claim. *Node set $\{v_l, v_{l+1}, v\}$ is a cutset of G separating H.*

Proof of Claim. Let (H, v) be chosen so that the claim is contradicted and $|V(H)|$ is as small as possible. Then there exists a chordless path $P' = u_i, x_1, \ldots, x_n, u_j, u_i \in V(S_l)\backslash\{v_l, v_{l+1}\}$ and $u_j \in V(H)\backslash V(S_l)$ such that $v \notin V(P')$. By picking P' minimal with this property, we may assume that no node of $V(P')\backslash\{u_i, x_1, x_n, u_j\}$ is adjacent to a node of $V(H)\backslash\{v_l, v_{l+1}\}$. Furthermore, if both v_l and v_{l+1} have neighbors in $V(P')\backslash\{u_i, x_1, x_n, u_j\}$, then a subpath of P' contradicts Remark 2.1. Consequently no node of $V(P')\backslash\{u_i, x_1, x_n, u_j\}$ is strongly adjacent to H. By Theorem 2.4, the graph G contains no cube. Now, by Theorem 2.5, if x_1 is strongly adjacent to H, then all its neighbors are contained in sector S_l and, by Remark 2.2, x_1 has at least three neighbors in H. But then there exists a hole H' containing node x_1 that is shorter than H, contradicting the choice of H. The same argument shows that x_n is not strongly adjacent to H. But then by Lemma 2.7, P' is an ear of H. Since the attachments of P' are in distinct sectors of (H, v), Lemma 2.9 is contradicted and the proof of the claim is complete.

The claim shows that no two nodes, belonging to distinct sectors of (H, v), are in the same connected component of $G\backslash\{v, v_1, \ldots, v_n\}$. So $G\backslash\{v, v_1, \ldots, v_n\}$ contains at least n connected components. Let $Q'_1, \ldots, Q'_n$ be the connected components containing the intermediate nodes of the sectors $S_1, \ldots, S_n$ and assume $G\backslash\{v, v_1, \ldots, v_n\}$ contains an additional connected component Q^*. Since G contains no K_1 or K_2 cutset, there exist i and $j, i \neq j$, such that Q^* contains a node adjacent to v_i and a node adjacent to v_j. Since no node of Q^* is adjacent to a node in $V(H)\backslash\{v, v_1, \ldots, v_n\}$, Theorem 2.5 and Remark 2.2 show that no node of Q^* is strongly adjacent to H. So let v_i and $v_j, i \neq j$, be chosen so that the path P connecting them with intermediate nodes in Q^* is the shortest. Since G is TF, v_i and v_j are nonadjacent and P contradicts Remark 2.1.

The claim also shows that no node of Q'_i is adjacent to $v_j, j \neq i, i+1$. This completes the proof of the theorem. ∎

Corollary 2.12. *Let G be a connected TF odd-signable graph which is not a cube and contains no K_1 or K_2 cutset and let (H, v) be a wheel in G. For any two neighbors v_i, v_j of v in H, the nodes v, v_i, v_j form a cutset that separates H. Furthermore, the graph $G\backslash\{v, v_i, v_j\}$ contains exactly two connected components.*

As an application of Theorem 2.11, we prove an extension of a theorem of Markossian Gasparian and Reed [7].

Theorem 2.13. *Let G be a TF odd-signable graph containing no cube. Let x be a node of G. Then either all other nodes of G are neighbors of x or G contains a node y, which is not adjacent to x, whose degree is at most two.*

Proof. A *mate* of x is a node y satisfying the theorem. Let G be a counterexample with the smallest number of nodes. Then G is connected. Since the theorem obviously holds when G contains at most two nodes or is a hole, Theorem 2.11 shows that G has a K_1 or a K_2 cutset or contains a wheel (H, v).

Let u be the node in a K_1 cutset of G and $G_1, \dots, G_n$ be the blocks of the corresponding decomposition of G. Since G is not the star of u, one block, say G_1, is not an edge and, by the minimality of G, u has a mate y in G_1. Then y is a mate of all nodes in G, except possibly the nodes in $V(G_1)\backslash\{u\}$. Similarly, any node of degree at most two in $V(G_2)\backslash\{u\}$ is a mate of all the nodes in $V(G_1)\backslash\{u\}$.

Assume G has no K_1 cutset and let $\{u, v\}$ be a K_2 cutset of G. Let $G_1, \dots, G_n$ be the blocks of this K_2 decomposition of G. Note that it is not possible that u is adjacent to all the nodes in $G_1\backslash\{u, v\}$, since otherwise v would not have a neighbor in $G_1\backslash\{u, v\}$ and hence $\{u\}$ would be a K_1 cutset. Hence, by minimality of G, u has a mate y in G_1. Since $y \in V(G_1)\backslash\{u, v\}$, y is a mate of all nodes in G, except possibly the nodes in $V(G_1)\backslash\{u\}$. The same argument shows how to find mates of the nodes in $V(G_1)\backslash\{u\}$.

Assume G has no K_1 or K_2 cutset and let $Q_1, \dots, Q_n$ be the blocks of a decomposition of G with (H, v). Let v_i and v_{i+1} be the neighbors of v in $V(H) \cap V(Q_i)$ and let $y \neq v_i, v_{i+1}$ be a mate of v in Q_i. Such a node exists by the minimality of G. Then y is a mate of all nodes in G, except possibly the nodes in $V(Q_i)\backslash\{v\}$. Since $n \geq 3$, for every node x of G there exists an index i such that $x \notin V(Q_i)\backslash\{v\}$ and, therefore, every node of G has a mate. ■

Markossian Gasparian and Reed [7] prove the above theorem for TF graphs containing no even hole. They use it to show that graphs containing no even hole and no even cycle with a unique short chord are β-perfect.

Lemma 2.14. *Let G be a connected TF graph that contains a wheel (H, v). Assume that G can be decomposed with (H, v) and that some block Q_i contains a hole C. If (C, u) is a wheel of G for some $u \neq v$, then (C, u) is a wheel in Q_i.*

Proof. If $u \notin V(Q_i)$, its only possible neighbors in Q_i are v, v_i or v_{i+1} (with the notation used in and after Definition 2.10). Since G is TF, if u is adjacent to v, then u cannot be adjacent to v_i or v_{i+1}. Since u has at least three neighbors in C and C is in Q_i, u must belong to Q_i. ■

Lemma 2.15. *Let G be a connected TF graph that contains a wheel (H, v). Assume G can be decomposed with (H, v) and let Q_1 be a block of a* *decomposition of G with* (H, v)*. If G contains no* K_1 *or* K_2 *cutset, then* Q_1 *contains no* K_1 *or* K_2 *cutset.*

Proof. Let v_1 and v_2 be the endnodes of the sector S_1 of (H, v) contained in Q_1. Let K be a clique cutset in Q_1 separating node u from w. If K is not a clique cutset in G, there exists a chordless path P from u to w in $G \backslash K$. P uses nodes in $G \backslash Q_1$, else it is a path in Q_1 contradicting the assumption that K is a clique cutset. But then P must contain both v_1 and v_2. The path v_1, v, v_2 is in Q_1 and it can be substituted for the subpath of P outside Q_1 unless $v \in K$ (note that v_1 and v_2 are not in K since P avoids K in G). So assume that $v \in K$. Nodes v_1 and v_2 are the only nodes of S_1 which are adjacent to v, and hence no node of S_1 is contained in K. But then S_1 together with $V(P) \cap V(Q_1)$ contains a path connecting u to w in Q_1, a contradiction. ∎

3. COMPOSING TF ODD-SIGNABLE GRAPHS

Assume that a connected graph G is not a cube and contains a wheel (H, v) but no K_1 or K_2 cutset. Then Theorem 2.11 shows that G can be decomposed with wheel (H, v) when G is TF odd-signable. Similar wheel decomposition theorems were obtained when G is linear balanced or, more generally, when G is balanced [6, 5]. In this section, we give sufficient conditions for G to be TF odd-signable when all the blocks of the wheel decomposition are. No such conditions are known for linear balanced or balanced graphs.

Theorem 3.1. *Let* $Q_1, \ldots, Q_n, n \geq 3$ *odd, be node-disjoint connected TF odd-signable graphs. Let each* Q_i *contain nodes labeled* v, v_i, v_{i+1} *such that* vv_i, vv_{i+1} *are edges and nodes* v_i *and* v_{i+1} *are connected by a path* P_i *in* $Q_i \backslash \{v\}$ *not containing other neighbors of* v*. Let G be obtained by identifying the n copies of* v *(one in each of the* Q_i*'s) and, for every i, the node* $v_{i+1} \in V(Q_i)$ *with node* $v_{i+1} \in V(Q_{i+1})$*. Then the followings are equivalent.*

(1) G is a TF odd-signable graph.

(2) For every i,

  (i) Q_i *contains no wheel* (H, v)*, where* v_i, v_{i+1} *are neighbors of* v *in H, and*

  (ii) Q_i *contains no wheel* (H, u)*, where* v_i, v, v_{i+1} *are consecutive in H and* v *is a neighbor of* u*.*

(3) There exists no chordless path P in $Q_i \backslash \{v\}$ *with endnodes* v_i *and* v_{i+1} *that has an intermediate node adjacent to* v*.*

Proof. Throughout the proof we assume that the paths P_i are chosen to be the shortest satisfying the condition of the theorem.

We first prove that (1) implies (2). If Q_1 contains a wheel (H, v) where v_1 and v_2 are neighbors of v in H, then since v has an odd number of neighbors in H (by

Remark 2.2), Q_1 contains a chordless v_1v_2-path R with an odd number of neighbors of v. Let H' be the hole of G made up by R and $P_2, \dots, P_n$. Then (H', v) is an even wheel of G.

If Q_1 contains a wheel (H, u) where v_1, v, v_2 belong to H and v is a neighbor of u, let R be a shortest v_1v_2-path containing node u, with nodes in $V(H) \cup \{u\} \setminus \{v\}$. Note that such a path R exists since u has an odd number of neighbors in H, by Remark 2.2. Then the neighbors of v in R are v_1, u, and v_2. Let H' be the hole of G made up by R and $P_2, \dots, P_n$. Then (H', v) is an even wheel of G.

We now prove that (2) implies (3). Let H_i be the hole induced by $V(P_i) \cup \{v\}$.

Claim. *Let R be any chordless path with endnodes v_i and v_{i+1} in $V(Q_i) \setminus \{v\}$. Then one of the following holds:*

(a) R is an ear of the hole H_i.

(b) $V(R) \setminus \{v_i, v_{i+1}\}$ contains a node u adjacent to v and strongly adjacent to P_i.

(c) Node v is not adjacent to any intermediate node of R.

Proof of Claim. Suppose R does not satisfy the claim. Since R does not satisfy (c) then $R = v_i, x_1, \dots, x_k, v_{i+1}$, a node x_l is adjacent to node $v, R \neq P_i$ and since G is TF, $k > 1$. Furthermore by Remark 2.2, if x_l has a neighbor in P_i, then x_l has at least three neighbors in H_i, so x_l is strongly adjacent to P_i and R satisfies (b). Therefore, x_l has no neighbors in P_i. Let R_1 and R_2 be the x_lv_i and x_lv_{i+1}-subpaths of R, respectively, and let x_s and x_t be the nodes in R_1 and R_2 adjacent to a node in P_i, and closest to x_l in R_1 and R_2, respectively. Again, by Remark 2.2, applied to x_s and H_i, if x_s is adjacent to v, then R satisfies (b). Now by the minimality of P_i and Remark 2.2, x_s is not strongly adjacent to H_i. The same argument shows that x_t is not strongly adjacent to H_i. Let $y \in V(P_i)$ be the unique neighbor of x_s in H_i, let R_s be the x_sx_l-subpath of R_1 and x_m the neighbor of v closest to x_s in R_s (possibly $x_m = x_l$). Let $P_s = y, x_s, \dots, x_m, v$. Now P_s satisfies Remark 2.1 with respect to the hole H_i, and hence $y = v_i$. Similarly, v_{i+1} is the unique neighbor of x_t in H_i. Since R is chordless, $x_s = x_1$ and $x_t = x_k$. So R is an ear of H_i and the proof of the claim is complete.

If the alternative (a) of the claim holds, then by Remark 2.2 (H, v) is a wheel, where H is the hole induced by the node set $V(R) \cup V(P_i)$. If the alternative (b) of the claim holds, then (H_i, u) is a wheel. Hence if (2) holds, then alternatives (a) and (b) of the claim are not possible. So every chordless v_iv_{i+1}-path satisfies (c) and hence (3) holds.

Finally we show that (3) implies (1). Suppose it does not. Since G is TF, it contains an even wheel or a $3PC$, by Theorem 1.1. Suppose first that G contains an even wheel (H, w), $v \neq w$, with $w \in V(Q_1)$. Lemma 2.14 shows that H is not contained in any block Q_i and therefore H contains nodes $v_1, \dots, v_n$. Let P^i be the

v_iv_{i+1}-subpath of H contained in Q_i. By (3), v is not adjacent to any intermediate node in P^i and so $V(P^1) \cup \{v\}$ together with node w is a wheel in Q_1. So the graph induced by $V(P^1) \cup \{w\}$ contains a chordless v_1v_2-path that contains w. If w is adjacent to v then (3) is contradicted; hence Q_1 contains an even wheel. This contradicts the assumption that Q_1 is odd-signable.

If G contains an even wheel (H, v) then, since Q_i is odd-signable, H is not contained in any of the Q_i's and so it contains $v_1, \ldots, v_n$. Since (H, v) is an even wheel, v has a neighbor in H distinct from $v_1, \ldots, v_n$. But then, for some i, the v_iv_{i+1}-subpath of H in Q_i contains an intermediate node adjacent to v, contradicting (3).

Suppose now that G contains a $3PC(x, y)$. Let R_1, R_2, R_3 be the three xy-paths in the $3PC(x, y)$. Note that x and y must belong to the same Q_i and none of the three paths can contain v. We can assume w.l.o.g. that R_1, R_2 belong to Q_1 and the v_1v_2-subpath R made up by $P_2, \ldots, P_n$ belongs to R_3. We also assume that v_1 is encountered before v_2 when traversing R_3 from x. By (3), v is not adjacent to any node of $V(R_1) \cup V(R_2) \cup V(R_3) \backslash V(R)$. But then there exists a $3PC(x,y)$ in Q_1 obtained by replacing the subpath R by v_1, v, v_2. This completes the proof of the theorem. ■

4. TESTING WHETHER A TF GRAPH IS ODD-SIGNABLE

In this section, we give a polytime algorithm to test whether a TF graph is odd-signable. A step of the algorithm is the decomposition of a graph G^* with a wheel (H, v). Let $v_1, \ldots, v_n$ be the neighbors of v in H, appearing in this order when traversing H, and let $Q_1, \ldots, Q_n$ be the blocks of this decomposition. The n pairs of edges v_iv and $v_{i+1}v$ in each of the blocks Q_i are declared *linked pairs* and will remain so throughout the algorithm.

Input: *A connected TF graph G.*

Output: *YES if G is odd-signable, NO otherwise.*

Step 1: *If G has no K_1 of K_2 cutsets, set $\mathcal{L} = \{G\}$. Otherwise decompose G with K_1 and K_2 cutsets, until no such cutset exists, and let $\mathscr{L}$ be the set of blocks thus obtained.*

Step 2: *If every graph in $\mathcal{L}$ has one or two nodes, is a hole or a cube, return YES. Otherwise go to Step 3.*

Step 3: *Remove a graph G^* from $\mathcal{L}$ which has more than two nodes and is neither a hole nor a cube. Identify a hole H' in G^*.*

If no node has at least 3 neighbors in H' and H' has no ear, output NO. If a node v has at least 3 neighbors in H', then let (H, v) be a wheel with $H = H'$. Otherwise, let P be an ear of H'. Let H be the hole obtained by augmenting H' with P and let $\{v\} = V(H') \backslash V(H)$. If (H, v) is not a wheel then output NO. Let $v_1, \ldots, v_n, n \geq 3$, be the neighbors of v in H, appearing in this order when traversing H.

If G^* cannot be decomposed with wheel (H, v), output NO. Otherwise let $Q_1, \ldots, Q_n$ be the blocks of this decomposition. If one of the following three alternatives occurs:

(a) n is even,

(b) a block Q_i is a cube,

(c) a linked pair is separated in the decomposition of G^* with (H, v),

then output NO. Otherwise declare all pairs vv_i, vv_{i+1} linked, add the n blocks of the decomposition to $\mathcal{L}$ and go to Step 2.

Theorem 4.1. *The above algorithm correctly tests in polytime whether a TF graph is odd-signable.*

Proof. We first prove the correctness of the above algorithm. Corollary 1.2 shows that if G^* contains a K_1 or a K_2 cutset, then G^* is odd-signable if and only if all the blocks are. When a wheel decomposition of G^* is found in Step 3, Lemma 2.15 shows that no block of this decomposition contains a K_1 or a K_2 cutset.

Corollary 2.8 shows that, in Step 3, if no node has at least 3 neighbors in H' and H' contains no ear, then G^* is correctly rejected. If H is obtained by augmenting H' with P, and (H, v) is not a wheel, then by Remark 2.2, G is correctly rejected. Theorem 2.11 shows that if G^* cannot be decomposed with wheel (H, v) in Step 3, then G^* is not odd-signable. If n is even, (H, v) is an even wheel and G^* is safely rejected in (a). If some Q_i is a cube, G^* is correctly rejected in (b) by Theorem 2.4.

Assume a linked pair uu_1 and uu_2 is separated in the decomposition of G^* with (H, v) and G is TF odd-signable. Then there exists a wheel (C,u) in G such that u_1 and u_2 are consecutive neighbors of u in C and we can assume w.l.o.g. that G has been decomposed in Step 3 with wheel (C, u) at a previous stage. Let $u_1, \ldots, u_m$ be the neighbors of u in C, let $U_1, \ldots, U_m$ be the blocks of the decomposition of G with (C, u). Let S_1 be the sector of (C, u) with endnodes u_1 and u_2. Since the decomposition of G^* with wheel (H, v) separates u_1u, u_2u, either u and v coincide or u is a neighbor of v in H, say v_1. The first case is not possible, since U_1 contains a path, namely S_1 connecting u_1 and u_2, not containing a neighbor of $u = v$. In the second case, v must be adjacent to a node of S_1. Then v is strongly adjacent to the hole H_1 induced by $V(S_1) \cup \{u\}$ and Remark 2.2 shows that (H_1, v) is a wheel. Now in the decomposition of G with (C, u), Condition (2) (ii) of Theorem 3.1 is contradicted by block U_1. This proves the validity of (c).

To complete the correctness proof of the algorithm, it only remains to show that if G is accepted in Step 2, then G is a TF odd-signable graph. Let G^* be a TF graph that is decomposed in Step 3 with wheel (H, v). Assume G^* is not odd-signable while all the blocks Q_i are odd-signable. Theorem 3.1 (2) shows that a block, say Q_1, either contains a wheel (C, v) and v_1, v_2 are neighbors of v in C or Q_1 contains a wheel (C, u), where v_1, v, v_2 are consecutive in C and v is a neighbor of u.

In the first case, since Q_1 is a TF odd-signable graph, Corollary 2.12 applied to the decomposition of Q_1 with the wheel (C, v) shows that when removing nodes v_1, v, v_2 the two v_1v_2-subpaths of C are separated. Therefore, $G^*\backslash\{v, v_1, \ldots, v_n\}$ contains at least two components that have nodes adjacent to v_1 and v_2. But then $G^*\backslash\{v, v_1, \ldots, v_n\}$ has at least $n+1$ connected components, which contradicts the assumption that G^* can be decomposed with the wheel (H, v).

In the second case, let C' be the smallest hole in Q_1 containing the path v_1, v, v_2. By Theorem 2.11, in the decomposition of Q_1 with the wheel (C, u), v_1 and v_2 are separated. So u must be strongly adjacent to C' and hence by Remark 2.2, (C', u) is a wheel in Q_1. We now show that in all further decompositions of Q_1 by the algorithm, (C', u) is contained in some block, which contradicts the assumption that the graphs in $\mathscr{L}$ are decomposed until all blocks are holes.

Suppose that C' is separated in a decomposition by a wheel (H', w). Then w is not a node of C' and at least two neighbors of w in H' belong to C'. So, by Remark 2.2 applied to Q_1, (C', w) is a wheel. If w is not adjacent to v, then there exists a smaller hole than C' that contains the path v_1, v, v_2 and the node w. By the choice of C' this is not possible, and hence w is adjacent to v. Let w_1 (respectively w_2) be a neighbor of w in $V(C') \cap V(H')$ such that w_1v-subpath (respectively w_2v-subpath) of C' that contains v_1 (respectively v_2) has no intermediate node in $N(w) \cap V(C') \cap V(H')$. Let P be the w_1w_2-subpath of C' that contains v. Since the linked pair v_1v, v_2v is not separated in the decomposition by (H', w), P is contained in some block. But then, since w is adjacent to v, Theorem 3.1 (3) is contradicted. Hence C' is not separated by any further decomposition of Q_1. Since u has at least three neighbors in C', in a decomposition by a wheel (H', w), the block that contains C' also contains u, and the proof of the correctness of the algorithm is complete.

To prove polynomiality of the above procedure, observe that when decomposing G with either a K_1 or K_2 cutset or a wheel (H, v) the total number of nonadjacent pairs of nodes within connected components strictly decreases. This is due to the fact that at least one such pair is separated and no new pair is created. (This idea is borrowed from [2].) ■

5. FINDING AN EVEN HOLE IN A TF GRAPH

We show how the algorithm from the previous section can be used to check in polytime whether a TF graph contains a hole of even cardinality and, in fact, how to find such a hole if one exists.

As a first step, we show that, if we can check whether a graph is odd-signable, then we can also check whether a signed graph is odd-signed.

Let G be a connected signed graph and let G' be a signed graph obtained from G by switching labels on all the edges of a cut of G. Since cuts and cycles of G have even intersections, it follows that the cycles of G have the same parity in G and G'. So G is odd-signed if and only if G' is odd-signed. Since every edge of a spanning tree T of G is contained in a cut of G that does not contain any other edge of T, then, if there exists an odd-signing of G, there exists one in which the edges of T have any specified (arbitrary) signing.

This implies that, if a connected graph G is odd-signable, one can produce such a signing as follows.

Signing Algorithm

Input: *A connected odd-signable graph G, a spanning tree T, and an arbitrary signing of the edges of T.*

Output: *The unique odd-signing of G such that the edges of T are signed as specified in the input.*

Index the edges of G $e_1, \ldots, e_n$, *so that the edges of T are the first* $|V(G)| - 1$, *and every edge* $e_j, j \geq |V(G)|$, *together with edges having smaller indices, closes a chordless cycle* H_j *of G. For* $j = |V(G)|, \ldots, n$, *sign* e_j *so that* H_j *is odd-signed.*

The fact that there exists an indexing of the edges of G as required in the signing algorithm follows from the following observation. For $j \geq |V(G)|$, we can select e_j so that the path connecting the endnodes of e_j in the subgraph $(V(G), \{e_1, \ldots, e_{j-1}\})$ is the shortest possible. The chordless cycle H_j identified this way is also a chordless cycle in G. This forces the signing of e_j, since all the other edges of H_j are signed already. So, once the (arbitrary) signing of T has been chosen, the signing of G is unique.

Assume that we have an algorithm to check odd-signability. Then, given a connected signed graph G, we can check whether G is odd-signed as follows. Let G' be an unsigned copy of G. Test whether G' is odd-signable. If it is not, then G is not odd-signed. Otherwise, let T be a spanning tree of G'. Run the signing algorithm on G' with the edges of T signed as they are in G. Then G is odd-signed if and only if the signing of G' equals the signing of G.

Now, using the result of Section 4, it follows that we can decide in polytime whether a signed TF graph is odd-signed. As a special case, consider a TF graph with all edges signed odd: this yields a polytime algorithm for deciding whether G has a hole of even cardinality.

To actually find a hole of even cardinality in G when one exists, let $v_1, \ldots, v_n$ denote the nodes of G and let $H = G$. In iteration i, test whether $H \backslash v_i$ contains a hole of even cardinality. If the answer is yes, set $H = H \backslash v_i$ and otherwise keep H unchanged. Perform n iterations. At termination, the graph H is the desired hole of even cardinality.

6. CONSTRUCTING ALL TF ODD-SIGNABLE GRAPHS

In this section, we give a procedure to construct all TF odd-signable graphs. Starting with a hole, we obtain every connected TF odd-signable graph that is not a cube and contains no K_1 of K_2 cutset, by a sequence of "good ear additions."

Definition 6.1. *A graph G is said to be obtained from a graph G' by an ear addition if the nodes of $G\backslash G'$ are the intermediate nodes of an ear of some hole H in G', say an ear P with attachments x and z, and the graph G contains no edge connecting a node of $V(P)\backslash\{x,z\}$ to a node of $V(G')\backslash\{x,y,z\}$, where $y \in V(H)$ is adjacent to x and z. An ear addition is said to be* good *if*

- *y has an odd number of neighbors in P,*
- *G' contains no wheel (H_1, v) where $x, y, z \in V(H_1)$ and v is adjacent to y, and*
- *G' contains no wheel (H_2, y), where x,z are neighbors of y in H_2.*

Remark 6.2. *Assume G is a connected TF graph obtained from G' by an ear addition P. Then G can be decomposed with a wheel (H', y) such that G' is a block of this decomposition and the other blocks are all the distinct holes in the subgraph induced by $V(P) \cup \{y\}$.*

Lemma 6.3. *Let G be a TF graph obtained from a connected TF odd-signable graph G' by an ear addition. Then G is odd-signable if and only if the ear addition is good.*

Proof. Let $H = x, y, z, \ldots, x$ be a hole of G' such that $P = x, u_1, \ldots, u_k, z$ is an ear of H. Suppose first that y has an even number of neighbors in P. In this case, the ear addition is not good. Furthermore, either $V(P) \cup V(H)$ induces a 3*PC(x,z)* or an even wheel, and G is not TF odd-signable by Theorem 1.1. So the lemma holds in this case. Suppose now that y has an odd number of neighbors in P. Let $Q_1 = G'$ and let $Q_2, \ldots, Q_n, n \geq 3$ odd, denote the distinct holes in the subgraph induced by $V(P) \cup \{y\}$. The conditions of Theorem 3.1 hold since the required xz-path P_1 can be taken as the xz-subpath of H avoiding node y. Now the lemma follows from the equivalence of (1) and (2) in Theorem 3.1. ■

Theorem 6.4. *Let G be a connected TF graph with at least three nodes which is not a cube and contains no K_1 or K_2 cutset. Then, G is odd-signable if and only if G can be obtained, starting from a hole, by a sequence of good ear additions.*

Proof. By Lemma 6.3, if G is obtained from a hole by a sequence of good ear additions, then G is TF odd-signable. To prove the converse it is enough to show, by Lemma 6.3, that, if G is not a hole, then it is obtained from some graph G' by an ear addition, where G' has no K_1 or K_2 cutset. By Theorem 2.11, G contains a wheel (H, v).

Claim. *Let Q_1 be a block of a decomposition of G with (H, v). Let (C, u) (possibly $u = v$) be a wheel of Q_1 and let $W_1^1, \ldots, W_m^1$ and $W_1^2, \ldots, W_m^2$ be the* *blocks of the decomposition of* Q_1 *and G with* (C, u)*, respectively, such that* u_i *and* u_{i+1} *belong to both* W_i^1 *and* W_i^2*. Then for some* $j \in \{1, \ldots, m\}$, $W_j^1 \subset W_j^2$ *and* $W_i^1 = W_i^2$ *whenever* $i \neq j$.

Proof of Claim. Assume that G is decomposed with wheel (H, v) using the algorithm of Section 4. Then vv_1 and vv_2 are declared a linked pair and Theorem 4.1 shows that this pair is not separated in the decomposition of Q_1 with (C,u). Let W_1^1 be the block of such a decomposition, containing the linked pair vv_1, vv_2. Every node of $V(G)\backslash V(Q_1)$ is connected to v_1 and v_2 by paths not containing nodes of Q_1 and, therefore, these paths do not contain u or a neighbor of u in C. Therefore, all these nodes belong to the same block of the decomposition of G with (C,u). So $j = 1$, $W_1^1 \subset W_1^2$ and $W_i^1 = W_i^2$ whenever $i \neq 1$. This completes the proof of the claim.

Choose the wheel (H, v) so that, among all decompositions of G with wheels, the largest block of the decomposition (in terms of number of nodes) is largest for the decomposition with (H, v). Let the blocks of the decomposition of G with (H, v) be $U_1, \ldots, U_n$ with block U_1 being the largest. By the claim and the choice of (H, v), none of the blocks $U_i, i \neq 1$, contains a wheel. By Lemma 2.15, none of the blocks U_i contains a K_1 or K_2 cutset and Corollary 2.8 shows that all the blocks $U_i, i \neq 1$ are in fact holes. The theorem now follows by choosing $G' = U_1$. ∎

It follows that every connected TF odd-signable graph with more than one node can be obtained starting from cubes, edges, and graphs constructed according to Theorem 6.4 by recursively identifying nodes or edges, thus creating K_1 or K_2 cutsets. Initially we thought all TF odd-signable graphs were planar. This is false as shown by the graph in Fig. 1.

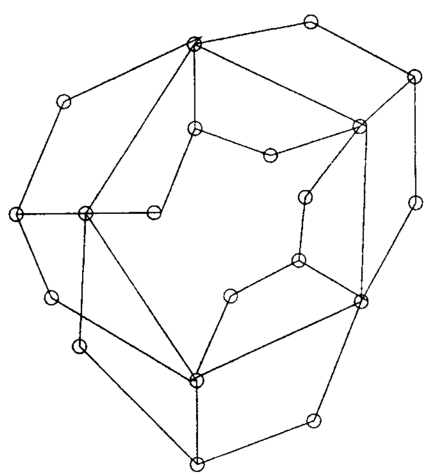

Figure 1. Non-planar TF odd-signable graph.

ACKNOWLEGMENT

Ajai Kapoor was supported in part by a grant from Gruppo Nazionale Delle Ricerche-CNR. Part of the work was completed while Kristina Vušković was at the Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, on an NSERC Canada International Fellowship. We also acknowledge the support of Laboratoire ARTEMIS, Université de Grenoble.

References

[1] D. Bienstock, On the complexity of testing for odd holes and induced odd paths. Discrete Math 90 (1991), 85–92, MR 92m:68040a. Corrigendum in Discrete Math 102 (1992), 109.

[2] V. Chvátal, J. Fonlupt, L. Sun, and A. Zemirline, Recognizing dart-free perfect graphs, Report no. 92778, University of Bonn, 1992.

[3] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković, Even and odd holes in cap-free graphs, J Graph Theory 30 (1993) 289–308.

[4] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković, Even-hole-free graphs: Parts I–II, J Graph Theory, to appear.

[5] M. Conforti, G. Cornuéjols, and M. R. Rao. Decomposition of balanced matrices, J Combin Theory B 77 (1999), 292–406.

[6] M. Conforti and M. R. Rao, Structural properties and decomposition of linear balanced matrices, Math Programming 55 (1992), 129–169.

[7] S. E. Markossian, G. S. Gasparian, and B. A. Reed, β-perfect graphs, J Combin Theory B 67 (1996), 1–11.

[8] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids. J Combin Theory B 32 (1982), 112–139.