A class of three-colorable triangle-free graphs

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Abstract

The chromatic number of a triangle-free graph can be arbitrarily large. In this paper we show that if all subdivisions of $K_{2,3}$ are also excluded as induced subgraphs, then the chromatic number becomes bounded by 3. We give a structural characterization of this class of graphs, from which we derive an $\mathcal{O}(nm)$ coloring algorithm, where n denotes the number of vertice and m the number of edges of the input graph.

Key words: coloring; decomposition; clique cutsets; star cutsets; triangle-free graphs; induced subdivisions of $K_{2,3}$.

AMS Subject Classification (2010): 05C20, 05C75, 05C85

1 Introduction

Throughout the paper all graphs are finite and simple. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G, and it is F*-free* if it does not contain F. For a family of graphs $\mathcal{F}$ we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$.

It is a well known fact that triangle-free graphs can have arbitrarily large chromatic number. The coloring problem remains difficult even when

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seemingly a lot of structure is imposed on a triangle-free graph. For example determining whether a graph is 3-colorable remains NP-complete for triangle-free graphs with maximum degree 4 [14].

A family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function f* if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$ (where χ denotes the chromatic number of a graph and ω the size of its larges clique). This concept was introduced by Gyárfás [12] as a natural extension of perfect graphs, that are a χ-bounded family of graphs with χ-binding function $f(x) = x$. A natural question to ask is: what choices of forbidden induced subgraphs guarantee that a family of graphs is χ-bounded? Much research has been done in this area, for a survey see [15]. We note that most of that research has been done on classes of graphs obtained by forbidding a finite number of graphs. Since there are graphs with an arbitrarily large chromatic number and girth [11], in order for a family of graphs defined by forbidding a finite number of graphs (as induced subgraphs) to be χ-bounded, at least one of these forbidden graphs needs to be acyclic. In this paper we consider a class of graphs defined by excluding only acyclic graphs, namely the class of .graphs that do not contain triangles nor subdivisions of $K_{2,3}$ as induced subgraphs. (A $K_{2,3}$ is the complete bipartite graph with 2 nodes on one side of bipartition and 3 nodes on the other side, and a subdivision of a graph is obtained by subdividing its edges into paths of arbitrary length). We show that the chromatic number for this class is bounded by 3, and we give an $\mathcal{O}(nm)$ algorithm for coloring graphs in this class.

Subdivisions of $K_{2,3}$ are in fact one of the configurations that appear in a theorem of Truemper [17] that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities. The characterization states that this can be done for a graph G if and only if it can be done for all induced subgraphs of G that are of few specific types, depicted in Figure 1, which we will call *Truemper configurations*. (In all figures a solid line denotes an edge and a dashed line denotes a chordless path containing one or more edges). In Section 1.1 we define these configurations and explain their wider significance.

In Section 1.2 we state the key results of this paper about the class of graphs defined by excluding triangles and subdivisions of $K_{2,3}$ as induced subgraphs, whose proofs are given in Section 2. In Section 1.3 we show how it follows from the work of Kühn and Osthus [13] that the class of graphs that do not contain subdivisions of $K_{2,3}$ is χ-bounded. The method relies on Ramsey numbers, and so the bound is quite large.

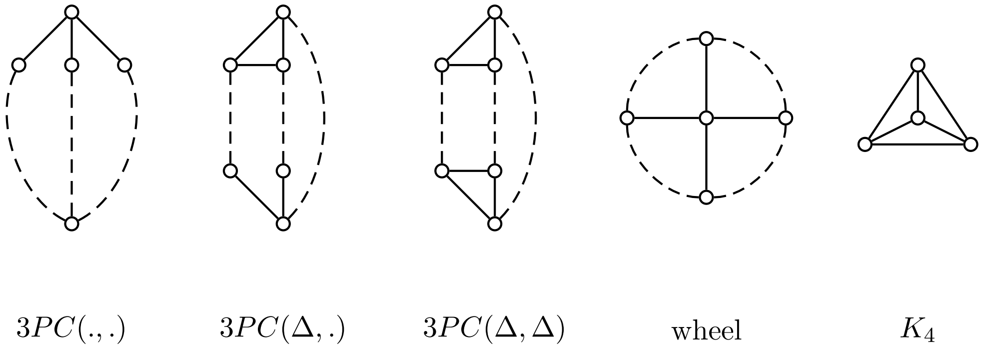

Figure 1: Truemper configurations

1.1 Truemper configurations

A *hole* in a graph is an induced cycle of length at least 4. For $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A. A *clique* is a graph in which every pair of nodes are adjacent. A clique on k nodes is denoted by K_k. A K_3 is also referred to as a *triangle*, and is denoted by Δ. A $K_{s,t}$ is a complete bipartite graph with s nodes on one side of the bipartition and t nodes on the other.

The first three configurations in Figure 1 are referred to as *3-path configurations* ($3PC$'s). They are structures induced by three paths P_1, P_2 and P_3, in such a way that the nodes of $P_i \cup P_j$, $i \neq j$, induce a hole. More specifically, a $3PC(x,y)$ is a structure induced by three paths that connect two nonadjacent nodes x and y; a $3PC(x_1x_2x_3, y)$, where $x_1x_2x_3$ is a triangle, is a structure induced by three paths having endnodes x_1, x_2 and x_3 respectively and a common endnode y; a $3PC(x_1x_2x_3, y_1y_2y_3)$, where $x_1x_2x_3$ and $y_1y_2y_3$ are two node-disjoint triangles, is a structure induced by three paths P_1, P_2 and P_3 such that, for $i = 1,2,3$, path P_i has endnodes x_i and y_i. We say that a graph G contains a $3PC(.,.)$ if it contains a $3PC(x,y)$ for some $x, y \in V(G)$, a $3PC(\Delta,.)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$, and it contains a $3PC(\Delta,\Delta)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. Note that the condition that nodes of $P_i \cup P_j$, $i \neq j$, must induce a hole, implies that all paths of a $3PC(.,.)$ have length greater than one, and at most one path of a $3PC(\Delta,.)$ has length one.

Note that a $3PC(.,.)$ is in fact a subdivision of $K_{2,3}$. In literature $3PC(.,.)$ is also referred to as *theta* [2], $3PC(\Delta,.)$ as *pyramid* [1], and $3PC(\Delta,\Delta)$ as *prism* [2].

A *wheel* (H, x) consist of a hole H and a node x called the *center* that has at least three neighbors on the hole H. Finally, a K_4 is a clique on four vertices. We note that in [17] K_4's are also referred to as wheels, but in this paper we choose to separate these two structures.

Theorem 1.1 (Truemper [17]) *Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C \bmod 2$ for all chordless cycles C of G, if and only if every induced subgraph G' of G that is a Truemper configuration, there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C \bmod 2$, for all chordless cycles C of G'.*

Truemper's original motivation for Theorem 1.1 was to obtain a co-NP characterization of bipartite graphs that are signable to be balanced (i.e. bipartite graphs whose node-node incidence matrices are balanceable matrices, a class of matrices with important polyhedral properties). Configurations that Truemper identified in his theorem ended up playing a key role in understanding the structure of several seemingly diverse classes of objects, such as regular matroids, balanceable matrices and perfect graphs. For example, $3PC(\Delta, .)$'s and wheels that induce an odd number of triangles (i.e. *odd wheels*) are excluded structures for perfect graphs (and more generally, for odd-hole-free graphs), that are convenient to work with when trying to characterize the class. Similarly, $3PC(., .)$'s, $3PC(\Delta, \Delta)$'s and wheels whose center has an even number of neighbors on the hole (i.e. *even wheels*) are excluded structures for even-hole-free graphs. All these classes of graphs were characterized by decomposition theorems ([7, 10, 9, 4]), and in all these decomposition theorems Truemper configurations (especially the 3-path configurations and wheels) were the key structures around which the decompositions took place and from which the basic (undecomposable) classes were built.

In a connected graph G, a subset S of nodes and edges is a *cutset* if its removal disconnects G. A cutset S is a *clique cutset* if S induces a clique, and it is a *star cutset* if S contains a node that is adjacent to all other nodes of S. Star cutsets and their generalizations were necessary in the above mentioned decomposition theorems. The problem with star cutsets (and their generalizations) is that it is difficult to make use of them in decomposition based algorithms, especially optimization algorithms. They were used in constructing polynomial time recognition algorithm for even-hole-free graphs [8, 10] and perfect graphs [1], and the proof of the famous Strong Perfect Graph Conjecture [4]. In contrast to that, if clique cutsets are used to decompose graphs in a class $\mathcal{C}$ down to some basic graphs that are simple to handle in terms of coloring, for example, then one easily obtains a polynomial time decomposition based coloring algorithm for class $\mathcal{C}$. Triangulated graphs (i.e. hole-free graphs) are an example of such a class. Here is another example introduced in [5], that generalizes triangulated graphs.

A graph is *universally signable* if for all choices of a vector γ (that is in one-to-one correspondence with the holes of a graph G), there exists a subset F of the edge set of G such that $|F \cap H| \equiv \gamma_H (\bmod 2)$, for all holes H of G. From Theorem 1.1 it is easy to obtain the following characterization of universally signable graphs in therms of forbidden induced subgraphs.

Theorem 1.2 [5] *A graph is universally signable if and only if it is* $(3PC(.,.), 3PC(\Delta,.), 3PC(\Delta,\Delta), wheel)$*-free.*

This characterization of universally signable graphs is then used to obtain the following decomposition theorem, from which one can derive efficient algorithms for finding the size of a largest clique, or independent set, or coloring the class.

Theorem 1.3 [5] *A connected* $(3PC(.,.), 3PC(\Delta,.), 3PC(\Delta,\Delta), wheel)$*-free graph is either a clique or a hole or has a clique cutset.*

We are interested in generalizing this class further by considering the class of $(3PC(.,.), 3PC(\Delta,.), 3PC(\Delta,\Delta))$-free graphs. As a first step to understanding their structure, in this paper we analyze $(\Delta, 3PC(.,.))$-free graphs.

1.2 $(\Delta, 3PC(.,.))$-free graphs

In this paper we obtain the following characterizations of $(\Delta, 3PC(.,.))$-free graphs, that lead to a coloring algorithm for this class. Our results generalize the work in [6] where the class of $(\Delta, 3PC(.,.), even\ wheel)$-free graphs is considered.

The complete bipartite graph $K_{4,4}$ with a perfect matching removed is called a *cube*. This graph is indeed the skeleton of a 3-dimensional cube.

Theorem 1.4 *A connected* $(\Delta, 3PC(.,.))$*-free graph that has a cube is either equal to that cube or has a* K_1 *or* K_2 *cutset.*

Analogous result is proved in [6] for $(\Delta, 3PC(.,.), even\ wheel)$-free graphs. From Theorem 1.4 it follows, by the discussion above, that if we know how to color $(\Delta, 3PC(.,.), cube)$-free graphs, then we can color the entire class of $(\Delta, 3PC(.,.))$-free graphs. Next we concentrate on $(\Delta, 3PC(.,.), cube)$-free graphs.

For $x \in V(G)$, $N(x)$ is the set of all neighbors of x in G, and $N[x] = N(x) \cup \{x\}$. A cutset S is a *full star cutset* of G, if for some $x \in V(G)$, $S = N[x]$.

Theorem 1.5 *If a connected $(\Delta, 3PC(.,.), cube)$-free graph contains a wheel, then it has a full star cutset.*

In [6] it is shown that if a $(\Delta, 3PC(.,.), even\ wheel, cube)$-free graph contains a wheel (H, x), then for any two distinct neighbors x_i and x_j of x on H, $\{x, x_i, x_j\}$ is a cutset. In the case of $(\Delta, 3PC(.,.), cube)$-free graphs, this is not true, since the wheels interact in more complex ways. To decompose wheels we need to use stronger cutsets, as well as be careful about the order in which the wheels are considered for decomposition.

The following decomposition theorem immediately follows from Theorem 1.4, Theorem 1.5 and Theorem 1.3.

Theorem 1.6 *A connected $(\Delta, 3PC(.,.))$-free graph is either a K_1, a K_2, a hole or a cube, or it has a K_1 or K_2 cutset or a full star cutset.*

This decomposition theorem does not help in coloring $(\Delta, 3PC(.,.))$-free graphs, since it is not clear how to use star cutsets in a decomposition based coloring algorithm. Instead we use Theorem 1.6 to prove the existence of a node of small degree, which is a characterization that can be used in a coloring algorithm.

Theorem 1.7 *If G is a $(\Delta, 3PC(.,.))$-free graph, then G has a vertex of degree at most 3.*

We note that this result is best possible since a cube is an example of a $(\Delta, 3PC(.,.))$-free graph all of whose vertices have degree 3. It follows from Theorem 1.7 that $(\Delta, 3PC(.,.))$-free graphs G can be 4-colored in time $\mathcal{O}(n^2)$ by coloring greedily on a sequence of nodes $x_1, \dots, x_n$ such that for every $i = 1, \dots, n$, x_i is of degree at most 3 in $G[\{x_1, \dots, x_i\}]$. We can do better than that by considering cube-free graphs.

Theorem 1.8 *If G is a $(\Delta, 3PC(.,.), cube)$-free graph, then G has a vertex of degree at most 2.*

Analogous result is proved in [6] for $(\Delta, 3PC(.,.), even\ wheel)$-free graphs. By Theorem 1.8 we can color $(\Delta, 3PC(.,.), cube)$-free graphs with at most 3 colors, by constructing a sequence of nodes $x_1, \ldots, x_n$ such that for every $i = 1, \ldots, n$, x_i is of degree at most 2 in $G[\{x_1, \ldots, x_i\}]$ and coloring greedily on this sequence. Putting all the results together we obtain the following theorem that will be proved in Section 2.

Theorem 1.9 *If G is a $(\Delta, 3PC(.,.))$-free graph, then $\chi(G) \leq 3$. Furthermore, there exists an $\mathcal{O}(nm)$ algorithm for coloring graphs in this class, where n denotes the number of vertices and m the number of edges of the input graph.*

Observe that this bound on the chromatic number is tight. We note that although the class of $(\Delta, 3PC(.,.))$-free graphs can be recognized in $\mathcal{O}(n^{11})$ time, since $3PC(.,.)$'s can be detected in that time by the algorithm of Chudnovsky and Seymour [3], it is in fact not necessary to recognize the class before applying the coloring algorithm. The algorithm given in the proof of Theorem 1.9 is *robust* in the following sense: given any graph G, the algorithm either verifies that G is not in our class, or it properly colors the graph. This means that the algorithm will properly color all graphs in our class, as well as some graphs that are not in our class. In case a proper coloring is not returned, we are given a certificate that the input graph is not in our class.

We close this section by observing that there are $(\Delta, 3PC(.,.))$-free graphs that are not planar, as shown in Figure 2.

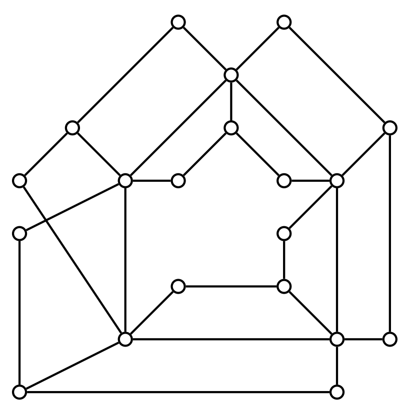

Figure 2: A $(\Delta, 3PC(.,.))$-free graph that has a K_5-minor.

1.3 χ-boundedness of $3PC(.,.)$-free graphs

We now show how it can be derived from the following theorem of Kühn and Osthus that $3PC(.,.)$)-free graphs are χ-bounded. This was pointed out to us by Trotignon, and it was pointed out to him by Scott.

Theorem 1.10 (Kühn and Osthus [13]) *For every graph H and every $s \in \mathbb{N}$ there exists $d = d(H, s)$ such that every graph G of average degree at least d contains either a $K_{s,s}$ as a subgraph or a subdivision of H as an induced subgraph.*

Corollary 1.11 $3PC(.,.)$*-free graphs are χ-bounded.*

PROOF — Let G be a $3PC(.,.)$-free graph. Let s be the Ramsey number $R(\omega(G)+1, 3)$, and let $c = d(K_{2,3}, s)$ be the constant from Theorem 1.10 (with $H = K_{2,3}$). We now show that G is c-colorable. Suppose not. Without loss of generality we may assume that $\chi(G) > c$ and for every proper induced subgraph H of G, $\chi(H) \leq c$.

We prove that the degree of every node of G is at least c. Suppose on the contrary that $\deg(v) \leq c - 1$ for some $v \in V(G)$. By the choice of G, $\chi(G - v) \leq c$, and therefore $\chi(G) \leq \max\{\chi(G - v), \deg(v) + 1\} \leq c$, a contradiction. So every node of G has degree at least C, and therefore G has average degree at least c.

Since G is $3PC(.,.)$-free, it cannot contain a subdivision of $K_{2,3}$ as an induced subgraph, and so by Theorem 1.10 G contains a $K_{s,s}$ as a subgraph. By the choice of s, both sides of the bipartition of $K_{s,s}$ contain a stable set of size 3. In particular, G contains a $K_{2,3}$ as an induced subgraph, a contradiction. □

We note that the bound one gets for the chromatic number in this corollary, is rather large. It follows from the proof of Theorem 1.10 that it is at least $\max\{2^{2^{2^{25}}+1}, 2^{2^{8R(\omega(G)+1,3)}}\}$.

2 Proofs

A *path* P is a sequence of distinct nodes $p_1, \ldots, p_k$, $k \geq 1$, such that $p_i p_{i+1}$ is an edge, for all $1 \leq i < k$. These are called the *edges* of the path P. Nodes p_1 and p_k are the *endnodes* of the path. The nodes of $V(P)$ that are not endnodes are called the *intermediate* nodes of P. Let p_i and p_l be two nodes of P, such that $l \geq i$. The path $p_i, p_{i+1}, \ldots, p_l$ is called the $p_i p_l$-subpath of P. A *cycle* C is a sequence of nodes $c_1, \ldots, c_k, c_1$, $k \geq 3$, such that the nodes

$c_1, \ldots, c_k$ form a path and c_1c_k is an edge. The edges of the path $c_1, \ldots, c_k$ together with edge c_1c_k are called the *edges* of cycle C. The *length* of a path P (resp. cycle C) is the number of edges in P (resp. C).

Given a path or a cycle Q in a graph G, any edge of G between nodes of Q that is not an edge of Q is called a *chord* of Q. Q is *chordless* if no edge of G is a chord of Q. As mentioned earlier a *hole* is a chordless cycle of length at least 4.

Let A and B be two disjoint node sets such that no node of A is adjacent to a node of B. A path $P = p_1, \ldots, p_k$ *connects* A *and* B if either $k = 1$ and p_1 has neighbors in both A and B, or $k > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between* A *and* B if in $G[V(P) \cup A \cup B]$ no path connecting A and B is shorter than P. The direct connection P is said to be *from* A *to* B if p_1 is adjacent to a node of A and p_k is adjacent to a node of B.

Throughout the paper, for a wheel (H, x) we will denote the neighbors of x in H with $x_1, \ldots, x_h$ assuming that they appear in this order when traversing H clockwise. For $i = 1, \ldots, h$, the subpath of H from x_i to x_{i+1} (where index $h+1$ is taken to be 1) that does not contain an interior node adjacent to x is called a *sector* of (H, x) and is denoted by S_i. We denote by x_i' (respectively x_i'') the neighbor of x_i in S_i (respectively S_{i-1}).

For a subgraph F of G, we say that a node $u \in V(G) \setminus V(F)$ is *strongly adjacent* to F if u has at least two neighbors in F.

Proof of Theorem 1.4: Assume that G contains a cube M induced by the nodes $u_1, \ldots, u_4, v_1, \ldots, v_4$ where u_i is adjacent to v_j whenever $i \neq j$ and no other edges exist. Also assume that G does not have a K_1 or K_2 cutset.

We first show that no node of G is strongly adjacent to M. Assume a node w is strongly adjacent to M. W.l.o.g. w is adjacent to u_1, and hence since G is Δ-free, w is not adjacent to v_2, v_3 nor v_4. If w is adjacent to u_i, then w.l.o.g $i = 2$, and hence the node set $\{u_1, u_2, v_3, v_4, w\}$ induces a $3PC(u_1, u_2)$. Therefore, $N(w) \cap M = \{u_1, v_1\}$. But then $(M \setminus \{u_4, v_4\}) \cup \{w\}$ induces a $3PC(u_1, v_1)$. Therefore, no node of G is strongly adjacent to M.

Assume $G \neq M$ and let C be a connected component of $G \setminus M$. Note that, since no node is strongly adjacent to M, the nodes of C that have a neighbor in M, have a unique neighbor in M. Since G has no K_1 nor K_2 cutset, nodes of C must have two nonadjacent neighbors in M. Therefore, C contains a chordless path $P = p_1, \ldots, p_k$, $k \geq 2$, such that the neighbors of p_1 and p_k in M are two nonadjacent nodes. Among all such paths in C, let P be minimal. Therefore, at most one node of M is adjacent to an intermediate node of P, and if such a node exists, then it is adjacent to both of the neighbors of p_1 and p_k in M. We now consider the following two cases.

Case 1: No node of M is adjacent to a node p_i, $2 \leq i \leq k-1$.
By symmetry we may assume that p_1 is adjacent to u_1 and that p_k is adjacent to either u_2 or v_1. If p_k is adjacent to u_2, then the node set $V(P) \cup \{u_1, u_2, v_3, v_4\}$ induces a $3PC(u_1, u_2)$. Otherwise, p_k is adjacent to v_1, and hence the node set $V(P) \cup \{u_1, u_2, u_4, v_1, v_2, v_4\}$ induces a $3PC(u_1, v_1)$.

Case 2: One node of M is adjacent to an intermediate node of P.
W.l.o.g. we may assume that p_1 is adjacent to u_1, p_k to u_2, and v_3 has a neighbor in the interior of P. Note that the nodes of $M \setminus \{u_1, u_2, v_3\}$ have no neighbor in P, and hence the node set $V(P) \cup \{v_1, v_2, v_4, u_1, u_2, u_4\}$ induces a $3PC(u_1, u_2)$. □

Lemma 2.1 *Let G be a $(\Delta, 3PC(.,.), cube)$-free graph. Let (H, x) be a wheel of G such that out of all wheels of G, (H, x) has the fewest number of edges. Then no node is strongly adjacent to (H, x).*

PROOF — Assume that y is strongly adjacent to (H, x). We consider the following cases.

Case 1: y is adjacent to x.
Since G is Δ-free, x and y do not have a common neighbor in H. If y has a unique neighbor y' in H, say in sector S_i, then $V(S_i) \cup \{x, y\}$ induces a $3PC(x, y')$. If y has exactly two neighbors in H, say y' and y'', then since G is Δ-free, $y'y''$ is not an edge, and hence $V(H) \cup \{y\}$ induces a $3PC(y', y'')$. Therefore (H, y) is also a wheel. Let S be a sector of (H, y) that contains a neighbor of x. If S contains exactly one neighbor of x, say x_i, then $V(S) \cup \{x, y\}$ induces a $3PC(y, x_i)$. Otherwise, $V(S) \cup \{x, y\}$ induces a wheel with center x that contradicts our choice of (H, x).

Case 2: y is not adjacent to x.
As in Case 1, y cannot have exactly two neighbors in H, and hence (H, y) is a wheel. Let S be a sector of (H, y) that contains a neighbor of x. If S contains exactly two neighbors of x, say x_i and x_{i+1}, then $V(S) \cup \{x, y\}$ induces a $3PC(x_i, x_{i+1})$. If S contains at least three neighbors of x, then $V(S) \cup \{x, y\}$ induces a wheel with center x that contradicts our choice of (H, x). Therefore, each sector of (H, y) contains at most one neighbor of x. If x and y have three common neighbors in H, say x_i, x_j and x_k, then $\{x, y, x_i, x_j, x_k\}$ induces a $3PC(x, y)$. Therefore some sector S of (H, y) contains exactly one neighbor of x, say x_i, and x_i is in the interior of S.

Let y_1 and y_2 be the endnodes of S. For $j = 1,2$, let y'_j be the neighbor of y_j in $H \setminus S$. Since G is Δ-free, y cannot be adjacent to y'_1 nor y'_2, and in particular, y has a neighbor in $V(H) \setminus (V(S) \cup \{y'_1, y'_2\})$. If x has a neighbor in $V(H) \setminus (V(S) \cup \{y'_1, y'_2\})$, then $G[(V(H) \setminus (V(S) \cup \{y'_1, y'_2\})) \cup \{x, y\}]$ contains a chordless path P from x to y, and hence $V(S) \cup V(P)$ induces a $3PC(x_i, y)$. Therefore $N(x) \cap V(H) = \{x_i, y'_1, y'_2\}$. If $x_i y_1$ is not an edge, then $V(S) \cup \{x, y, y'_1\}$ induces a $3PC(x_i, y_1)$. So $x_i y_1$ is an edge, and by symmetry so is $x_i y_2$. Let y' be the neighbor of y in $H \setminus S$ that is closest to y'_2. If $y'_1 y'$ is not an edge, then the subpath of $H \setminus S$ from y' to y'_2 together with the node set $\{x, y, x_i, y_1, y'_1\}$ induces a $3PC(x, y_1)$. So $y'_1 y'$ is an edge. Since $V(H) \cup \{x, y\}$ cannot induce a cube, $y' y'_2$ is not an edge. But then $\{x, y, x_i, y_2, y'_2, y'_1, y'\}$ induces a $3PC(x, y_2)$. □

A chordless path $P = p_1, \ldots, p_k$ of $G \setminus (H \cup \{x\})$ is an *ear* of (H, x) if for some $i \in \{1, \ldots, h\}$, no node of P is adjacent to x, $N(p_1) \cap V(H) = \{x''_i\}$, $N(p_k) \cap V(H) = \{x'_i\}$, and no intermediate node of P has a neighbor in $H \setminus \{x_i\}$. In this case we say that P is an *x_i-ear*.

Lemma 2.2 *Let G be a $(\Delta, 3PC(.,.), cube)$-free graph. Let (H, x) be a wheel of G such that out of all wheels of G, (H, x) has the fewest number of edges. Then there are nodes x_i and x_j, $i \neq j$, such that there is no x_i-ear and no x_j-ear.*

PROOF — Assume not and w.l.o.g. let P_i be an x_i-ear, for $i = 2, \ldots, h$. Let G' be the subgraph of G induced by $\cup_{i=3}^{h} V(P_i) \cup (V(H) \setminus (V(S_1) \cup \{x_3, \ldots, x_h\})) \cup \{x_1, x_2\}$. Clearly G' is connected. Let P be a chordless path from x_1 to x_2 in G'. By definition of ears, no node of $(V(S_1) \setminus \{x_1, x_2\}) \cup \{x\}$ has a neighbor in P, and hence $V(P) \cup V(S_1) \cup \{x\}$ induces a $3PC(x_1, x_2)$. □

Theorem 2.3 *Let G be a $(\Delta, 3PC(.,.), cube)$-free graph. Let (H, x) be a wheel of G such that out of all wheels of G, (H, x) has the fewest number of edges. Then for some $i, j \in \{1, \ldots, h\}$, $i \neq j$, $S = N[x] \setminus (\{x_1, \ldots, x_h\} \setminus \{x_i, x_j\})$ is a star cutset separating the interior nodes of the two $x_i x_j$-subpaths of H.*

PROOF — By Lemma 2.1 no node is strongly adjacent to (H, x). By Lemma 2.2 there are some $1 \leq i < j \leq h$ such that there is no x_i-ear and no x_j-ear. Let S' (resp. S'') be the $x_i x_j$-subpath of H that contains S_i (resp. S_{i-1}). We will show that $S = N[x] \setminus (\{x_1, \ldots, x_h\} \setminus \{x_i, x_j\})$ is a star cutset separating $S' \setminus \{x_i, x_j\}$ from $S'' \setminus \{x_i, x_j\}$. Assume not and let $P = p_1, \ldots, p_k$ be a direct connection from $S' \setminus \{x_i, x_j\}$ to $S'' \setminus \{x_i, x_j\}$ in $G \setminus S$. Let s' (resp. s'') be the neighbor of p_1 (resp. p_k) in $S' \setminus \{x_i, x_j\}$ (resp. $S'' \setminus \{x_i, x_j\}$). Note that the only nodes of (H, x) that may have a neighbor in $P \setminus \{p_1, p_k\}$ are x_i and x_j.

A node of $\{x_i, x_j\}$ must have a neighbor in $P \setminus \{p_1, p_k\}$, since otherwise $V(H) \cup V(P)$ induces a $3PC(s', s'')$. If both x_i and x_j have a neighbor in $P \setminus \{p_1, p_k\}$, then there is a subpath P' of $P \setminus \{p_1, p_k\}$ such that x_i, P', x_j is a chordless path, and hence $V(H) \cup V(P')$ induces a $3PC(x_i, x_j)$. So w.l.o.g. we may assume that x_i has a neighbor in $P \setminus \{p_1, p_k\}$, and x_j does not. If both $x_i s'$ and $x_i s''$ are edges, then P is an x_i-ear, contradicting our assumption. So w.l.o.g. $x_i s'$ is not an edge. Let p_l be the node of $P \setminus \{p_1, p_k\}$ with smallest index adjacent to x_i. Then $V(H) \cup \{p_1, \ldots, p_l\}$ induces a $3PC(x_i, s')$. □

Proof of Theorem 1.5: Follows from Theorem 2.3 and the assumption of being Δ-free. □

Theorem 2.4 *If G is a $(\Delta, 3PC(.,.), cube)$-free graph, then for every $x \in V(G)$, either $V(G) = N[x]$ or G contains a vertex $y \in V(G) \setminus N[x]$ whose degree is at most 2.*

PROOF — Assume not and let G be a counterexample with fewest number of nodes. Observe that since G is Δ-free, if C is a connected component of G that is a star, i.e. $V(C) = N[x]$ for some $x \in V(G)$, then either $|V(C)| = 1$ (and hence x is of degree 0) or every node of $N(x)$ has degree 1. So by minimality of G, it follows that G is connected. Also G is not a star. We say that a node y is a *mate* of node x, if y is not adjacent to x and is of degree at most 2. Since the theorem obviously holds when G has at most two nodes or is a hole, by Theorem 1.6, since G is cube-free, it follows that G has a K_1 or K_2 cutset, or a full star cutset.

First suppose that G has a K_1 cutset, say $\{u\}$. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus \{u\}$, and for $i = 1, \ldots, k$, let $G_i = G[V(C_i) \cup \{u\}]$. Since $V(G) \neq N[u]$, w.l.o.g. $V(G_1) \neq N_{G_1}[u]$. By minimality of G it follows that some $c_1 \in V(G_1) \setminus N[u]$ has degree at most 2 in G_1 (and hence in G as well). So for every $x \in V(G) \setminus V(C_1)$, c_1 is a mate of x. If $|V(C_2)| = 1$ then the node of C_2 is of degree 1 in G, and otherwise by the same argument C_2 contains a node of degree at most 2. So C_2 contains a node c_2 of degree at most 2 in G. But then c_2 is a mate of every node of C_1, a contradiction. Therefore G cannot have a K_1 cutset.

Next assume that $\{u, v\}$ is a K_2 cutset of G. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus \{u,v\}$, and for $i = 1,\dots,k$, let $G_i = G[V(C_i) \cup \{u,v\}]$. Since neither $\{u\}$ nor $\{v\}$ can be a K_1 cutset, for $i = 1,\dots,k$, both u and v have a neighbor in C_i. So since G is Δ-free $V(G_1) \neq N_{G_1}[u]$, and hence by minimality of G, u has a mate c_1 in G_1. Since $c_1 \in V(C_1)$, node c_1 is of degree at most 2 in G as well, and hence it is a mate in G of all nodes of $V(G) \setminus (V(C_1) \cup \{v\})$. By analogous argument v has a mate c_2 in G_2, that is a mate in G of all nodes of $V(C_1) \cup \{v\}$, a contradiction. Therefore G cannot have a K_2 cutset.

So G has a full star cutset $S = N[x]$. Let $C_1,\dots,C_k$ be the connected components of $G \setminus S$, and for $i = 1,\dots,k$, let G_i be the subgraph of G induced by $V(C_i) \cup \{x\}$ and all the nodes of $N(x)$ that have a neighbor in C_i. By minimality of G, for $i = 1,\dots,k$, x has a mate c_i in G_i, and since $c_i \in V(C_i)$, it is a mate of x in G as well. Then c_1 is a mate in G of all nodes of $V(G) \setminus (V(C_1) \cup N(x))$, and c_2 is a mate in G of all nodes of $V(C_1)$. Hence all nodes of G except possibly the nodes of $N(x)$ have a mate in G. Since G is a counterexample, some $x' \in N(x)$ does not have a mate in G, and hence x' is adjacent to c_i for every $i = 1,\dots,k$. Since $\{x,x'\}$ cannot be a cutset of G separating C_1 from the rest of G, G_1 cannot be a star, and hence by minimality of G, x' has a mate x'' in G_1. Node $x'' \notin V(C_1)$, since otherwise it would be a mate of x' in G as well. So $x'' \in N(x)$, and since $x'' \in V(G_1)$, it follows that x'' has a neighbor in C_1. Since x'' is not a mate of x' in G, w.l.o.g. x'' has a neighbor in C_2. For $i = 1,2$, let P_i be a chordless path from x' to x'' in $G[V(C_i) \cup \{x',x''\}]$ (since both x' and x'' have a neighbor in C_i, and C_i is connected, such a path exists). But then $V(P_1) \cup V(P_2) \cup \{x\}$ induces a $3PC(x',x'')$, a contradiction. $\square$

Proof of Theorem 1.8: Follows from Theorem 2.4 and the assumption of being Δ-free. $\square$

Proof of Theorem 1.7: The proof is obtained in analogous way to the proof of Theorem 2.4. One just needs to replace all "degree at most 2" with "degree at most 3" and observe that if G is a cube then for every $x \in V(G)$ there exists a node $y \in V(G)$ that is not adjacent to x and is of degree at most 3. $\square$

Proof of Theorem 1.9: Let G be a $(\Delta, 3PC(.,.))$-free graph. Since graphs whose chromatic number is at most two can be recognized and properly colored in linear time, it suffices to show how to 3-color G. Clearly we may also assume that G is connected.

Let S be a clique cutset of a graph G, and let C_1, C_2, S be a vertex partition so that no node of C_1 is adjacent to a node of C_2. We define the *blocks of decomposition* by clique cutset S to be graphs $G_i = G[V(C_i) \cup S]$, for $i = 1, 2$. The first step of our algorithm constructs a *decomposition tree* T using clique cutsets as follows. The root of T is the input graph G, for every internal node G' of T, the children of G' are the blocks of decomposition of G' with respect to some clique cutset such that at least one of the children has no clique cutset, and all the leaves of T are graphs that have no clique cutset. Such a decomposition tree can be constructed in $\mathcal{O}(nm)$ time and it has at most $n-1$ leaves [16].

The second step of our algorithm 3-colors the leaves of T as follows. Let L be a leaf of T. By Theorem 1.4, L is either a cube or is cube-free. If L is a cube, then it is bipartite and hence can be 2-colored. Otherwise, by Theorem 1.8, there is an ordering $x_1, \dots, x_l$ of vertices of L such that for every $i = 1, \dots, l$, x_i is of degree at most 2 in $G[\{x_1, \dots, x_i\}]$, and hence L can be 3-colored by coloring greedily on this ordering of vertices. Such an ordering can be constructed in $\mathcal{O}(|V(L)|^2)$ time. Let $L_1, \dots, L_k$ be the leaves of T, and for $i = 1, \dots, k$ let $n_i = |V(L_i)|$. Since the clique cutsets in Δ-free graphs are of size at most 2, the sum of the nodes of children of an internal node G' of T is at most 2 greater than $|V(G')|$. It follows that $\Sigma_{i=1}^k n_i \leq 3n$, and hence, since $\Sigma_{i=1}^k n_i^2 \leq (\Sigma_{i=1}^k n_i)^2$, step 2 can be implemented to run in $\mathcal{O}(n^2)$ time.

In the third step of our algorithm we backtrack along T to obtain a 3-coloring of G from the 3-colorings of leaves of the decomposition tree as follows. Let H be an internal node of T, and $H_1, \dots, H_k$ its children in T. So $H_1, \dots, H_k$ are blocks of decomposition of H with respect to some clique cutset S. Since S is a clique, nodes of S must have different colors in all of these colorings. So we can permute the colors of the colorings of H_i's so that they all agree on the colors of the nodes of S, and by putting together such colorings we get a 3-coloring of H.

This algorithm can clearly be implemented to run in $\mathcal{O}(nm)$ time. □

Observe that the algorithm given in the proof of Theorem 1.9 can easily be turned into a robust algorithm as discussed in Section 1.2. In step 1, if a clique of size greater than 2 is used in the construction of the decomposition tree, then output "G is not $(\Delta, 3PC(.,.))$-free" and stop. In step 2, in case a leaf that is considered is not a cube and does not have the desired ordering of vertices, then output "G is not $(\Delta, 3PC(.,.))$-free" and stop.

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