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The graph sandwich problem for 1-join composition is NP-complete

Celina M.H. de Figueiredo[a,*], Sulamita Klein[a], Kristina Vušković[b]

[a] IM and COPPE, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil

[b] School of Computing, University of Leeds, Leeds LS2 9JT, UK

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Abstract

A graph is a 1-join composition if its vertex set can be partitioned into four nonempty sets A_L, A_R, S_L and S_R such that: every vertex of A_L is adjacent to every vertex of A_R; no vertex of S_L is adjacent to vertex of $A_R \cup S_R$; no vertex of S_R is adjacent to a vertex of $A_L \cup S_L$. The graph sandwich problem for 1-join composition is defined as follows: Given a vertex set V, a forced edge set E^1, and a forbidden edge set E^3, is there a graph $G=(V,E)$ such that $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$, which is a 1-join composition graph? We prove that the graph sandwich problem for 1-join composition is NP-complete. This result stands in contrast to the case where $S_L = \emptyset$ ($S_R = \emptyset$), namely, the graph sandwich problem for homogeneous set, which has a polynomial-time solution.

Keywords: Computational complexity; Graph algorithms; Sandwich problems; Perfect graphs

1. Introduction

We say that a graph $G^1=(V,E^1)$ is a *spanning* subgraph of $G^2=(V,E^2)$ if $E^1 \subseteq E^2$; we say that a graph $G=(V,E)$ is a *sandwich* graph for the pair G^1, G^2 if $E^1 \subseteq E \subseteq E^2$. For notational simplicity in the sequel, we let E^3 be the set of all edges in the complete graph with vertex set V which are not in E^2. Thus every sandwich graph for the pair G^1, G^2 satisfies $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$. We call E^1 the *forced edge set*, and E^3 the *forbidden edge set*. The GRAPH SANDWICH PROBLEM FOR PROPERTY Φ is defined as follows [11]:

GRAPH SANDWICH PROBLEM FOR PROPERTY Φ

Instance: Vertex set V, forced edge set E^1, forbidden edge set E^3.

* Corresponding author.

E-mail addresses: celina@cos.ufrj.br (C.M.H. de Figueiredo), sula@cos.ufrj.br (S. Klein), vuskovi@comp.leeds.ac.uk (K. Vušković).

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Question: Is there a graph $G=(V,E)$ such that $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$ that satisfies property Φ?

Graph sandwich problems have attracted much attention lately arising from many applications and as a natural generalization of recognition problems [4,9–15]. The recognition problem for a class of graphs $\mathscr{C}$ is equivalent to the graph sandwich problem in which the forced edge set $E^1 = E$, the forbidden edge set $E^3 = \emptyset$, $G=(V,E)$ is the graph we want to recognize, and property Φ is "to belong to class $\mathscr{C}$".

Perfect graphs were introduced by Berge [1] as those graphs for which, in every induced subgraph, the size of a largest clique is equal to the chromatic number. The recognition problem for the class of perfect graphs is a famous open problem in computational complexity [16].

Golumbic et al. [11] have considered sandwich problems with respect to several subclasses of perfect graphs, and proved that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains in P. On the other hand, they proved that the GRAPH SANDWICH PROBLEM FOR PERMUTATION GRAPHS turns out to be NP-complete.

We are interested in graph sandwich problems for properties Φ related to compositions arising in perfect graph theory. Several compositions of graphs are known to preserve perfection: graph substitution [17], join composition [3,6,8], clique identification [2].

A graph $G=(V,E)$ has a *homogeneous set* H, $H \subset V$, if each vertex of $V \setminus H$ is either adjacent to all vertices of H or to none of the vertices of H, $|H| \geqslant 2$ and $|V \setminus H| \geqslant 1$. Polynomial algorithms for finding homogeneous sets are given in [5,18–21]. A graph $G=(V,E)$ is a 1-*join partition* (or *join partition*) if its vertex set V can be partitioned into sets V_L and V_R so that $|V_L| \geqslant 2$ and $|V_R| \geqslant 2$, V_L contains a nonempty set A_L, V_R contains a nonempty set A_R, with the property that every node of A_L is adjacent to every node of A_R, no node of $V_L \setminus A_L$ is adjacent to a node of V_R, and no node of $V_R \setminus A_R$ is adjacent to a node of V_L. Let $S_L = V_L \setminus A_L$ and $S_R = V_R \setminus A_R$. If $S_L = \emptyset$, then A_L is a homogeneous set in G, and if $S_R = \emptyset$, then A_R is a homogeneous set in G. In [4], a polynomial-time algorithm is given for solving the graph sandwich problem when property Φ is to contain a homogeneous set. To distinguish the property of containing a homogeneous set from the property of being a 1-join partition graph, we further impose the condition that $S_L \neq \emptyset$ and $S_R \neq \emptyset$. In this case, we say that $V_L|V_R$ is a 1-*join composition* of G, with *left side* V_L, with *right side* V_R, and with *special sets* S_L, A_L, S_R, A_R. A polynomial-time algorithm, of complexity $O(n^3)$, for 1-join composition recognition is given in [7]. In this paper, we prove that the graph sandwich problem when property Φ is "to be a 1-join composition" is NP-complete, even when restricted to instances that admit a solution for the homogeneous set sandwich problem.

2. Proof of NP-completeness

In this section, we prove that the GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION is NP-complete by reducing the NP-complete problem 3-SATISFIABILITY to GRAPH SAND-

WICH PROBLEM FOR 1-JOIN COMPOSITION. These two decision problems are defined as follows:

3-SATISFIABILITY (3SAT)

Instance: Set $X = \{x_1, \dots, x_n\}$ of variables, collection $C = \{c_1, \dots, c_m\}$ of clauses over X such that each clause $c \in C$ has $|c| = 3$ literals.

Question: Is there a truth assignment for X such that each clause in C has at least one true literal?

GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION

Instance: Vertex set V, forced edge set E^1, forbidden edge set E^3.

Question: Is there a graph $G(V, E)$, such that $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$ which is a 1-join composition?

Theorem 1. *The* GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION *is* NP-*complete.*

Proof. In order to reduce 3SAT to GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION, we need to construct a particular instance (V, E^1, E^3) of GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION from a generic instance (X, C) of 3SAT, such that C is satisfiable if and only if (V, E^1, E^3) admits a sandwich graph $G = (V, E)$ which is a 1-join composition. First, we describe the construction of a particular instance (V, E^1, E^3) of GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION; second, we prove in Lemma 2 that every 1-join composition $G = (V, E)$ satisfying $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$ defines a truth assignment for (X, C); third, we prove in Lemma 3 that every truth assignment for (X, C) defines a 1-join composition $G = (V, E)$ satisfying $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$. These steps are explained in detail below. □.

Construction of particular instance of GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION.

The vertex set V contains: an auxiliary set of vertices $\{v_L, v, a_L, a_R, s_L, s_R\}$; for each SAT variable x_k, $1 \leqslant k \leqslant n$, one corresponding vertex x_k; for each SAT clause c_i, $1 \leqslant i \leqslant m$, three vertices z_1^i, z_2^i and z_3^i, and three sets of vertices V_1^i, V_2^i and V_3^i. As the notation of the auxiliary vertices suggests, the role of those vertices is to define the sides of the remaining vertices in a 1-join composition of the graph we are about to construct. As we shall see, in any 1-join composition for this graph, vertices a_L and s_L have to be placed on the same side as vertex v_L, vertices a_R and s_R have to be opposite to vertex v_L. This property of the auxiliary vertices with respect to any 1-join composition will define a forcing for the sides (left or right) of the remaining vertices of the graph.

See Figs. 1–4, where solid edges denote forced E^1-edges and dashed edges denote forbidden E^3-edges.

The forced edge set E^1 is the union of sets of edges: $\{a_L a_R, s_L a_L, a_R s_R\}$, $\{v_L w\colon w \in \bigcup_{i=1}^m \{z_3^i\} \cup \{x_1, \dots, x_n, s_L, a_R, v\}\}$, $\{vw\colon w \in \cup_{i=1}^m \{z_1^i, z_2^i, z_3^i\} \cup \{x_1, \dots, x_n, a_L, a_R, v_L\}\}$, $\cup_{i=1}^m \{x_i a_L, x_i a_R, z_2^i z_3^i, a_R z_1^i, a_R z_2^i, a_R z_3^i, a_L z_3^i\}$, $\bigcup_{i=1}^m (B_1^i \cup B_2^i \cup B_3^i)$, where $B_1^i \cup B_2^i \cup B_3^i$ consists of auxiliary forced edges corresponding to each clause.

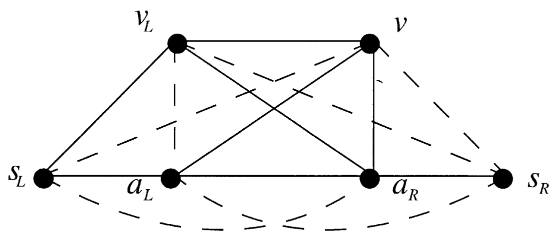

Fig. 1. Auxiliary graph for special instance (V, E^1, E^3).

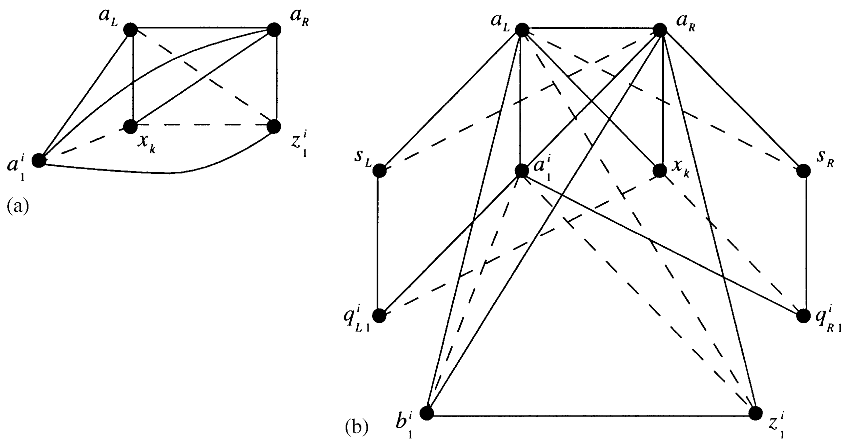

Fig. 2. Same side (a) and different side (b) gadgets respectively, for $l_1^i = x_k$ and for $l_1^i = \bar{x}_k$.

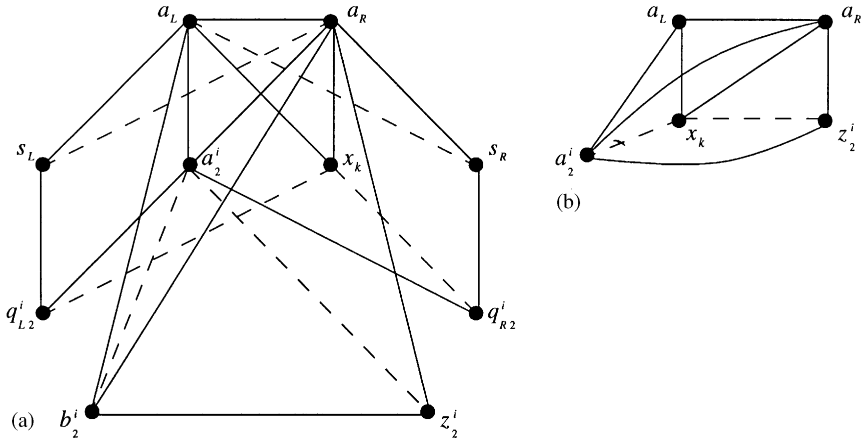

Fig. 3. Different side (a) and same side (b) gadgets respectively for $l_2^i = x_k$ and for $l_2^i = \bar{x}_k$.

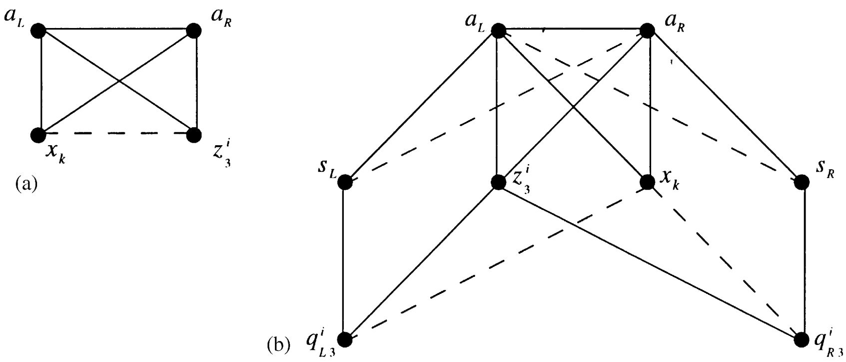

Fig. 4. Same side (a) and different side (b) gadgets respectively, for $l_3^i = x_k$ and for $l_3^i = \bar{x}_k$.

The forbidden edge set E^3 is the union of sets of edges: $\{v_L s_R, v s_L, v s_R, a_R s_L, a_L s_R, a_L v_L\}$, $\bigcup_{i=1}^m \{z_1^i z_2^i, a_L z_1^i\}$, $\bigcup_{i=1}^m (D_1^i \cup D_2^i \cup D_3^i)$, where $D_1^i \cup D_2^i \cup D_3^i$ consists of auxiliary forbidden edges corresponding to each clause.

In the sequel, for $j = 1, 2, 3$, the auxiliary forced edges B_j^i and the auxiliary forbidden edges D_j^i will be detailed by considering six different cases.

Let $c_i = (l_1^i \vee l_2^i \vee l_3^i)$ be a SAT clause. We have two kinds of gadgets which we call, respectively, same side gadget and different side gadget. The name and role of these gadgets will become clear when we prove Claims 2 and 3 for Lemma 2 below.

If $l_1^i = x_k$, then the following nodes and edges are added to construct a *same side* gadget: $V_1^i = \{a_1^i\}$, $B_1^i = \{a_1^i a_L, a_1^i a_R, z_1^i a_1^i, v_L a_1^i, v a_1^i\}$, $D_1^i = \{z_1^i x_k, a_1^i x_k\}$. We say that x_k and z_1^i are *connected* by a same side gadget.

On the other hand, if $l_1^i = \bar{x}_k$, where $\bar{x}_k$ is the negation of variable x_k, then the following nodes and edges are added to construct a *different side* gadget: $V_1^i = \{a_1^i, b_1^i, q_{L1}^i, q_{R1}^i\}$, $B_1^i = \{s_L q_{L1}^i, s_R q_{R1}^i, a_L a_1^i, a_R a_1^i, a_L b_1^i, a_R b_1^i, a_1^i q_{L1}^i, a_1^i q_{R1}^i, b_1^i z_1^i, v_L a_1^i, v_L b_1^i, v_L q_{L1}^i, v a_1^i, v b_1^i, v q_{R1}^i\}$, $D_1^i = \{q_{L1}^i x_k, q_{R1}^i x_k, z_1^i a_1^i, a_1^i b_1^i\}$. We say that x_k and z_1^i are *connected* by a different side gadget.

If $l_2^i = x_k$, then the following nodes and edges are added to construct a *different side* gadget: $V_2^i = \{a_2^i, b_2^i, q_{L2}^i, q_{R2}^i\}$, $B_2^i = \{s_L q_{L2}^i, s_R q_{R2}^i, a_L a_2^i, a_R a_2^i, a_L b_2^i, a_R b_2^i, a_2^i q_{L2}^i, a_2^i q_{R2}^i, b_2^i z_2^i, v_L a_2^i, v_L b_2^i, v_L q_{L2}^i, v a_2^i, v b_2^i, v q_{R2}^i\}$, $D_2^i = \{q_{L2}^i x_k, q_{R2}^i x_k, z_2^i a_2^i, a_2^i b_2^i\}$.

We say that x_k and z_2^i are *connected* by a different side gadget. On the other hand, if $l_2^i = \bar{x}_k$, then the following nodes and edges are added to construct a *same side* gadget: $V_2^i = \{a_2^i\}$, $B_2^i = \{a_L a_2^i, a_R a_2^i, z_2^i a_2^i, v_L a_2^i, v a_2^i\}$, $D_2^i = \{z_2^i x_k, x_k a_2^i\}$. We say that x_k and z_2^i are *connected* by a same side gadget.

Finally, if $l_3^i = x_k$, then the following nodes and edges are added to construct a *same side* gadget: $V_3^i = \emptyset$, $B_3^i = \emptyset$, $D_3^i = \{z_3^i x_k\}$. We say that x_k and z_3^i are *connected* by a same side gadget. On the other hand, if $l_3^i = \bar{x}_k$, then the following nodes and edges are added to construct a *different side* gadget: $V_3^i = \{q_{L3}^i, q_{R3}^i\}$, $B_3^i = \{s_L q_{L3}^i, s_R q_{R3}^i, z_3^i q_{L3}^i, z_3^i q_{R3}^i, v_L q_{L3}^i, v q_{R3}^i\}$, $D_3^i = \{x_k q_{L3}^i, x_k q_{R3}^i\}$. We say that x_k and z_3^i are *connected* by a different side gadget.

Lemmas 2 and 3 prove the required equivalence for establishing Theorem 1.

Lemma 2. *If the particular instance* (V,E^1,E^3) *of* GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION *constructed above admits a* 1*-join composition* $G=(V,E)$ *such that* $E^1 \subseteq E$ *and* $E\cap E^3=\emptyset$, *then there exists a truth assignment that satisfies* (X,C).

Proof. Suppose there exists a sandwich graph $G=(V,E)$ that has a 1-join composition $V_L|V_R$, where V_L denotes the side of the partition that contains the node v_L. Let S_L, A_L, S_R and A_R be the special sets of this 1-join composition.

Claim 1. $s_L\in S_L$, $a_L\in A_L$, $s_R\in S_R$, *and* $a_R\in A_R$.

Proof. We first show that $s_L,a_L\in V_L$ and $s_R,a_R\in V_R$ by considering two cases.

Case 1: $v\in V_L$. Since $\{w\colon wv_L\in E^1\}\cup\{w\colon wv\in E^1\}=V\setminus\{s_R\}$ and $S_R\neq\emptyset$, we have $S_R=\{s_R\}$. Since $s_R\in S_R$ and $s_Ra_R\in E^1$, we have $a_R\in V_R$. Since $\{w\colon wa_R\in E^1\}\cup\{w\colon ws_L\in E^1\}\cup\{w\colon ws_R\in E^1\}=V\setminus\{s_L\}$ and $S_L\neq\emptyset$, we have $s_L\notin V_R$ and hence $s_L\in V_L$. Since $a_Rv, a_Rv_L\in E^1$, we have v_L, $v\in A_L$. Since $a_Lv\in E^1$ and $a_Lv_L\in E^3$, we have $a_L\notin V_R$ and hence $a_L\in V_L$.

Case 2: $v\in V_R$. Since $v_Lv\in E^1$, we have $v_L\in A_L$ and $v\in A_R$. First suppose that $s_L\in V_R$. Since $\{w\colon wv\in E^1\}\cup\{w\colon ws_L\in E^1\}=V\setminus\{v,s_L,s_R\}$ and $S_L\neq\emptyset$, we have $S_L=\{s_R\}$. Since $s_Ra_R\in E^1$, we have $a_R\in V_L$. Since $a_Rv\in E^1$, we have $a_R\in A_L$. But then $s_Lv_L\in E^1$ and $s_La_R\in E^3$ contradict the assumption that $V_L|V_R$ is a 1-join composition. Therefore, $s_L\in V_L$. Since $s_Lv\in E^3$, we have $s_L\in S_L$. Since $a_Ls_L\in E^1$, we have $a_L\in V_L$. Suppose that $a_R\in V_L$. Since $a_Rv\in E^1$, we have $a_R\in A_L$. Since $s_Ra_R\in E^1$ and $s_Rv_L\in E^3$, we have $s_R\notin V_R$ and hence $s_R\in V_L$. But then, $\{w\colon wa_R\in E^1\}\cup\{w\colon ws_L\in E^1\}\cup\{w\colon ws_R\in E^1\}=V\setminus\{s_L\}$ implies that $S_R=\emptyset$, which contradicts the assumption that $V_L|V_R$ is a 1-join composition. Therefore $a_R\in V_R$. Since $a_Rv_L\in E^1$, we have $a_R\in A_R$. Since $s_Ra_R\in E^1$ and $s_Rv\in E^3$, we have $s_R\notin V_L$ and hence $s_R\in V_R$.

Therefore, $a_L,s_L\in V_L$ and $a_R,s_R\in V_R$. Since $a_La_R\in E^1$, we have $a_L\in A_L$ and $a_R\in A_R$. Since $s_La_R\in E^3$, we have $s_L\in S_L$. Since $s_Ra_L\in E^3$, we have $s_R\in S_R$. This ends the proof of Claim 1. □

Claim 2. *For* $i=1,\dots,m$ *and* $j=1,2,3$, *if* z_j^i *and* x_k, *for some* $k\in\{1,\dots,n\}$, *are connected by a same side gadget, then* z_j^i *and* x_k *are on the same side of* 1*-join composition* $V_L|V_R$ *of* G.

Proof. By Claim 1, $a_L\in A_L$ and $a_R\in A_R$. Since x_ka_L, $x_ka_R\in E^1$, we have $x_k\in A_L\cup A_R$. First suppose that $j=3$. Since $z_3^ia_L$, $z_3^ia_R\in E^1$, we have $z_3^i\in A_L\cup A_R$. Since $z_3^ix_k\in E^3$, we have that z_3^i and x_k must be on the same side of 1-join composition $V_L|V_R$. Now assume that $j=1$ or 2. Since $a_j^ia_L, a_j^ia_R\in E^1$, we have $a_j^i\in A_L\cup A_R$. Since $a_j^ix_k\in E^3$, we have that a_j^i and x_k must be on the same side of 1-join composition $V_L|V_R$. Since $z_j^ia_j^i\in E^1$ and $z_j^ix_k\in E^3$, we conclude that z_j^i must be on the same side of 1-join composition $V_L|V_R$ as a_j^i and x_k. This ends the proof of Claim 2. □

Claim 3. *For* $i=1,\ldots,m$ *and* $j=1,2,3$, *if* z^i_j *and* x_k, *for some* $k\in\{1,\ldots,n\}$, *are connected by a different side gadget, then* z^i_j *and* x_k *are on different sides of* 1-*join composition* $V_L|V_R$ *of* G.

Proof. By Claim 1, $s_L\in S_L$, $a_L\in A_L$, $s_R\in S_R$ and $a_R\in A_R$. Since $s_L\in S_L$ and $q^i_{Lj}s_L\in E^1$, we have $q^i_{Lj}\in V_L$. Since $s_R\in S_R$ and $q^i_{Rj}s_R\in E^1$, we have $q^i_{Rj}\in V_R$. Since $x_k a_L$, $x_k a_R\in E^1$, we have $x_k\in A_L\cup A_R$. First suppose that $j=3$. Since $z^i_3 a_L, z^i_3 a_R\in E^1$, we have $z^i_3\in A_L\cup A_R$. Since $q^i_{R3}z^i_3\in E^1$ and $q^i_{R3}x_k\in E^3$, it follows that z^i_3 and x_k cannot both be in V_L. Since $q^i_{L3}z^i_3\in E^1$ and $q^i_{L3}x_k\in E^3$, it follows that z^i_3 and x_k cannot both be in V_R. Hence, z^i_3 and x_k are on different sides of 1-join composition $V_L|V_R$. Now assume that $j=1$ or 2. By a similar argument as above, x_k and a^i_j must be on different sides of 1-join composition $V_L|V_R$. By a similar argument as in Claim 2 applied to vertices a^i_j, b^i_j and z^i_j, we have that a^i_j and z^i_j must be on the same side of 1-join composition $V_L|V_R$. Hence, x_k and z^i_j must be on different sides of 1-join composition $V_L|V_R$. This ends the proof of Claim 3. □

We now define the following truth assignment for (X,C): for $k=1,\ldots,n$, x_k is false if and only if vertex $x_k\in V_L$. By the construction of (V,E^1,E^3) and by Claims 2 and 3, for $i=1,\ldots,m$, literal l^i_1 is false if and only if $z^i_1\in V_L$, literal l^i_2 is false if and only if $z^i_2\in V_R$, and literal l^i_3 is false if and only if $z^i_3\in V_L$. Suppose that for some $i\in\{1,\ldots,m\}$, the clause $c_i=(l^i_1\vee l^i_2\vee l^i_3)$ is false. Then $z^i_1,z^i_3\in V_L$ and $z^i_2\in V_R$. By Claim 1, $a_L\in A_L$ and $a_R\in A_R$. Since $z^i_1 a_R$, $z^i_3 a_R\in E^1$, we have $z^i_1,z^i_3\in A_L$. But then, since $z^i_1 z^i_2\in E^3$ and $z^i_2 z^i_3\in E^1$, the assumption that $V_L|V_R$ is a 1-join composition of G is contradicted. Hence, the above defined truth assignment satisfies (X,C). This ends the proof of Lemma 2. □

The converse of Lemma 2 is given by Lemma 3.

Lemma 3. *If there exists a truth assignment that satisfies* (X,C), *then the particular instance* (V,E^1,E^3) *of* GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION *constructed above admits a* 1-*join composition* $G=(V,E)$ *such that* $E^1\subseteq E$ *and* $E\cap E^3=\emptyset$.

Proof. Suppose there is a truth assignment that satisfies (X,C). We shall define a partition of V into sets V_L,V_R that in turn defines a solution G for the particular instance (V,E^1,E^3) of GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION associated with the 3SAT instance (X,C).

Place vertices v_L, a_L and s_L in V_L, and vertices v, a_R and s_R in V_R. For $k=1,\ldots,n$, if x_k is false, then place it in V_L, and otherwise in V_R. For each clause c_i, $i=1,\ldots,m$, place the remaining vertices as follows: If $l^i_1=x_k$, then place z^i_1 and a^i_1 on the same side as x_k. If $l^i_1=\bar{x}_k$, then place q^i_{L1} in V_L, q^i_{R1} in V_R, and a^i_1,b^i_1 and z^i_1 on a different side from x_k. If $l^i_2=x_k$, then place q^i_{L2} in V_L, q^i_{R2} in V_R, and a^i_2,b^i_2 and z^i_2 on a different side from x_k. If $l^i_2=\bar{x}_k$, then place z^i_2 and a^i_2 on the same side as x_k. If $l^i_3=x_k$, then place z^i_3 on the same side as x_k. If $l^i_3=\bar{x}_k$, then place q^i_{L3} in V_L, q^i_{R3} in

$V_{\rm R}$ and z_3^i on a different side from x_k. Note that the above placement of vertices z_1^i, z_2^i and z_3^i, $i=1,\ldots,m$, implies that for each clause c_i, $i=1,\ldots,m$, literal l_1^i is false if and only if $z_1^i \in V_{\rm L}$, literal l_2^i is false if and only if $z_2^i \in V_{\rm R}$, and literal l_3^i is false if and only if $z_3^i \in V_{\rm L}$.

Sets $S_{\rm L}$ and $S_{\rm R}$ are defined as follows: Place $s_{\rm L}$ in $S_{\rm L}$ and $s_{\rm R}$ in $S_{\rm R}$. For $i=1,\ldots,m$ and $j=1,2,3$: if z_j^i is connected by a different side gadget, then if z_j^i is in $V_{\rm L}$ place $q_{{\rm L}j}^i$ in $S_{\rm L}$, and if z_j^i is in $V_{\rm R}$ place $q_{{\rm R}j}^i$ in $S_{\rm R}$. For $i=1,\ldots,m$, if z_1^i is in $V_{\rm R}$, then place z_1^i in $S_{\rm R}$, and if z_2^i and z_3^i are both in $V_{\rm R}$, then place z_2^i in $S_{\rm R}$.

Let $A_{\rm L} = V_{\rm L} \setminus S_{\rm L}$ and $A_{\rm R} = V_{\rm R} \setminus S_{\rm R}$. Let $E = E^1 \cup \{uy\colon u \in A_{\rm L} \text{ and } y \in A_{\rm R}\}$. To show that the graph $G=(V,E)$ is a sandwich graph with 1-join composition $V_{\rm L}|V_{\rm R}$, we need to show that the following types of edges do not exist: $uy \in E^3$ such that $u \in A_{\rm L}$ and $y \in A_{\rm R}$, $uy \in E^1$ such that $u \in S_{\rm L}$ and $y \in V_{\rm R}$, $uy \in E^1$ such that $u \in V_{\rm L}$ and $y \in S_{\rm R}$.

Let $e \in E^3$ and suppose that one endnode of e is in $A_{\rm L}$ and the other is in $A_{\rm R}$. Then neither $s_{\rm L}$ nor $s_{\rm R}$ is an endnode of e. Edge e cannot be of the form $a_{\rm L} z_1^i$, for some $i \in \{1,\ldots,m\}$, because $a_{\rm L} \in A_{\rm L}$ and if z_1^i is in $V_{\rm R}$ then z_1^i is in $S_{\rm R}$. Edge e cannot be of the form $z_1^i z_2^i$, for some $i \in \{1,\ldots,m\}$, because that would imply that z_1^i is in $V_{\rm L}$ (since if z_1^i is in $V_{\rm R}$ then it is in $S_{\rm R}$) and so z_2^i is in $A_{\rm R}$, which means that z_3^i is in $V_{\rm L}$ (since z_2^i and z_3^i are both in $V_{\rm R}$ then z_2^i is in $S_{\rm R}$), which would imply that all three literals in c_i are false. Let $i \in \{1,\ldots,m\}$. If z_1^i and x_k are connected by a same side gadget, then $e \notin D_1^i$ since x_k, z_1^i and a_1^i are all placed on the same side. If z_1^i and x_k are connected by a different side gadget, then a_1^i, b_1^i and z_1^i are all placed on the same side (hence $e \neq a_1^i b_1^i, z_1^i a_1^i$), z_1^i and x_k are placed on different sides, so if $x_k \in V_{\rm L}$ then $q_{\rm R1}^i \in S_{\rm R}$ (hence $e \neq x_k q_{\rm L1}^i, x_k q_{\rm R1}^i$), and if $x_k \in V_{\rm R}$ then $q_{\rm L1}^i \in S_{\rm L}$ (hence $e \neq x_k q_{\rm L1}^i, x_k q_{\rm R1}^i$). Therefore $e \notin D_1^i$. Similarly $e \notin D_2^i \cup D_3^i$.

Let $e \in E^1$ and suppose one endnode of e is in $S_{\rm L}$ and the other is in $V_{\rm R}$. If $s_{\rm L} u \in E^1$, then $u = a_{\rm L}, v_{\rm L}$ or $q_{{\rm L}j}^i$ for some $i \in \{1,\ldots,m\}$ and $j \in \{1,2,3\}$. Since all these nodes are in $V_{\rm L}$, edge e cannot have $s_{\rm L}$ as an endnode. Thus an endnode of e is $q_{{\rm L}j}^i$ for some $i \in \{1,\ldots,m\}$ and $j \in \{1,2,3\}$. But then $q_{{\rm L}j}^i \in S_{\rm L}$, hence $z_j^i \in V_{\rm L}$ and so $e \neq q_{\rm L3}^i z_3^i, q_{\rm L2}^i a_2^i, q_{\rm L1}^i a_1^i$.

Let $e \in E^1$ and suppose that one endnode of e is in $S_{\rm R}$ and the other is in $V_{\rm L}$. If $s_{\rm R} u \in E^1$, then $u = a_{\rm R}$ or $q_{{\rm R}j}^i$ for some $i \in \{1,\ldots,m\}$ and $j \in \{1,2,3\}$. Since all these nodes are in $V_{\rm R}$, edge e cannot have $s_{\rm R}$ as an endnode. Suppose that for some $i \in \{1,\ldots,m\}$ and $j \in \{1,2,3\}$, $q_{{\rm R}j}^i$ is an endnode of e. Then $q_{{\rm R}j}^i \in S_{\rm R}$, hence $z_j^i \in V_{\rm R}$ and so $e \neq q_{\rm R1}^i a_1^i, q_{\rm R2}^i a_2^i, q_{\rm R3}^i z_3^i$. Suppose that for some $i \in \{1,\ldots,m\}$, $z_1^i \in S_{\rm R}$ and that z_1^i is an endnode of e. Then $e \neq z_1^i a_{\rm R}, z_1^i v$, since $a_{\rm R}, v \in V_{\rm R}$. If z_1^i and x_k are connected by a same side gadget, then $e \neq z_1^i a_1^i$, since z_1^i and a_1^i are placed on the same side. If z_1^i and x_k are connected by a different side gadget, then $e \neq z_1^i b_1^i$, since z_1^i and b_1^i are placed on the same side. Thus we may assume that for some $i \in \{1,\ldots,m\}$, $z_2^i \in S_{\rm R}$ and that z_2^i is an endnode of e. Then $z_3^i \in V_{\rm R}$ and hence $e \neq z_2^i z_3^i$. Since $a_{\rm R}, v \in V_{\rm R}$, we have $e \neq z_2^i a_{\rm R}, z_2^i v$. If z_2^i and x_k are connected by a same side gadget, then $e \neq z_2^i a_2^i$, since z_2^i and a_2^i are placed on the same side. If z_2^i and x_k are connected by a different side gadget, then $e \neq z_2^i b_2^i$, since z_2^i and b_2^i are placed on the same side. □

Note that the particular instances (V, E^1, E^3) of GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION constructed in the proof of Theorem 1 always admit a solution for GRAPH SANDWICH PROBLEM FOR HOMOGENEOUS SET by defining $H = \{v_L, a_L\}$. Therefore, we have established that GRAPH SANDWICH PROBLEM FOR 1-JOIN COMPOSITION is NP-complete, even when restricted to instances that admit a solution for the homogeneous set sandwich problem.

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