# Perfect, Ideal and Balanced Matrices

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**Abstract**

In this paper, we survey results and open problems on perfect, ideal and balanced matrices. These matrices arise in connection with the set packing and set covering problems, two integer programming models with a wide range of applications. We concentrate on some of the beautiful polyhedral results that have been obtained in this area in the last thirty years. This survey first appeared in *Ricerca Operativa.*

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## 1 Introduction

The integer programming models known as set packing and set covering have a wide range of applications. As examples, the set packing model occurs in the phasing of traffic lights (Stoffer [38]), in pattern recognition (Lee, Shan and Yang [26]), and the set covering model in scheduling crews for railways, airlines, buses (Caprara, Fischetti and Toth [6]), location theory and vehicle routing. Sometimes, due to the special structure of the constraint matrix, the natural linear programming relaxation yields an optimal solution that is integer, thus solving the problem. We investigate conditions under which this integrality property holds.

Let $A$ be a $0,1$ matrix. This matrix is *perfect* if the fractional set packing polytope $\{x \geq 0 : \ Ax \leq 1\}$ only has integral extreme points. It is *ideal* if the fractional set covering polyhedron $\{x \geq 0 : \ Ax \geq 1\}$ only has integral extreme points. It is *balanced* if no square submatrix of odd order contains exactly two 1's per row and per column. The concepts of perfection and balancedness are due to Berge [1], [2]. The concept of idealness was introduced by Lehman [27] under the name of width-length property. Berge [3] proved that a matrix is balanced if and only if it and all its submatrices are perfect or, equivalently, if and only if it and all its submatrices are ideal.

When a polyhedron $Q \subseteq R^n_+$ only has integral extreme points, the linear program max $\{cx : \ x \in Q\}$ has an integral optimal solution $x$ for all $c \in R^n$ for which it has an optimal solution. Therefore, perfect, ideal and balanced matrices give rise to integer programs that can be solved as linear programs, for all objective functions.

Another interesting situation occurs when the dual linear program has an integral optimal solution for all integral objective functions for which it has an optimal solution: let $B$ be a matrix with integral entries. A linear system $Bx \leq b, \ x \geq 0$ is *totally dual integral* (TDI) if the linear program max $\{cx : \ Bx \leq b, \ x \geq 0\}$ has an integral optimal dual solution $y$ for all $c \in Z^n$ for which it has an optimal solution. Edmonds and Giles [18] proved that, if the linear system $Bx \leq b, \ x \geq 0$ is TDI, then the polyhedron $\{x : \ Bx \leq b, \ x \geq 0\}$ only has integral extreme points.

## 2 Perfect Matrices

**Theorem 2.1** *(Lovász [29])*
*For a 0,1 matrix $A$, the following statements are equivalent:*

*(i)* *the linear system* $Ax \leq 1,\ x \geq 0$ *is TDI,*

*(ii)* *the matrix $A$ is perfect,*

*(iii)* *max* $\{cx :\ Ax \leq 1,\ x \geq 0\}$ *has an integral optimal solution $x$ for all* $c \in R^n$*,*

*(iv)* *max* $\{cx :\ Ax \leq 1,\ x \geq 0\}$ *has an integral optimal solution $x$ for all* $c \in \{0,1\}^n$*.*

Clearly (i) implies (ii) implies (iii) implies (iv), where the first implication is the Edmonds-Giles property and the other two are immediate. What is surprising is that (iv) implies (i) and, in fact, that (ii) implies (i).

Graph theory is very relevant to the study of perfect matrices. In fact, historically, this is where the motivation for the study of perfection originated. In 1961, Berge [1] proposed a conjecture about graphs. In the same paper, Berge also proposed an elegant weakening of this conjecture, to entice researchers to investigate the topic. The weaker conjecture was proved in 1972 by Lovász and is known as the perfect graph theorem. This result implies Theorem 2.1 and, in fact, is equivalent to it, since the key step of the proof amounts to showing that (iv) implies (i) above. Berge's conjecture, known as the strong perfect graph conjecture, has been thoroughly investigated but is still unsettled.

In a graph, a *clique* is a set of pairwise adjacent nodes. The *chromatic number* is the smallest number of colours needed to colour the nodes so that adjacent nodes have distinct colours. Since all nodes of a clique must have a distinct colour, the chromatic number is always at least as large as the size of a largest clique. A graph $G$ is *perfect* if, for every node induced subgraph of $G$, the chromatic number equals the size of a largest clique. The perfect graph theorem states that a graph $G$ is perfect if and only if its complement $\bar{G}$ is perfect. (The *complement* of graph $G$ is the graph $\bar{G}$ having same node set and complement edge set.)

The connection between perfect graphs and perfect 0,1 matrices was studied by Fulkerson [20]. See also Chapter 9 in Schrijver [34]. The next theorem, due to Chvátal [7], gives a crisp statement of this connection. It is obtained through the concept of clique-node matrix of a graph. A *clique-node matrix* of a graph $G$ is a $0,1$ matrix whose columns are indexed by the nodes of $G$ and whose rows are the incidence vectors of the maximal cliques of $G$.

**Theorem 2.2** *(Chvátal [7])*
*A 0,1 matrix is perfect if and only if its nonredundant rows form the clique-node matrix of a perfect graph.*

A major open question is to characterize the graphs that are not perfect but all their proper node induced subgraphs are. These graphs are called *minimally imperfect.* A *hole* is a chordless cycle of length greater than three and it is *odd* if it contains an odd number of edges. Odd holes are minimally imperfect since their chromatic number is three and the size of the largest clique is two, but all proper induced subgraphs are bipartite and therefore perfect. If $G$ is minimally imperfect, then so is its complement, by Lovász's perfect graph theorem. In particular, complements of odd holes are minimally imperfect.

**Conjecture 2.3** *(Berge [1])*
*The odd holes and their complements are the only minimally imperfect graphs.*

Several classes of graphs that arise in applications, such as comparability graphs and interval graphs, are perfect. In these special cases, fast combinatorial algorithms have been developed for solving the set packing and related problems. A comprehensive collection of papers on perfect graphs can be found in the book edited by Berge and Chvátal [4].

# 3 Ideal Matrices

In a network with source $s$ and destination $t$, a path from $s$ to $t$ is an *$st$-path* and an edge set disconnecting $s$ from $t$ is an *$st$-cut.* It is easy to see that the product of the minimum number of edges in an $st$-path by the minimum number of edges in an $st$-cut is at most equal to the total number of edges in the network. This length-width inequality can be generalized to any nonnegative edge lengths $\ell_e$ and widths $w_e$: the minimum length of an $st$-path times the minimum width of an $st$-cut is at most equal to the scalar product $\ell w$. This length-width inequality was observed by Moore and Shannon [31] and Duffin [17]. Construct two 0,1 matrices $A$ and $B$ as follows: the columns are indexed by the edges of the network, the rows of $A$ are all the incidence vectors of minimal $st$-paths and the rows of $B$ are all the incidence vectors of minimal $st$-cuts. Moore, Shannon and Duffin show that the width-length inequality implies that both $A$ and $B$ are ideal. Lehman [27] showed that ideal 0,1 matrices always come in pairs and that the width-length inequality is in fact a characterization of idealness. Another important result of Lehman about ideal 0,1 matrices is the following.

**Theorem 3.1** *(Lehman [28])*
*For a 0,1 matrix A, the following statements are equivalent:*

*(i) the matrix $A$ is ideal,*

*(ii) min $\{cx: \ Ax \geq 1, \ x \geq 0\}$ has an integral optimal solution $x$ for all $c \in R^n$,*

*(iii) min $\{cx: \ Ax \geq 1, \ x \geq 0\}$ has an integral optimal solution $x$ for all $c \in \{0, 1, +\infty\}^n$.*

Statements (i) and (ii) are equivalent by the definition of idealness and (ii) implies (iii) is immediate. The difficult part of Lehman's theorem is that (iii) implies (ii). Here, contrary to the situation for perfection, idealness of $A$ does not imply TDIness of the linear system $Ax \geq 1, \ x \geq 0$. This can be seen using $A = Q_6$ defined as follows:

$$Q_6 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Choosing $c = (1, 1, 1, 1, 1, 1)$, the unique optimal dual solution is $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Yet, it is easy to check that the polyhedron $\{x \geq 0 : \ Ax \geq 1\}$ only has integral extreme points.

Therefore, we are led to define another class of matrices: a $0, 1$ matrix $A$ has the *max flow min cut property* (MFMC property) if the linear system $Ax \geq 1, \ x \geq 0$ is TDI. Why is this property called MFMC? The reason is that, when $A$ is the incidence matrix of $st$-paths versus edges, min $\{cx : Ax \geq 1, x \geq 0\}$ is the classical minimum $st$-cut problem. Its dual is the maximum flow problem. The max flow min cut theorem of Ford and Fulkerson [19] states that, for all nonnegative integral vectors $c$, there exists a maximum flow that is all integral. By analogy, Seymour [36] coined the term MFMC property for any $0,1$ matrix $A$ with such a dual integrality property. By the theorem of Edmonds and Giles [18], if a $0,1$ matrix has the MFMC property, then it is ideal.

**Conjecture 3.2** *(Conforti and Cornuéjols [8])*
*For a 0,1 matrix A, the following statements are equivalent:*

*(i) the matrix A has the MFMC property,*

*(ii) min* $\{cx : Ax \geq 1, x \geq 0\}$ *has an integral optimal dual solution y for all* $c \in Z^n_+$*,*

*(iii) min* $\{cx : Ax \geq 1, x \geq 0\}$ *has an integral optimal dual solution y for all* $c \in \{0, 1, +\infty\}^n$*.*

Statements (i) and (ii) are equivalent by the definition of the MFMC property and (ii) implies (iii) is immediate. The difficult part of the conjecture is to show that (iii) implies (ii). This holds in an important special case, namely when $A$ has the *binary property*. This happens when any minimal $0,1$ vector that satisfies $Ax \geq 1$ has an odd intersection with every minimal row of $A$. Seymour [36] shows that the matrix $Q_6$ is the unique minimal violator of the MFMC property that has the binary property. It follows easily from this deep theorem that, when $A$ has the binary property, (iii) implies (ii) in Conjecture 3.2.

Similarly, to get a better understanding of idealness, one might try to list (or at least describe constructively) the minimal violators of this property. Specifically, a $0,1$ matrix $A$ is said to be *minimally nonideal* if the polyhedron $\{x \geq 0 : Ax \geq 1\}$ has a fractional extreme point but all polyhedra obtained from it by setting a variable $x_j$ equal to 0 or to 1 only have integral vertices. Lehman [27] gives three infinite classes of minimally nonideal matrices. But, as for minimally imperfect graphs, it is an open problem to list all the minimally nonideal matrices. In fact, the situation appears more complicated than for minimally imperfect graphs since, in addition to Lehman's three infinite classes, there is a host of small examples that are known. Cornuéjols and Novick [16] proved that there are exactly 10 minimally nonideal circulant matrices with $k$ consecutive 1's, $k \geq 3$ (a matrix is *circulant* if it is square and its rows are all the cyclic shifts of the first row). A minimally nonideal circulant matrix where the 1's are not consecutive is the Fano matrix:

$$F_7 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This example was already known to Lehman [27]. Recently, Lütolf and Margot [30] have enumerated all minimally nonideal matrices with up to 12 columns. They also found quite a number of examples with 14 and with 17 columns. However, only three 0,1 matrices with the binary property are known to be minimally nonideal: $F_7$, the incidence matrix $O_{K_5}$ of odd cycles versus edges in the complete graph $K_5$, and the incidence matrix $b(O_{K_5})$ of complements of cuts versus edges in $K_5$.

**Conjecture 3.3** *(Seymour [37])*
*Up to permutation of rows and columns, $F_7$, $O_{K_5}$ and $b(O_{K_5})$ are the only minimally nonideal matrices with the binary property.*

An important special case of this conjecture was solved recently by Guenin [22], namely when the matrix $A$ is the incidence matrix of odd cycles versus edges of a graph. Another recent attempt at Seymour's conjecture was made in [15].

Although a complete list of all minimally nonideal matrices currently seems out of reach, Lehman [28] gives striking properties of minimally nonideal matrices. First, he proves that their fractional set covering polyhedron has a unique fractional extreme point. He also proves that, except for the matrices in one of his infinite classes, the unique fractional extreme point for all other minimally nonideal matrices is of the form $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$. A similar theorem holds for minimally imperfect 0,1 matrices (see Padberg [33]).

## 4 Perfect and Ideal $0, \pm 1$ Matrices

The concepts of perfect and of ideal 0,1 matrices can be extended to $0, \pm 1$ matrices. Given a $0, \pm 1$ matrix $A$, denote by $n(A)$ the column vector whose $i^{th}$ component is the number of -1's in the $i^{th}$ row of matrix $A$. The $0, \pm 1$ matrix $A$ is *perfect* if its fractional generalized set packing polytope $\{x : Ax \leq 1 - n(A),\ 0 \leq x \leq 1\}$ only has integral extreme points. Similarly, the $0, \pm 1$ matrix $A$ is *ideal* if its fractional generalized set covering polytope $\{x :\ Ax \geq 1 - n(A),\ 0 \leq x \leq 1\}$ only has integral extreme points.

A matrix is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$. In particular, all entries are $0, \pm 1$. A milestone paper in the study of integer polyhedra is that of Hoffman and Kruskal [23], which characterizes totally unimodular matrices, see also Chapter 19 in Schrijver [34]. It follows from this characterization that a totally unimodular matrix is both perfect and ideal.

It is well known that several problems in propositional logic, such as SAT, MAXSAT and logical inference, can be written as integer programs of the form

$$min\{cx :\ Ax \geq 1 - n(A),\ x \in \{0,1\}^n\}.$$

These problems are NP-hard in general but they can be solved in polytime by linear programming when the corresponding $0, \pm 1$ matrix $A$ is ideal. In fact, in this case, SAT and logical inference can be solved very fast by unit resolution (see Section 3 of [10]).

Hooker [24] was the first to relate idealness of a $0, \pm 1$ matrix to that of a family of 0,1 matrices. A similar result for perfection was obtained by Conforti, Cornuéjols and de Francesco [11]. These results were strengthened by Guenin [21] for perfection and by Nobili and Sassano [32] for idealness. We present Guenin's theorem. The other results have a similar flavor.

Given a $0, \pm 1$ matrix $A$, its *completion* $A^*$ is the matrix obtained as follows: if two rows $a_i$ and $a_k$ satisfy $a_{ij} = -a_{kj} \neq 0$ for some $j$, and $a_{il}a_{kl} = 0$ for all $l \neq j$, then add row $a_i + a_k$ to $A$ (if it is not already present). Repeat the process until no more rows can be added. By construction $A^*$ is a $0, \pm 1$ matrix. Now construct

$$A^{**} = \begin{pmatrix} P & N \\ I & I \end{pmatrix}$$

where $P$ and $N$ are 0,1 matrices of the same dimensions as $A^*$ defined by $p_{ij} = 1$ if and only if $a^*_{ij} = 1$ and $n_{ij} = 1$ if and only if $a^*_{ij} = -1$, and $I$ is the identity matrix. Finally, we say that $A$ is *irreducible* if the polytope $\{x : \ Ax \leq 1 - n(A), \ 0 \leq x \leq 1\}$ is not entirely contained in one of the hyperplanes $x_j = 0$ or $x_j = 1$.

**Theorem 4.1** *(Guenin [21])*

*Let $A$ be an irreducible $0, \pm 1$ matrix. Then $A$ is perfect if and only if the 0,1 matrix $A^{**}$ is perfect.*

There is a connection between perfect $0, \pm 1$ matrices and perfect bidirected graphs, a concept introduced by Johnson and Padberg [25]. In this paper, they also propose a conjecture relating perfect bidirected graphs to perfect graphs. This conjecture has recently been proved by Sewell [35].

# 5 Balanced Matrices

Balanced 0,1 matrices come up in various ways in the context of facility location on trees (see Tamir [39], [40]).

Berge's motivation for introducing balancedness was to extend to hypergraphs the notion of bipartite graph. A $0, 1$ matrix $A$ can be viewed as the *node-edge matrix* of a hypergraph $H$: the nodes of the hypergraph $H$ correspond to the rows of $A$ and the edges correspond to the columns with edge $j$ containing node $i$ if and only if $a_{ij} = 1$. A hypergraph is *balanced* if its node-edge matrix is balanced. Therefore, a hypergraph is balanced if it has no odd cycle in which every edge contains exactly two nodes of the cycle. Berge defined a hypergraph to be *bicolorable* if its nodes can be partitioned into two classes, say red and blue, in such a way that every edge of cardinality two or greater contains at least one blue and at least one red node.

**Theorem 5.1** *(Berge [2])*

*A hypergraph is balanced if and only if all its node induced subhypergraphs are bicolorable.*

When specializing this theorem to graphs we get the well known and easy to prove result that a graph has no odd cycle if and only if it is bipartite. Several properties of bipartite graphs extend to balanced hypergraphs. For example, Berge and Las Vergnas [5] generalize König's theorem (see, for example, Theorem 3.1.11 in West [41]). A *matching* is a set of pairwise nonintersecting edges and a *transversal* is a node set that intersects all the edges.

**Theorem 5.2** *(Berge and Las Vergnas [5])*
*In a balanced hypergraph, the maximum cardinality of a matching equals the minimum cardinality of a transversal.*

Recently, Conforti, Cornuéjols, Kapoor and Vušković [13] have shown that the celebrated theorem of Hall (Theorem 3.1.7 in west [41]), about the existence of a perfect matching in a bipartite graph, also extends to balanced hypergraphs. (A matching is *perfect* if every node belongs to an edge of the matching.)

**Theorem 5.3** *(Conforti, Cornuéjols, Kapoor and Vušković [13])*
*A balanced hypergraph has no perfect matching if and only if there exist disjoint node sets $R$ and $B$ with $|R| > |B|$ and every edge contains at least as many nodes in $B$ as in $R$.*

Interestingly, the notion of balanced $0, 1$ matrix (motivated as a generalization of bipartite graphs) can itself be extended to $0, \pm 1$ matrices, while still preserving many important properties.

Specifically, a $0, \pm 1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and per column, the sum of the entries is a multiple of four. The class of balanced $0, \pm 1$ matrices also properly includes totally unimodular $0, \pm 1$ matrices.

A $0, \pm 1$ matrix $A$ is *bicolorable* if its rows can be partitioned into blue rows and red rows in such a way that every column with two or more nonzero entries, either contains two entries of opposite signs in rows of the same colour, or contains two entries of the same sign in rows of different colours. For a $0, 1$ matrix, this definition coincides with Berge's notion of bicolorable hypergraph.

**Theorem 5.4** *(Conforti and Cornuéjols [9])*
*A $0, \pm 1$ matrix is balanced if and only if all its row submatrices are bicolorable.*

Berge's theorem [3] stated in the introduction also extends.

**Theorem 5.5** *(Conforti and Cornuéjols [9])*
*Let $A$ be a $0, \pm 1$ matrix. Then the following statements are equivalent.*

*(i) the matrix $A$ is balanced,*

*(ii) every submatrix of $A$ is perfect,*

*(iii) every submatrix of $A$ is ideal.*

# 6 Recognition

Given a $0, 1$ matrix $A$, are there polytime algorithms to recognize whether $A$ is perfect, ideal or balanced? No such algorithm is known for perfection or idealness. However, Conforti, Cornuéjols and Rao [14] give a polytime recognition algorithm for balancedness. The situation is the same for $0, \pm 1$ matrices: no algorithm is known for checking that a $0, \pm 1$ matrix is perfect or ideal, but Conforti, Cornuéjols, Kapoor and Vušković [12] give a polytime algorithm for checking balancedness. The algorithm is complicated and its computational complexity, although polynomial, is rather high. Nevertheless, the basic idea underlying it is very simple. The algorithm is based on a theorem stating that, if a matrix is balanced, then either it belongs to a "basic class" or else it can be decomposed into two smaller matrices using a well-defined "decomposition operation". Based on this theorem, the recognition algorithm recursively decomposes the matrix until no further decomposition exists. Then each of the final blocks is checked for balancedness. For this approach to work, one must be able to recognize "basic" balanced matrices in polytime and the "decomposition operation" must have three properties: (i) the fact that a matrix can be decomposed must be detected in polytime, (ii) the two blocks of the decomposition should be balanced if and only if the original matrix is balanced, and (iii) the total number of blocks generated in the algorithm must be polynomial.

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