Chromatic index of graphs with no cycle with a unique chord

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A B S T R A C T

The class C of graphs that do not contain a cycle with a unique chord was recently studied by Trotignon and Vušković (in press) [23], who proved for these graphs strong structure results which led to solving the recognition and vertex-colouring problems in polynomial time. In the present paper, we investigate how these structure results can be applied to solve the edge-colouring problem in the class. We give computational complexity results for the edge-colouring problem restricted to C and to the subclass C′ composed of the graphs of C that do not have a 4-hole. We show that it is NP-complete to determine whether the chromatic index of a graph is equal to its maximum degree when the input is restricted to regular graphs of C with fixed degree Δ ≥ 3. For the subclass C′, we establish a dichotomy: if the maximum degree is Δ = 3, the edge-colouring problem is NP-complete, whereas, if Δ ≠ 3, the only graphs for which the chromatic index exceeds the maximum degree are the odd holes and the odd order complete graphs, a characterization that solves edge-colouring problem in polynomial time. We determine two subclasses of graphs in C of maximum degree 3 for which edge-colouring is polynomial. Finally, we remark that a consequence of one of our proofs is that edge-colouring in NP-complete for r-regular tripartite graphs of degree Δ ≥ 3, for r ≥ 3.

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1. Motivation

Let G = (V, E) be a simple graph. The degree of a vertex v in G is denoted by deg G(v), and the maximum degree of a vertex in G is denoted by Δ(G). An edge-colouring of G is a function π : E → C such that no two adjacent edges receive the same colour c ∈ C. If C = {1, 2, . . . , k}, we say that π is a k-edge-colouring. The chromatic index of G, denoted by χ′(G), is the least k for which G has a k-edge-colouring.

Vizing’s theorem [24] states that χ′(G) = Δ(G) or Δ(G) + 1, defining the classification problem: graphs with χ′(G) = Δ(G) are said to be Class 1, while graphs with χ′(G) = Δ(G) + 1 are said to be Class 2. The edge-colouring problem or chromatic index problem is the problem of determining the chromatic index of a graph. Edge-colouring is a challenging topic in graph theory and the complexity of the problem is unknown for several important well-studied classes. Edge-colouring is NP-complete for regular graphs [13,17] of degree Δ ≥ 3. The problem is NP-complete also for the following classes [5]:

- r-regular comparability (hence perfect) graphs, for r ≥ 3;
- r-regular line graphs of bipartite graphs (hence line graphs and clique graphs), for r ≥ 3;
- r-regular k-hole-free graphs, for r ≥ 3, k ≥ 3;
- cubic graphs of girth k, for k ≥ 4.
Graph classes for which edge-colouring is polynomially solvable include the following:

- bipartite graphs [14];
- split-indifference graphs [19];
- series–parallel graphs (hence outerplanar) [14];
- k-outerplanar graphs [2], for k ≥ 1.

The complexity of edge-colouring is unknown for several well-studied strong structured graph classes, for which only partial results have been reported, such as cographs [1], join graphs [10,11,18], cobipartite graphs [18], planar graphs [21,25], chordal graphs, and several subclasses of chordal graphs such as indifference graphs [8], split graphs [7] and interval graphs [3].

Given a graph F, we say that a graph G contains F if graph F is isomorphic to an induced subgraph of G. A graph G is F-free if G does not contain F. A cycle C in a graph G is a sequence of vertices v1v2 . . . vnvn, that are distinct except for the first and the last vertex, such that for i = 1, . . . , n − 1, vi+1vi is an edge and vnvn is an edge — we call these edges the edges of C. An edge of G with both endvertices in a cycle C is called a chord of C if it is not an edge of C. One can similarly define a path and a chord of a path. A hole is a chordless cycle of length at least four and an ℓ-hole is a hole of length ℓ. A triangle is a cycle of length 3 and a square is a 4-hole.

Trotignon and Vušković [23] studied the class C̄ of graphs that do not contain a cycle with a unique chord. The main motivation to investigate this class was to find a structure theorem for it, a kind of result which is not very frequent in the literature. Basically, this structure result states that every graph in C̄ can be built starting from a restricted set of basic graphs and applying a series of known “gluing” operations. Another interesting property of this class is that it belongs to the family of the χ-bounded graphs, introduced by Gyárfás [12] as a natural extension of perfect graphs. A family of graphs Ḡ is χ-bounded with χ-binding function f if, for every induced subgraph G of Ḡ, χ(G) ≤ f(ω(G)), where χ(G) denotes the chromatic number of G and ω(G) denotes the size of a maximum clique in G. The research in this area is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded, see [20] for a survey. Note that perfect graphs are a χ-bounded family of graphs with χ-binding function f(x) = x, and perfect graphs are characterized by excluding odd holes and their complements. Also, by Vizing’s theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function f(x) = x + 1 [this special upper bound is known as the Vizing bound] and line graphs are characterized by nine forbidden induced subgraphs [26]. The class C̄ is also χ-bounded with the Vizing bound [23]. Also in [23] the following results are obtained for graphs in C̄: an O(nm) algorithm for optimal vertex-colouring, an O(n + m) algorithm for maximum clique, an O(nm) recognition algorithm, and the NP-completeness of the maximum stable set problem.

In the present paper we consider the complexity of determining the chromatic index of graphs in C̄. In particular, we investigate how structure results can be used to solve the edge-colouring problem. We also investigate the subclasses obtained from C̄ by forbidding 4-holes and/or 6-holes. Tables 1 and 2 summarize the main results achieved in the present work.

The results of Tables 1 and 2 show that, even for graph classes with strong structure and powerful decompositions, the edge-colouring problem may be difficult.

The class initially investigated in this work is the class C of graphs with no cycle with a unique chord. Each non-basic graph in this class can be decomposed [23] by special cutsets: 1-cutsets, proper 2-cutsets or proper 1-joins. We prove that edge-colouring is NP-complete for graphs in C. We consider, then, a subclass C′ ⊂ C whose graphs are the graphs in C that do not have a 4-hole. By forbidding 4-holes we avoid decompositions by joins, which are difficult to deal with in edge-colouring [1,10,11]. That is, each non-basic graph in C′ can be decomposed of 1-cutsets and proper 2-cutsets. For this class C′ we establish a dichotomy: edge-colouring is NP-complete for graphs in C′ with maximum degree 3 and polynomial for graphs in C′ with maximum degree not 3. We determine also a necessary condition for a graph G ∈ C′ of maximum degree 3 to be Class 1. This condition is having graph P∗ – a subgraph of the Petersen graph – as a basic block in the decomposition

### Table 1

<table>
<thead>
<tr>
<th>Class</th>
<th>Δ = 3</th>
<th>Δ ≥ 4</th>
<th>Regular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphs of C</td>
<td>NP-complete</td>
<td>NP-complete</td>
<td>NP-complete</td>
</tr>
<tr>
<td>4-hole-free graphs of C</td>
<td>NP-complete</td>
<td>Polynomial</td>
<td>Polynomial</td>
</tr>
<tr>
<td>6-hole-free graphs of C</td>
<td>NP-complete</td>
<td>NP-complete</td>
<td>NP-complete</td>
</tr>
<tr>
<td>[4-hole, 6-hole]-free graphs of C</td>
<td>Polynomial</td>
<td>Polynomial</td>
<td>Polynomial</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Class</th>
<th>k ≤ 2</th>
<th>k ≥ 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>k-partite graphs</td>
<td>Polynomial</td>
<td>NP-complete</td>
</tr>
</tbody>
</table>
tree. As a consequence, if both 4-holes and 6-holes are forbidden, the chromatic index of graphs with no cycle with a unique chord can be determined in polynomial time. The results achieved in this work have connections with other areas of research in edge-colouring, as we describe in the following three observations.

The first observation refers to the complexity dichotomy result found for class $C'$. This dichotomy presents great interest since, to the best of our knowledge, this is the first class for which edge-colouring is NP-complete for graphs with a given fixed maximum degree $\Delta$ and is polynomial for graphs with maximum degree $\Delta' > \Delta$, as the reader may verify in the NP-completeness results reviewed in the beginning of the present section. It is interesting to observe that the only regular graphs in $C'$ are the Petersen graph, the Heawood graph, the complete graphs, and the holes. As a consequence, edge-colouring is NP-complete when restricted to $C'$, but polynomial when restricted to regular graphs in $C'$.

The second observation is related to the study of snarks [22]. A snark is a cubic bridgeless graph with chromatic index 4. In order to avoid trivial (easy) cases, snarks are commonly restricted to have girth 5 or more and not to contain three edges whose deletion results in a disconnected graph, each of whose components is non-trivial. The study of snarks is closely related to the Four Colour Theorem. By the result of Lemma 8, the only non-trivial snark which has no cycle with a unique chord is the Petersen graph.

Finally, the third observation refers to the problem of determining the chromatic index of a $k$-partite graph, that is, a graph whose vertices can be partitioned into $k$ stable sets. The problem is known to be polynomial [14,16] for $k = 2$ and for complete multipartite graphs. However, there is no explicit result in the literature regarding the complexity of determining the chromatic index of a $k$-partite graph for $k \geq 3$. From the proof of Theorem 2 we can observe that edge-colouring is NP-complete for $k$-partite $r$-regular graphs, for each $k \geq 3$, $r \geq 3$.

The remainder of the paper is organized as follows. In Section 2, we prove NP-completeness results regarding edge-colouring in the classes $C$ and $C'$, and in Section 3, we review known results on the structure of graphs in $C$ and obtain stronger structure results for graphs in $C'$. In Section 4 we show how to determine in polynomial time the chromatic index of a graph in $C'$ with maximum degree $\Delta \geq 4$. In Section 5 we further investigate graphs in $C'$ with maximum degree 3: we show that edge-colouring can be solved in polynomial time if the inputs are restricted to regular graphs of $C'$ and to 6-hole-free graphs of $C'$.

2. NP-completeness results

In this section, we state NP-completeness results on the edge-colouring problem restricted to the class $C$ of graphs that do not contain a cycle with a unique chord and to the class $C'$ composed of the graphs in $C$ that do not contain a 4-hole. First, we prove that edge-colouring is NP-complete for regular graphs of $C$ with fixed degree $\Delta \geq 3$. We observe that it can be shown that the construction of Cai and Ellis [5] which proves the NP-completeness of $r$-regular $k$-hole-free graphs, for $r \geq 3$ and $k \neq 4$, creates a graph with no cycle with a unique chord. Nevertheless, in the present section, we give a simpler construction. Second, we prove that edge-colouring is NP-complete for graphs in $C'$ with maximum degree $\Delta = 3$. For the proof of this second result, we construct a replacement graph which is not present in any edge-colouring NP-completeness proof we could find in the literature.

We use the term $\text{CHRIND}(P)$ to denote the problem of determining the chromatic index restricted to graph inputs with property $P$. For example, $\text{CHRIND}(\text{graph of } C')$ denotes the following problem:

**INSTANCE:** a graph $G$ of $C$.

**QUESTION:** is $\chi' (G) = \Delta (G)$?

The following theorem [13,17] establishes the NP-completeness of determining the chromatic index of $\Delta$-regular graphs of fixed degree $\Delta$ at least 3:

**Theorem 1** [13,17]. For each $\Delta \geq 3$, $\text{CHRIND}(\Delta$-regular graph) is NP-complete.

Please refer to Fig. 1. Graph $Q_n$, for $n \geq 3$, is obtained from the complete bipartite graph $K_{n,n}$ by removing an edge $xy$, by adding new pendant vertices $u$ and $v$, and by adding pendant edges $ux$ and $vy$. Graph $Q'_n$ is obtained from $Q_n$, by identifying vertices $u$ and $v$ into a vertex $w$. Observe that $Q'_n$ is a graph of maximum degree $n$, and has $2n + 1$ vertices and $n^2 + 1$ edges. So, $Q'_n$ is overfull and, hence [9], Class 2. Lemma 1 investigates the properties of graph $Q_n$, which is used as “gadget” in the NP-completeness proof of Theorem 2.

**Lemma 1.** Graph $Q_n$ is $n$-edge-colourable, and in any $n$-edge-colouring of $Q_n$, edges $ux$ and $vy$ receive the same colour.

**Proof.** We use the notation from Fig. 1. First, we exhibit an $n$-edge-colouring of $Q_n$. Denote by $x_0, \ldots, x_{n-1}$ (resp. $y_0, \ldots, y_{n-1}$) the vertices of $Q_n$ which belong to the same partition as $x$ (resp. $y$), where $x = x_0$ (resp. $y = y_0$). An $n$-edge-colouring of $Q_n$ is constructed as follows: just let the colour of edge $x_0y_j$ be $(i + j \ mod \ n) + 1$ and let the colour of edges $x_0u$ and $y_0v$ be 1.

Now we prove that, in any $n$-edge-colouring of $Q_n$, edges $ux$ and $vy$ have the same colour. Suppose there is an $n$-edge-colouring $\pi$ of $Q_n$, where $ux$ and $vy$ have different colours. Consider the graph $Q'_n = (V', E')$ obtained from $Q_n$, by contracting vertices $u$ and $v$ into vertex $w$. Then we can construct an $n$-edge-colouring $\pi'$ of $Q'_n$ by setting $\pi'(e) = \pi(e)$ if $e \in E \setminus \{ux, vy\}, \pi'(ux) = \pi(ux)$ and $\pi'(vy) = \pi(vy)$, which is a contradiction to the fact that $Q'_n$ is Class 2. □

We prove in Theorem 2 the NP-completeness of edge-colouring regular graphs that do not contain a cycle with a unique chord for each fixed degree $\Delta \geq 3$. 
Theorem 2. For each $\Delta \geq 3$, $\text{CHRIND}(\Delta\text{-regular graph in } \mathcal{C})$ is NP-complete.

Proof. Let $G = (V, E)$ be an input of the NP-complete problem $\text{CHRIND}(\Delta\text{-regular graph}).$ Now, let $G'$ be the graph obtained from $G$ by removing each edge $pq \in E$ and adding a copy of $Q_\Delta$, identifying vertices $u$ and $v$ of $Q_\Delta$ with vertices $p$ and $q$ of $G$. For each edge $pq$ of $G$, denote $H_{pq}$ the subgraph of $G'$ isomorphic to $Q_\Delta$ whose pendant vertices are $p$ and $q$. Observe that $G'$ is also $\Delta$-regular.

Claim 1: $G'$ can be constructed in polynomial time from $G$. In fact, we make one substitution – by a copy of $Q_\Delta$ – for each edge of $G$, so that the construction time is linear on the number of edges of $G$.

Claim 2: if $G$ is $\Delta$-edge-colourable, then so is $G'$. Let $\pi'$ be a $\Delta$-edge-colouring of $G$. We construct a $\Delta$-edge-colouring $\pi'$ of $G'$ in the following way: for each edge $pq$ of $G$, let the edges of $H_{pq}$ in $G'$ be coloured in such a way that the pendant edges have the colour $\pi(pq)$ – this colouring exists and is described by Lemma 1.

Claim 3: if $G'$ is $\Delta$-edge-colourable, then so is $G$. Let $\pi'$ be a $\Delta$-edge-colouring of $G'$. We construct a $\Delta$-edge-colouring $\pi$ of $G$ as follows: let the colour in $\pi$ of each edge $pq$ of $G$ be equal to the colour in $\pi'$ of the pendant edges of $H_{pq}$ (by Lemma 1, these two pendant edges must receive the same colour).

Claim 4: $G' \in \mathcal{C}$. Suppose $G'$ has a cycle $C$ with a unique chord $\alpha \beta$. Observe that, by construction, every edge of $G'$ – and, in particular, chord $\alpha \beta$ – has both endvertices in the same copy of $Q_\Delta$. Denote by $H_{pq'}$ this copy and observe that cycle $C$, when restricted to $H_{pq'}$, is a path between $p'$ and $q'$, and that $\alpha \beta$ is a unique chord of this path. But there is no path with a unique chord between the pendant vertices of $Q_\Delta$, so that we have a contradiction. \hfill \Box

Observe that graph $G'$ in the proof of Theorem 2 is tripartite with vertex tripartition $(P_1, P_2, P_3)$ determined as follows:

- $P_1$ is the set whose elements are the original vertices of $G$ and the vertices denoted by $y_1, \ldots, y_\Delta$ in each copy of $Q_\Delta$;
- $P_2$ is the set whose elements are the vertices denoted by $x_0$ and $y_0$ in each copy of $Q_\Delta$;
- $P_3$ is the set whose elements are the vertices denoted by $x_1, \ldots, x_\Delta$ in each copy of $Q_\Delta$.

So, the following result holds:

Theorem 3. For each $k \geq 3, \Delta \geq 3$, $\text{CHRIND}(\Delta\text{-regular } k\text{-partite graph})$ is NP-complete.

We emphasize that $\mathcal{C}$ is a class with strong structure [23], yet, it is NP-complete for edge-colouring. We manage in Section 4 to define a subclass of $\mathcal{C}$ where edge-colouring is solvable in polynomial time. Consider the class $\mathcal{C}'$ as the subset of the graphs of $\mathcal{C}$ that do not contain a square. The structure of graphs in $\mathcal{C}'$ is stronger than that of graphs in $\mathcal{C}$, and is described in detail in Section 3. Yet, the edge-colouring problem is still NP-complete for inputs in $\mathcal{C}'$, as we prove next in Theorem 4. We recall that the proof of Cai and Ellis [5] for the NP-completeness of edge-colouring cubic square-free graphs generates a graph which has a cycle with a unique chord. In addition, remark that the gadget $Q_\Delta$ used in the proof of the NP-completeness of edge-colouring graphs with no cycle with a unique chord has a square. So, we need an alternative construction, which is based on the gadget $\bar{P}$ shown in Fig. 2. Graph $\bar{P}$ is constructed in such a way that the identification of its pendant vertices generates a graph isomorphic to $P^*$, the graph obtained from the Petersen graph by removing one vertex. Graph $P^*$ is a non-overfull Class 2 graph [15,6]. The properties of $\bar{P}$ with respect to edge-colouring are described in Lemma 2.

Lemma 2. Graph $\bar{P}$ is 3-edge-colourable, and in any 3-edge-colouring of $\bar{P}$, the edges $ux$ and $vy$ receive the same colour.
Theorem 5.1. \[ \text{CHRIND}(\text{graph in } C' \text{ with maximum degree } 3) \text{ is NP-complete.} \]

Proof. Fig. 2 shows a 3-edge-colouring of $\tilde{P}$ — observe that edges $ux$ and $vy$ receive the same colour.

The fact that edges $ux$ and $vy$ always receive the same colour is a consequence of $P^*$ being Class 2. The proof is similar to that of Lemma 1, except that gadget $\tilde{P}$ is used instead of $Q_3$. \[\□\]

3. Structure of graphs in $C$ and $C'$

The goal of the present section is to review structure results for the graphs in $C$ and obtain stronger results for the subclass $C'$. These results are used in Section 4 to edge-colour the graphs in $C'$ with maximum degree at least 4. In the present section we review the results of Trotignon and Vušković [23] on the structure of graphs in $C$ and obtain stronger results for graphs in $C'$.

Let $C$ be the class of the graphs that do not contain a cycle with a unique chord and let $C'$ be the class of the graphs of $C$ that do not contain a square. Trotignon and Vušković give a decomposition result [23] for graphs in $C$ and graphs in $C'$ in the following form: every graph in $C$ or in $C'$ either belongs to a basic class or has a cutset. Before we can state these decomposition theorems, we define the basic graphs and the cutsets used in the decomposition.

The Petersen graph is the graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that both $a_1a_2a_3a_5a_1$ and $b_1b_2b_3b_5b_1$ are chordless cycles, and such that the only edges between some $a_i$ and some $b_i$ are $a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3$. We denote by $P$ the Petersen graph and by $P^*$ the graph obtained from $P$ by removal of one vertex. Observe that $P \in C$.

The Heawood graph is a cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$ so that $a_1a_2 \ldots a_{14}a_1$ is a cycle, and such that the only other edges are $a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}$. We denote by $H$ the Heawood graph and by $H^*$ the graph obtained from $H$ by removal of one vertex. Observe that $H \in C$.

A graph is strongly 2-bipartite if it is square-free and bipartite with bipartition $(X, Y)$ where every vertex in $X$ has degree 2 and every vertex in $Y$ has degree at least 3. A strongly 2-bipartite graph is in $C$ because any chord of a cycle is an edge between two vertices of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

For the purposes of this work, a graph $G$ is called basic\(^1\) if

1. $G$ is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph; and
2. $G$ has no 1-cutset, proper 2-cutset or proper 1-join (all defined next).

We denote by $C_\emptyset$ the set of the basic graphs. Observe that $C_\emptyset \subseteq C$.

A cutset $S$ of a connected graph $G$ is a set of elements, vertices and/or edges, whose removal disconnects $G$. A decomposition of a graph is the removal of a cutset to obtain smaller graphs, called the blocks of the decompositions, by possibly adding some nodes and edges to connected components of $G \setminus S$. The goal of decomposing a graph is trying to solve a problem on the whole graph by combining the solutions on the blocks. For a graph $G = (V, E)$ and $V' \subseteq V$, $G[V']$ denotes the subgraph of $G$ induced by $V'$. The following cutsets are used in the known decomposition theorems of the class $C$ [23]:

\(^1\)By the definition of [23], a basic graph is not, in general, indecomposable. However, our slightly different definition helps simplifying some of our proofs.
A 1-cutset of a connected graph $G = (V, E)$ is a node $v$ such that $V$ can be partitioned into sets $X$, $Y$ and $\{v\}$, so that there is no edge between $X$ and $Y$. We say that $(X, Y, v)$ is a split of this 1-cutset.

A proper 2-cutset of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes $a$, $b$, both of degree at least three, such that $V$ can be partitioned into sets $X$, $Y$ and $\{a, b\}$ so that $|X| \geq 2$, $|Y| \geq 2$; there is no edge between $X$ and $Y$, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an $ab$-path. We say that $(X, Y, a, b)$ is a split of this proper 2-cutset.

A 1-join of a graph $G = (V, E)$ is a partition of $V$ into sets $X$ and $Y$ such that there exist sets $A$, $B$ satisfying:

- $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
- $|X| \geq 2$ and $|Y| \geq 2$;
- there are all possible edges between $A$ and $B$;
- there is no other edge between $X$ and $Y$.

We say that $(X, Y, A, B)$ is a split of this 1-join.

A proper 1-join is a 1-join such that $A$ and $B$ are stable sets of $G$ of size at least two.

We can now state a decomposition result for graphs in $\mathcal{C}$:

**Theorem 5 (Trotignon and Vušković [23]).** If $G \in \mathcal{C}$ is connected then either $G \in \mathcal{C}_B$ or $G$ has a 1-cutset, or a proper 2-cutset, or a proper 1-join.

The block $G_X$ (resp. $G_Y$) of a graph $G$ with respect to a 1-cutset with split $(X, Y, v)$ is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$). The block $G_X$ (resp. $G_Y$) of a graph $G$ with respect to a 1-join with split $(X, Y, A, B)$ is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node $y$ complete to $A$ (resp. $x$ complete to $B$). Nodes $x$, $y$ are called markers of their respective blocks.

The blocks $G_X$ and $G_Y$ of a graph $G$ with respect to a proper 2-cutset with split $(X, Y, a, b)$ are defined as follows. If there exists a node $c$ of $G$ such that $N_2(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block $G_X$ (resp. $G_Y$) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new node $c$ adjacent to $a$, $b$. Node $c$ is called the marker of the block $G_X$ (resp. $G_Y$).

The blocks with respect to 1-cutsets, proper 2-cutsets and proper 1-joins are constructed in such a way that they remain in $\mathcal{C}$, as shown by Lemma 3.

**Lemma 3 (Trotignon and Vušković [23]).** Let $G_X$ and $G_Y$ be the blocks of decomposition of $G$ with respect to a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.

Observe that the Petersen graph and the Heawood graph may appear as a block of decomposition with respect to a proper 1-join, as shown in Fig. 3. However, these graphs cannot appear as a block of decomposition with respect to a proper 2-cutset, because they have no vertex with degree 2 to play the role of a marker.

Despite the fact that the Petersen graph and the Heawood graph do not appear as a block of decomposition with respect to a proper 2-cutset, they must be listed as basic blocks, because these graphs, themselves, are in $\mathcal{C}$. So, the Petersen graph (resp. the Heawood graph) appears as a leaf of exactly one decomposition tree, namely, the decomposition tree of the Petersen graph (resp. the Heawood graph), itself — which is, actually, a trivial decomposition tree. Observe that graphs $P^*$ (Petersen graph minus one vertex) and $H^*$ (Heawood graph minus one vertex) may appear as a block with respect to a proper 2-cutset decomposition, as shown in Fig. 4.

We reviewed results that show how to decompose a graph of $\mathcal{C}$ into basic blocks: Theorem 5 states that each graph in $\mathcal{C}$ has a 1-cutset, a proper 2-cutset or a proper 1-join, while Lemma 3 states that the blocks generated with respect to any
Let \( G \) be a connected graph with a 1-cutset with split \( X, Y, v \). The chromatic index of \( G \) is \( \chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)\} \).

By Observation 1, if both blocks \( G_X \) and \( G_Y \) are \( \Delta(G) \)-edge-colourable, then so is \( G \). That is, once we know the chromatic index of the biconnected components of a graph, it is easy to determine the chromatic index of the whole graph. So, we may focus our investigation on the biconnected graphs of \( C' \).

**Theorem 6** (Trotignon and Vušković [23]). If \( G \in C' \) is biconnected, then either \( G \in C_G \) or \( G \) has a proper 2-cutset.

**Theorem 6** is an immediate consequence of **Theorem 5**: since \( G \) has no 4-hole, \( G \) cannot have a proper 1-join, and since \( G \) is biconnected, \( G \) cannot have a 1-cutset.

Next, in Lemma 4, we show that the blocks of decomposition of a biconnected graph of \( C' \) with respect to a proper 2-cutset, are also biconnected graphs of \( C' \). The proof of Lemma 4 is similar to that of Lemma 5.2 of [23]. For the sake of completeness, the proof, which uses the result of **Theorem 7** below, is included here.

**Theorem 7** (Trotignon and Vušković [23]). Let \( G \in C \) be a connected graph. If \( G \) contains a triangle then either \( G \) is a complete graph, or some vertex of the maximal clique that contains this triangle is a 1-cutset of \( G \).

**Lemma 4.** Let \( G \in C' \) be a biconnected graph and let \( (X, Y, a, b) \) be a split of a proper 2-cutset of \( G \). Then both \( G_X \) and \( G_Y \) are biconnected graphs of \( C' \).

**Proof.** We first prove that \( G \) is triangle-free. Suppose \( G \) contains a triangle. Then, by **Theorem 7**, either \( G \) is a complete graph, which contradicts the assumption that \( G \) has a proper 2-cutset, or \( G \) has a 1-cutset, which contradicts the assumption that \( G \) is biconnected. So \( G \) is triangle-free, and hence by construction, both of the blocks \( G_X \) and \( G_Y \) are triangle-free.

Now we show that \( G_X \) and \( G_Y \) are square-free. Suppose w.l.o.g. that \( G_X \) contains a square \( C \). Since \( G \) is square-free, \( C \) contains the marker node \( M \), which is not a real node of \( G \), and \( C = MabM \), for some node \( z \in X \). Since \( M \) is not a real node of \( G \), we have \( \text{deg}_G(z) = 2 \), otherwise, \( z \) would be a marker of \( G_X \). Let \( z' \) be a neighbor of \( z \) distinct of \( a \) and \( b \). Since \( G \) is triangle-free, \( z' \) is not adjacent to \( a \) nor \( b \). Since \( z \) is not a 1-cutset, there exists a path \( P \) in \( G[X \cup \{a, b\}] \) from \( z' \) to \( a \) or \( b \). We choose \( z' \) and \( P \) subject to the minimality of \( P \). So, w.l.o.g., \( z' Pa \) is a chordless path. Note that \( b \) is not adjacent to the neighbor of \( a \) along \( P \) because \( G \) is triangle-free and square-free, so that \( z \) is the unique common neighbor of \( a \) and \( b \) in \( G \). So, by the minimality of \( P \), vertex \( b \) does not have a neighbor in \( P \). Now let \( Q \) be a chordless path from \( a \) to \( b \) whose interior is in \( Y \). So, \( bzz' PaQb \) is a cycle of \( G \) with a unique chord (namely \( az \)), contradicting the assumption that \( G \in C' \).

By **Lemma 3**, \( G_X \) and \( G_Y \) both belong to \( C \), and since \( G_X \) and \( G_Y \) are both square-free, it follows that \( G_X \) and \( G_Y \) both belong to \( C' \).

Finally we show that \( G_X \) and \( G_Y \) are biconnected. Suppose w.l.o.g. that \( G_X \) has a 1-cutset with split \( (A, B, v) \). Since \( G \) is biconnected and \( G[X \cup \{a, b\}] \) contains an \( ab \)-path, we have that \( v \neq M \), where \( M \) is the marker of \( G_X \). Suppose \( v = a \). Then, w.l.o.g., \( b \in B \), and \( (A, B, Y, a) \) is a split of a 1-cutset of \( G \), with possibly \( M \) removed from \( B \cup Y \). If \( M \) is not a real node of \( G \), contradicting the assumption that \( G \) is biconnected. So \( v \neq a \) and by symmetry \( v \neq b \). So \( v \in X \setminus \{M\} \). W.l.o.g. \( \{a, b, M\} \subseteq B \). Then \( (A, B, Y, v) \) is a split of a 1-cutset of \( G \), with possibly \( M \) removed from \( B \cup Y \) if \( M \) is not a real node of \( G \), contradicting the assumption that \( G \) is biconnected. \( \square \)
Lemma 3 Consider a graph $G$. Every biconnected graph $G$ is obtained from a proper 2-cutset (a, b). Observe that the marker vertices and their incident edges – identified by dashed lines – do not belong to the original graph.

Observe that Lemma 3 is somehow stronger than Lemma 4. While Lemma 3 states that a graph is in $C$ if and only if the blocks with respect to any cutset are also in $C$, Lemma 4 establishes only one direction: if a graph is a biconnected graph of $C'$, then the blocks with respect to any cutset are also biconnected graph of $C'$. For our goal of edge-colouring, there is no need of establishing the "only if" part. Anyway, it is possible to verify that, if both blocks $G_x$ and $G_y$ generated with respect to a proper 2-cutset of a graph $G$ are biconnected graphs of $C'$, then $G$ itself is a biconnected graph of $C'$.

Next lemma shows that every non-basic biconnected graph in $C'$ has a decomposition such that one of the blocks is basic.

**Lemma 5.** Every biconnected graph $G \in C' \setminus C_B$ has a proper 2-cutset such that one of the blocks of decomposition is basic.

**Proof.** By Theorem 6 $G$ has a proper 2-cutset. Consider all possible 2-cutset decompositions of $G$ and pick a proper 2-cutset $S$ that has a block of decomposition $B$ whose size is smallest possible. By Lemma 4, $B \in C'$ and is biconnected. So by Theorem 6, either $B$ has a proper 2-cutset or it is basic. We now show that in fact $B$ must be basic.

Let $(X, Y, a, b)$ be a split with respect to $S$. Let $M$ be the marker node of $G_x$, and assume w.l.o.g. that $B = G_x$. Suppose $G_x$ has a proper 2-cutset with split $(X_1, X_2, u, v)$. By minimality of $B = G_x$, $\{a, b\} \neq \{u, v\}$. Assume w.l.o.g. $b \notin \{u, v\}$. Note that since $\deg_{G_x}(u) \geq 3$ and $\deg_{G_x}(v) \geq 3$, it follows that $M \notin \{u, v\}$. Suppose $a \notin \{u, v\}$. Then w.l.o.g. $\{a, b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with $M$ removed if $M$ is not a real node of $G$, is a proper 2-cutset of $G$ whose block of decomposition $G_{x_2}$ is smaller than $G_x$, contradicting the minimality of $G_x = B$. Therefore $a \in \{u, v\}$. Then w.l.o.g. $\{b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with $M$ removed if $M$ is not a real node of $G$, is a proper 2-cutset of $G$ whose block of decomposition $G_{x_2}$ is smaller than $G_x$, contradicting the minimality of $G_x = B$. Therefore $G_x$ does not have a proper 2-cutset, and hence it is basic. \[\square\]

4. Chromatic index of graphs in $C'$ with maximum degree at least 4

The first NP-completeness result of Section 2 proves that edge-colouring is difficult for the graphs in $C$. We consider, further, the subclass $C'$ and verify that the edge-colouring problem is still NP-complete when restricted to $C'$. In the present section we apply the structure results of Section 3 to show that edge-colouring graphs in $C'$ of maximum degree $\Delta \geq 4$ is polynomial by establishing that the only Class 2 graphs in $C'$ are the odd order complete graphs. Remark that the NP-completeness holds only for 3-edge-colouring restricted to graphs in $C'$ with maximum degree 3.

We describe, next, the technique applied to edge-colour a graph in $C'$ by combining edge-colourings of its blocks with respect to a proper 2-cutset. Observe that the fact that a graph $F$ is isomorphic to a block $B$ obtained from a proper 2-cutset decomposition of $G$ does not imply that $G$ contains $F$: possibly $B$ is constructed by the addition of a marker vertex. This is illustrated in the example of Fig. 5, where $G$ is $P^*$-free, yet, graph $P^*$ appears as a block with respect to a proper 2-cutset of $G$.

The reader will also observe that it is not necessary that a block of decomposition of $G$ is $\Delta (G)$-edge-colourable in order that $G$ itself is $\Delta (G)$-edge-colourable: graph $G$ in Fig. 5 is 3-edge-colourable, while block $P^*$ is not. This is an important observation: possibly, the edges adjacent to a marker vertex of a block of decomposition are not real edges of the original graph, or are already coloured by an edge-colouring of another block, so that these edges do not need to be coloured.

**Observation 2.** Consider a graph $G \in C'$ with the following properties:

- $(X, Y, a, b)$ is a split of a proper 2-cutset of $G$;
- $G_y$ is obtained from $G_y$ by removing its marker if this marker is not a real vertex of $G$;

thatby, \( G \) is a strongly 2-bipartite graph. Let \( G \) be a bipartite graph and \( \mathcal{L} = \{ \mathcal{L}_v \}_{v \in E} \) be a collection of lists of colours which associates to each edge \( uv \in E \) a list \( \mathcal{L}_u \) of colours. If, for each edge \( uv \in E \), \( |\mathcal{L}_u| \geq \max\{ \deg_c(u), \deg_c(v) \} \), then there is an edge-colouring \( \pi \) of \( G \) such that, for each edge \( uv \in E \), \( \pi(uv) \in \mathcal{L}_u \).

We investigate, now, how to extend (\( \Delta(G) \)-edge-colour) a graph \( G \in \mathcal{E} \) by combining \( \Delta(G) \)-edge-colourings of its blocks with respect to a proper 2-cutset. More precisely, Lemma 6 shows how this can be done if one of the blocks is basic. Subsequently, we obtain, in Theorem 9 and Corollary 1, a characterization for graphs in \( \mathcal{E} \) of maximum degree at least 4 of its Class 2 graphs which establishes that edge-colouring is polynomial for these graphs.

Lemma 6. Let \( G \in \mathcal{E} \) be a graph of maximum degree \( \Delta \geq 4 \) and let \((X, Y, a, b)\) be a split of proper 2-cutset, in such a way that \( G_X \) is basic. If \( G_Y \) is \( \Delta \)-edge-colourable, then \( G \) is \( \Delta \)-edge-colourable.

Proof. Denote by \( M \) the marker vertex of \( G_X \) and let \( G_Y \) be obtained from \( G_Y \) by removing its marker if this marker is not a real vertex of \( G \). Since \( G_Y \) is a subgraph of \( G_X \), \( \pi \) is a \( \Delta \)-edge-colourable. Let \( \pi_Y \) be a \( \Delta \)-edge-colouring of \( G_Y \), i.e. a partial-edge-colouring of \( G_Y \), and let \( F_a \) and \( F_b \) be the sets of the free colours of \( a \) and \( b \), respectively, with respect to the partial-edge-colouring \( \pi_Y \). We show how to extend the partial-edge-colouring \( \pi_Y \) to \( G \), as described in Observation 2, that is, by colouring the edges of \( G_X \setminus M \). Since \( a \) and \( b \) are not adjacent, \( G_X \) is not a complete graph. Moreover, the block \( G_X \) cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and \( G_X \) has a marker vertex \( M \) of degree 2. So, \( G_X \) is isomorphic to an induced subgraph of \( P^* \), or to an induced subgraph of \( H^* \), or to a strongly 2-bipartite graph, or is a hole.

Case 1. \( G_X \) is a strongly 2-bipartite graph.

Since \( \deg_{G_Y}(M) = 2 \), vertex \( M \) belongs to the bipartition of \( G_X \) whose vertices have degree 2. So, vertices \( a \) and \( b \) belong to the bipartition of \( G_X \) whose vertices have degree larger than 2, and \( |F_a| \geq \deg_{G_X \setminus M}(a) \geq 2 \) and \( |F_b| \geq \deg_{G_X \setminus M}(b) \geq 2 \). Associate to each edge \( v \in E \) incident to \( a \) (resp. \( b \)) a list of colours equal to \( \mathcal{L}_u \) (resp. \( \mathcal{L}_v \)). To each of the other edges of \( G_X \setminus M \), associate list \( \{1, \ldots, \Delta\} \). Now, to each edge \( uv \in E \), \( |\mathcal{L}_u| \geq \max\{ \deg_{G_X \setminus M}(u), \deg_{G_X \setminus M}(v) \} \) and, by Theorem 8, there is an edge-colouring \( \pi_1 \) of \( G_X \setminus M \) from these lists. Finally, set \( \pi := \pi_1 \) for the edges of \( G_X \setminus M \).

Case 2. \( G_X \) is a hole.

In this case, \( G_X \setminus M \) is a path. Denote the vertices of \( G_X \setminus M \) by \( a = x_1, x_2, \ldots, x_k = b \), in such a way that \( x_1x_2\ldots x_k \) is a path. We now show that \( k \geq 4 \). Since \( a \) and \( b \) are not adjacent, \( k \geq 3 \). Suppose that \( k = 3 \). If \( M \) is a real node of \( G \), then \( G_X \) is a square-free graph, and hence \( G_X \setminus M = X \). But, then, \( |X| = 1 \), contradicting the definition of a proper 2-cutset. Therefore, \( k \geq 4 \).

Observe that there is at least one colour \( c_a \) in \( F_a \) and one colour \( c_b \) in \( F_b \). We construct a \( \Delta \)-edge-colouring \( \pi \) of \( G_X \setminus M \) by setting \( \pi(x_ix_j) := c_a \) and \( \pi(x_{k-i}x_k) := c_b \), and by colouring the other edges of \( G_X \setminus M \) as follows. If \( k = 4 \), let \( \pi(x_2x_3) \) be some colour in \( \{1, 2, 3\} \setminus \{c_a, c_b\} \), which is clearly a non-empty set. If \( k \geq 5 \), let \( \mathcal{L}_2 = \{L_2, L_3, \ldots, L_{k-2}\} \) be a collection which associates to each edge \( x_{i-1}x_i \) a list of colours \( L_i \) such that:

- \( L_i = \{1, 2, 3\} \setminus \{c_a\} \), for \( i = 2, 3, \ldots, k-3 \), and
- \( L_{k-2} = \{1, 2, 3\} \setminus \{c_b\} \).

Observe that \( G_X \setminus M, a, b \) is a path, hence bipartite of maximum degree 2, and that \( |L_i| \geq 2 \) for each \( i = 2, \ldots, k-2 \), so that by, Theorem 8, there is an edge-colouring \( \pi_2 \) of \( G_X \setminus M, a, b \) from the lists \( L_2 \). Moreover, this colouring creates no conflicts with the colours \( c_a \) of \( x_1x_2 \) and \( c_b \) of \( x_{k-1}x_k \), so that we can set \( \pi := \pi_2 \) for edges \( x_2x_3, x_3x_4, \ldots, x_{k-2}x_{k-1} \).
Case 3. \(G_X\) is an induced subgraph of the Heawood graph.

Observe that \(a\) and \(b\) have only \(M\) as common neighbor in \(G_X\), otherwise \(G_X\) has a square (recall that Heawood graph is square-free). We construct a 4-edge-colouring of \(G_X \setminus M\). Denote the neighbors of \(a\) (resp. \(b\)) in \(G_X \setminus M\) by \(a_1, \ldots, a_k\) (resp. \(b_1, \ldots, b_l\)), where \(x = deg_{G_X \setminus M}(a)\) (resp. \(y = deg_{G_X \setminus M}(b)\)). Note that \(x, y \in \{1, 2\}\). Observe that \(F_a\) (resp. \(F_b\)) contains at least \(x\) (resp. \(y\)) colours, which we denote by \(c_{a_1}, \ldots, c_{a_k}\) (resp. \(c_{b_1}, \ldots, c_{b_l}\)). Set the colour \(\pi\) of edge \(aa_i\) (resp. \(bb_j\)), for \(i = 1, \ldots, x\) (resp. for \(j = 1, \ldots, y\)), to \(c_{a_i}\) (resp. \(c_{b_j}\)). Now, associate to each incident edge to \(a\) and different from \(aa\) a list of colours \(\{1, 2, 3, 4\} \setminus \{c_{a_i}\}\). Similarly, associate to each incident edge to \(b\) and different of \(bb\) a list of colours \(\{1, 2, 3, 4\} \setminus \{c_{b_j}\}\). Finally, associate to each of the other edges of \(G_X \setminus \{M, a, b\}\) the list of colours \(\{1, 2, 3, 4\}\). Observe that \(G_X \setminus \{M, a, b\}\) is bipartite of maximum degree at most 3 and that each of the lists has 3 or 4 colours, so that, by Theorem 8, there is an edge-colouring \(\pi_3\) of \(G_X \setminus \{M, a, b\}\) from these lists, and we set \(\pi := \pi_3\) for the edges of \(G_X \setminus M\).

Case 4.a: \(G_X = P^*\).

Observe that there are at least two colours \(c_{a_1}, c_{a_2}\) in \(F_a\) and two colours \(c_{b_1}, c_{b_2}\) in \(F_b\), and that exactly one of the following three possibilities holds:

- \(|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0|\)
- \(|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1\)
- \(|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 2\).

In the three cases, it is possible to extend the \(\Delta\)-edge-colouring \(\pi_Y\) to \(G\) by colouring the edges of \(G_X \setminus M\), as it is shown on Fig. 6.

Case 4.b: \(G_X\) is a proper induced subgraph of \(P^*\).

We need to investigate which are the proper induced subgraphs of \(P^*\). We invite the reader to verify that, except for graph \(P^{**}\) shown on the left of Fig. 7, each proper induced subgraph of \(P^*\) either has a 1-cutset or a proper 2-cutset, and we do not consider it because \(G_X\) is assumed basic, or is a hole, which is already considered in Case 2.

There is only one possible choice for the marker \(M\) of \(G_X = P^{**}\), in the sense that, for any other choice of marker \(M'\), we have \(G_X \setminus M' = G_X \setminus M\). As in Case 4a, there are at least two colours \(c_{a_1}, c_{a_2}\) in \(F_a\) and two colours \(c_{b_1}, c_{b_2}\) in \(F_b\), and \(|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0\) or \(2\). In Fig. 7 we exhibit three edge-colourings for \(P^{**} \setminus M\), one for each possibility. □

Using Lemma 6 we can determine in polynomial time the chromatic index of the graphs of \(C^*\), as we show in Theorem 9 and its Corollary 1.

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**Fig. 6.** Extending the colouring to the edges of \(G\).

**Fig. 7.** Graph \(P^{**}\) and three edge-colourings of \(P^{**} \setminus M\) subject to each possible free colour restriction.
Theorem 9. If \( \lambda \) is an integer at least 4 and \( G \) is a connected non-complete graph of \( \mathcal{C}' \) with maximum degree \( \Delta(G) \leq \lambda \), then \( G \) is \( \lambda \)-edge-colourable.

Proof. We prove the theorem by induction. Let \( G \in \mathcal{C}' \) be a connected graph with \( k \) vertices such that \( \Delta(G) \leq \lambda \) and \( G \) is not a complete graph. By Theorem 6 either \( G \) is basic, or \( G \) has a 1-cutset, or \( G \) is biconnected and has a proper 2-cutset.

Suppose \( G \) is basic. If \( G \) is strongly 2-bipartite, then \( G \) is \( \lambda \)-edge-colourable because bipartite graphs are Class 1 and \( \Delta(G) \leq \lambda \). If \( G \) is not strongly 2-bipartite, then \( G \) is a hole or a subgraph of the Petersen graph or of the Heawood graph, so that \( \Delta(G) \leq 3 \leq \lambda - 1 \) and \( G \) is \( \lambda \)-edge-colourable by Vizing’s theorem. Assume, as induction hypothesis, that every connected non-complete graph \( G' \in \mathcal{C}' \) with \( k' < k \) vertices such that \( \Delta(G') \leq \lambda \) is \( \lambda \)-edge-colourable.

Suppose \( G \) has a 1-cutset with split \((X, Y, v)\). Note that blocks of decomposition \( G_X \) and \( G_Y \) are induced subgraphs of \( G \) and hence both belong to \( \mathcal{C}' \). If \( G_X \) (resp. \( G_Y \)) is complete, then its maximum degree is at most \( \lambda - 1 \), so that \( G_X \) (resp. \( G_Y \)) is \( \lambda \)-edge-colourable by Vizing's theorem. If \( G_X \) (resp. \( G_Y \)) is not complete, \( G_X \) (resp. \( G_Y \)) is \( \lambda \)-edge-colourable by the induction hypothesis. In any case, both \( G_X \) and \( G_Y \) are \( \lambda \)-edge-colourable, and hence by Observation 1, graph \( G \) is \( \lambda \)-edge-colourable.

Finally, suppose \( G \) is biconnected and has a proper 2-cutset. Let \((X, Y, a, b)\) be a split of a proper 2-cutset such that block \( G_X \) is basic (note that such a cutset exists by Lemma 5). By Theorem 7, block \( G_X \) is not a complete graph. By Lemma 4, block \( G_Y \) is \( \in \mathcal{C}' \). By the induction hypothesis, block \( G_Y \) is \( \lambda \)-edge-colourable. By Lemma 6, graph \( G \) is \( \lambda \)-edge-colourable.

Corollary 1. A connected graph \( G \in \mathcal{C}' \) of maximum degree \( \Delta \geq 4 \) is Class 2 if and only if it is an odd order complete graph.

Proof. If \( G \) is complete, then the result clearly holds. So, we may assume \( G \) is not complete. Just choose \( \lambda = \Delta \) in Theorem 9 to prove that every connected non-complete graph of \( \mathcal{C}' \) with maximum degree \( \Delta(G) \geq 4 \) is \( \lambda \)-edge-colourable, hence Class 1.

5. Graphs of \( \mathcal{C}' \) with maximum degree 3

Class \( \mathcal{C}' \) has a stronger structure than \( \mathcal{C} \), yet, edge-colouring problem is NP-complete for inputs in \( \mathcal{C}' \). In fact, the problem is NP-complete for graphs in \( \mathcal{C}' \) with maximum degree \( \Delta = 3 \). In this section, we further investigate graphs in \( \mathcal{C}' \) with maximum degree \( \Delta = 3 \), providing two subclasses for which edge-colouring can be solved in polynomial time: cubic graphs of \( \mathcal{C}' \) and 6-hole-free graphs of \( \mathcal{C}' \).

5.1. Cubic graphs of \( \mathcal{C}' \)

In the present section, we prove the polynomiality of the edge-colouring problem restricted to cubic graphs of \( \mathcal{C}' \). This is a direct consequence of Lemma 7, that states that every non-biconnected cubic graph is Class 2, and Lemma 8, that states that the Petersen graph is the only biconnected cubic Class 2 graph in \( \mathcal{C}' \). We remark that the bipartite Heawood graph and the complete graph on four vertices are both cubic Class 1 graphs.

Lemma 7. Let \( G \) be a connected cubic graph. If \( G \) has a 1-cutset, then \( G \) is Class 2.

Proof. Denote by \((X, Y, v)\) a split of a 1-cutset of \( G \). Observe that \( v \) has degree 1 in exactly one of the blocks \( G_X \) and \( G_Y \); assume, w.l.o.g. that this block is \( G_X \). Let \( G'_X \) be the graph obtained from \( G_X \) by removing vertex \( v \). Observe that \( G'_X \) has exactly one vertex of degree 2 and each of the other vertices has degree 3. Since the sum of the degrees of the vertices is even, \( G'_X \) has an even number of vertices of degree 3, say \( n \). So, the number of edges in \( G'_X \) is \((3n + 2)/2\). Since \( (n + 1)/2 < 3n/2 < (3n + 2)/2 \), graph \( G'_X \) is overfull and hence Class 2. Since \( G_X \) is a subgraph of \( G \), and both \( G'_X \) and \( G \) have maximum degree 3, the graph \( G \) is itself Class 2.

Lemma 8. Let \( G \in \mathcal{C}' \) be biconnected graph. If \( G \) is cubic, then \( G \) is isomorphic to the Petersen graph or to the Heawood graph or is a complete graph on four vertices.

Proof. Suppose \( G \) is not basic. By Lemma 5, \( G \) has a proper 2-cutset such that one of the blocks is basic. Let \((X, Y, a, b)\) be a split of such cutset, in such a way that \( G_X \) is basic, and denote by \( M \) the marker vertex of \( G_X \). If \( deg(G_X)(a) = 1 \), vertex \( M \) is the only neighbor of \( a \) and, clearly, is a 1-cutset of \( G_X \). By Lemma 4, \( G_X \) is a biconnected graph of \( \mathcal{C}' \). Since \( G_X \) is biconnected \( deg(G_X) \geq 2 \). Let \( a' \) be a neighbor of \( a \) in \( G_X \) that is distinct from \( M \). Since \( M \), \( a \), and \( a' \) cannot induce a square, \( b \) is not adjacent to \( a' \), and hence (since \( G \) is cubic) \( a' \) has two neighbors in \( G_X \setminus \{a, b, M\} \). If \( deg(G_X)(a') = 2 \), then \( a' \) is a proper 2-cutset of \( G \), contradicting the assumption that \( G_X \) is basic. Hence \( deg(G_X)(a) \geq 3 \), and by symmetry \( deg(G_X)(b) \geq 3 \). Observe that each of the other vertices – different from \( a, b \) and \( M \) – has degree \( \Delta(G) \). In other words, \( G_X \) is a graph with exactly one vertex of degree 2, and each of the other vertices has degree 3. But there is no graph in \( \mathcal{C}' \) with this property, and we have a contradiction to the fact that \( G_X \) is basic. So, \( G \) is basic and the statement of the lemma clearly holds.

Theorem 10. Let \( G \in \mathcal{C}' \) be a connected cubic graph. Then \( G \) is Class 1 if and only if \( G \) is biconnected and is not isomorphic to the Petersen graph.

Proof. If \( G \) is not biconnected, then, by Lemma 7, \( G \) is Class 2. If \( G \) is biconnected, then, by Lemma 8, \( G \) is isomorphic to the Petersen graph \( P \) or to the Heawood graph \( H \) or is a complete graph \( K_4 \) on four vertices. Remark that \( H \) is Class 1, because it is bipartite, and \( K_4 \) is Class 1, because it is a complete graph with even number of vertices. Hence, \( G \) is Class 2 if and only if it is isomorphic to the Petersen graph.
Finally, the last non-basic proper subgraph of $H^*$ is the graph $H_3$.  

5.2. 6-hole-free graphs of $C'$

In the present section, we prove the polynomiality of the edge-colouring problem restricted to 6-hole-free graphs of $C'$. This is a consequence of Lemma 9, a variation for 3-edge-colouring of Lemma 6.

Lemma 9. Let $G \in C'$ be a graph of maximum degree at most 3 and $(X, Y, a, b)$ be a split of a proper 2-cutset, in such a way that $G_X$ is basic but not isomorphic to $P^*$. If $G_Y$ is 3-edge-colourable, then $G$ is 3-edge-colourable.

Proof. Assume $G_Y$ is 3-edge-colourable. Denote by $M$ the marker vertex of $G_X$ and let $G_Y \setminus M$ be obtained from $G_Y$ by removing its marker if this marker is not a real vertex of $G$. Since $G_Y$ is a subgraph of $G_Y$, graph $G_Y$ is 3-edge-colourable. Let $\pi_Y$ be a 3-edge-colouring of $G_Y$, i.e., a partial-edge-colouring of $G$, and let $F_a$ and $F_b$ be the sets of the free colours of $a$ and $b$, respectively, with respect to the partial-edge-colouring $\pi_Y$. We show how to extend the partial-edge-colouring $\pi_Y$ to $G$, as described in Observation 2, that is, by colouring the edges of $G_X \setminus M$. Since $a$ and $b$ are not adjacent, $G_X$ is not a complete graph. Moreover, the block $G_X$ cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and $G_X$ has a marker vertex $M$ of degree 2. Also, by assumption, block $G_X$ is not isomorphic to $P^*$. So, $G_X$ is isomorphic to a proper induced subgraph of $P^*$, or to an induced subgraph of $H^*$, or to a strongly 2-bipartite graph, or is a hole.

Case 1. $G_X$ is a strongly 2-bipartite graph.

Case 2. $G_X$ is a hole.

Case 3. $G_X$ is an induced subgraph of $H^*$.

First, observe that $\deg_{G_X \setminus M}(a) = 2$ and $\deg_{G_X \setminus M}(b) = 2$, otherwise $G_X$ has a decomposition by a 1-cutset or a proper 2-cutset and is not basic. Observe also, that there are at least two colours $c_a$, $c_b$ in $F_a$ and two colours $c_1$, $c_2$ in $F_b$, and that $|\{c_a, c_b\} \cap \{c_1, c_2\}| = 1$ or 2. We consider each case next.

If $|\{c_a, c_b\} \cap \{c_1, c_2\}| = 1$, we must exhibit a 3-edge-colouring $\pi$ of $G_X \setminus M$ such that the free colours at $a$ and $b$ are different. If $M$ is a real node of $G$, then $G_X$ is an induced subgraph of $G$, and hence $\Delta(G_X) \leq 3$. If $M$ is not a real node of $G$, then by definition of proper 2-cutset both $a$ and $b$ have a neighbor in $Y$, and hence $\Delta(G_X) \leq 3$. So $\Delta(G_X) \leq 3$. Since $G_X$ is bipartite, $G_X$ has a 3-edge-colouring $\pi'$. So, let $\pi$ be the restriction of $\pi'$ to $G_X \setminus M$.

If $|\{c_a, c_b\} \cap \{c_1, c_2\}| = 2$, we must exhibit a colouring of $G_X \setminus M$ such that the free colours at $a$ and $b$ are the same. We exhibit these colourings for each possible induced subgraph of the Heawood graph. First, consider the case $G_X = H^*$, whose colouring is given in Fig. 8.

Now, observe that each non-basic proper subgraph of $H^*$ is a subgraph of the graph $H_1$, which is obtained from $H^*$ by removing a vertex of degree 2. Graph $H_2$ of Fig. 9 is obtained from $H_1$ by removing one of the four vertices of degree 2 (any choice yields the same graph up to an isomorphism). Finally, the last non-basic proper subgraph of $H^*$ is the graph $H_3$.  

Fig. 8. A 3-edge-colouring of $H^* \setminus M$ such that the sets of the colours incident to vertex $a$ and vertex $b$ are the same.

Fig. 9. Non-basic proper induced subgraphs of $H^*$. 

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of Fig. 9. Observe that there is only one possible choice $M$ for the marker when $G_X = H_1$, in the sense that, for any other choice $M$, we have $G_X \setminus M = G_X \setminus M$. If $G_X = H_2$, there are two possible choices $M'$ and $M''$ for the marker, in the sense that, for any other choice $M'$, we have $G_X \setminus M' = G_X \setminus M'$ or $G_X \setminus M' = G_X \setminus M''$. We show, in Fig. 10, one edge-colouring of $H_1 \setminus M$, and two edge-colourings of $H_2 \setminus M$, one for each possible choice of marker $M$. We do not consider here that case $G_X = H_3$ because $H_3$ is a strongly 2-bipartite graph, considered in Case 1.

Case 4. $G_X$ is a proper subgraph of $P^∗$.

As we already discussed in Case 4 of Lemma 6, except for graph $P^{**}$ shown on the left of Fig. 7, each of the other proper induced subgraphs of $P^*$ either has a 1-cutset or a proper 2-cutset, and we do not consider because $G_X$ is basic, or is a hole, which are considered in Case 2. There is only one possible choice of marker $M_1$ for the case $G_X = P^{**}$, in the sense that for any other choice of marker $M'$, we have $G_X \setminus M' = G_X \setminus M_1$. Observe, also, that there are at least two colours $c_{a_1}, c_{a_2}$ in $P_5$ and two colours $c_{b_1}, c_{b_2}$ in $P_6$, and that $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$ or 2. These two possibilities are considered in the first two colourings of Fig. 7. □

Remark that the NP-Complete gadget $\bar{P}$ of Fig. 2 is constructed from $P^*$. The NP-completeness of edge-colouring graphs in $C'$ is obtained as a consequence of $P^* \in C'$. Using Lemma 9, we can prove that if the special graph $P^*$ does not appear as a leaf in the decomposition tree, i.e., as a basic block when we recursively apply the proper 2-cutset decomposition to a biconnected graph $G$ in $C'$ of maximum degree 3, then $G$ is Class 1.

Theorem 11. Let $G \in C'$ be a connected graph of maximum degree 3. If $G$ does not contain a 6-hole all of whose nodes are of degree 3, then $G$ is Class 1.

Proof. Assume the theorem does not hold and let $G$ be a counterexample with fewest number of nodes. So $G$ is a connected graph of maximum degree 3, it does not contain a 6-hole all of whose nodes are of degree 3, and it is not 3-edge-colourable. By Theorem 6 either $G$ is basic, or it has a 1-cutset, or it is biconnected and has a proper 2-cutset.

Suppose $G$ is basic. $G$ cannot be strongly 2-bipartite nor an induced subgraph of Heawood graph, since bipartite graphs are Class 1 [26]. Graph $G$ cannot be a complete graph on four vertices, because such a graph is 3-edge-colourable. $G$ cannot be a hole since it has maximum degree 3. So $G$ must be an induced subgraph of the Petersen graph. $G$ cannot be isomorphic to $P$ nor $P^*$, because both of these graphs contain a 6-hole all of whose nodes are of degree 3. But all the other induced subgraphs of the Petersen graph are in fact 3-edge-colourable [6]. Therefore $G$ cannot be basic.

Now suppose that $G$ has a 1-cutset with split $(X, Y, v)$. Note that blocks of decomposition are induced subgraphs of $G$, and hence both are connected graphs of $C'$ that do not contain a 6-hole all of whose nodes are of degree 3. If $\Delta(G_X) = 3$ then since $G$ is a minimum counterexample, $G_X$ is 3-edge-colourable. If $\Delta(G_X) \leq 2$ then $G_X$ is 3-edge-colourable by Vizing’s theorem. So $G_X$ is 3-edge-colourable, and similarly so is $G_Y$. But then by Observation 1, $G$ is also 3-edge-colourable, a contradiction.

Therefore $G$ is biconnected and has a proper 2-cutset. Let $(X, Y, a, b)$ be a split of a proper 2-cutset such that block $G_X$ is basic (note that such a cutset exists by Lemma 5). By Lemma 4 both of the blocks $G_X$ and $G_Y$ are biconnected graphs of $C'$. Since the marker node $M$ is of degree 2 in both $G_X$ and $G_Y$, and $G_X \setminus M$ and $G_Y \setminus M$ are both induced subgraphs of $G$, it follows that neither $G_X$ nor $G_Y$ can contain a 6-hole all of whose nodes are of degree 3. If $M$ is a real node of $G$, then $G_X$ and $G_Y$ are both induced subgraphs of $G$, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. If $M$ is not a real node of $G$, then by definition of proper 2-cutset both $a$ and $b$ have a neighbour in both $X$ and $Y$, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. Since both $G_X$ and $G_Y$ have fewer nodes than $G$, it follows either from minimality of counterexample $G$ or by Vizing’s theorem that both $G_X$ and $G_Y$ are 3-edge-colourable. Since $G_X$ does not contain a 6-hole all of whose nodes are of degree 3, $G_X$ is not isomorphic to $P^*$, and hence by Lemma 9, $G$ is 3-edge-colourable, a contradiction. □

Corollary 2. Every connected 6-hole-free graph of $C'$ with maximum degree 3 is Class 1.

A natural question in connection with Theorem 12 is whether forbidding 6-holes would make it easier to edge-colour graphs of $C$, and the answer is no. By observing graph $G'$ of the proof of Theorem 2, one can easily verify that this graph has no 6-hole, so that the following theorem holds:

Theorem 12. For each $\Delta \geq 3$, $CHRIND(\Delta$-regular 6-hole-free graph in $C)$ is NP-complete.
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