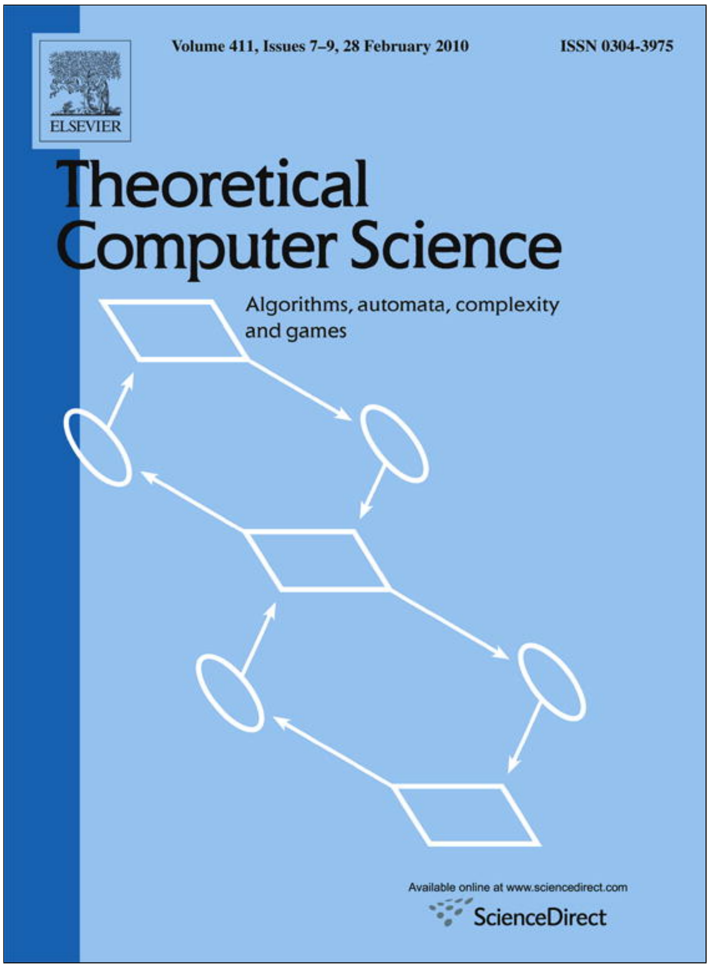
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Chromatic index of graphs with no cycle with a unique chord

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ABSTRACT

The class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord was recently studied by Trotignon and Vušković (in press) [23], who proved for these graphs strong structure results which led to solving the recognition and vertex-colouring problems in polynomial time. In the present paper, we investigate how these structure results can be applied to solve the edge-colouring problem in the class. We give computational complexity results for the edge-colouring problem restricted to $\mathcal{C}$ and to the subclass $\mathcal{C}'$ composed of the graphs of $\mathcal{C}$ that do not have a 4-hole. We show that it is NP-complete to determine whether the chromatic index of a graph is equal to its maximum degree when the input is restricted to regular graphs of $\mathcal{C}$ with fixed degree $\Delta \geq 3$. For the subclass $\mathcal{C}'$, we establish a dichotomy: if the maximum degree is $\Delta = 3$, the edge-colouring problem is NP-complete, whereas, if $\Delta \neq 3$, the only graphs for which the chromatic index exceeds the maximum degree are the odd holes and the odd order complete graphs, a characterization that solves edge-colouring problem in polynomial time. We determine two subclasses of graphs in $\mathcal{C}'$ of maximum degree 3 for which edge-colouring is polynomial. Finally, we remark that a consequence of one of our proofs is that edge-colouring in NP-complete for r-regular tripartite graphs of degree $\Delta \geq 3$, for $r \geq 3$.

1. Motivation

Let $G = (V, E)$ be a simple graph. The degree of a vertex v in G is denoted by $deg_G(v)$, and the maximum degree of a vertex in G is denoted by $\Delta(G)$. An *edge-colouring* of G is a function $\pi : E \to \mathbf{C}$ such that no two adjacent edges receive the same colour $c \in \mathbf{C}$. If $\mathbf{C} = \{1, 2, \ldots, k\}$, we say that π is a *k-edge-colouring*. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-colouring.

Vizing's theorem [24] states that $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$, defining the *classification problem*: graphs with $\chi'(G) = \Delta(G)$ are said to be *Class 1*, while graphs with $\chi'(G) = \Delta(G)+1$ are said to be *Class 2*. The *edge-colouring problem* or *chromatic index problem* is the problem of determining the chromatic index of a graph. Edge-colouring is a challenging topic in graph theory and the complexity of the problem is unknown for several important well-studied classes. Edge-colouring is NP-complete for regular graphs [13,17] of degree $\Delta \geq 3$. The problem is NP-complete also for the following classes [5]:

- r-regular comparability (hence perfect) graphs, for $r \geq 3$;
- r-regular line graphs of bipartite graphs (hence line graphs and clique graphs), for $r \geq 3$;
- r-regular k-hole-free graphs, for $r \geq 3$, $k \geq 3$;
- cubic graphs of girth k, for $k \geq 4$.

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Table 1
Complexity dichotomy for edge-colouring in the class of graphs with no cycle with a unique chord.

Class	$\Delta = 3$	$\Delta \geq 4$	Regular
Graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
4-hole-free graphs of $\mathcal{C}$	NP-complete	Polynomial	Polynomial
6-hole-free graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
{4-hole, 6-hole}-free graphs of $\mathcal{C}$	Polynomial	Polynomial	Polynomial

Table 2
Complexity dichotomy for edge-colouring in the class of multipartite graphs.

Class	$k \leq 2$	$k \geq 3$
k-partite graphs	Polynomial	NP-complete

Graph classes for which edge-colouring is polynomially solvable include the following:

- bipartite graphs [14];
- split-indifference graphs [19];
- series–parallel graphs (hence outerplanar) [14];
- k-outerplanar graphs [2], for $k \geq 1$.

The complexity of edge-colouring is unknown for several well-studied strong structured graph classes, for which only partial results have been reported, such as cographs [1], join graphs [10,11,18], cobipartite graphs [18], planar graphs [21,25], chordal graphs, and several subclasses of chordal graphs such as indifference graphs [8], split graphs [7] and interval graphs [3].

Given a graph F, we say that a graph G *contains* F if graph F is isomorphic to an induced subgraph of G. A graph G is F-*free* if G does not contain F. A *cycle* C in a graph G is a sequence of vertices $v_1 v_2 \dots v_n v_1$, that are distinct except for the first and the last vertex, such that for $i = 1, \dots, n-1$, $v_i v_{i+1}$ is an edge and $v_n v_1$ is an edge — we call these edges *the edges of* C. An edge of G with both endvertices in a cycle C is called a *chord* of C if it is not an edge of C. One can similarly define a path and a chord of a path. A *hole* is a chordless cycle of length at least four and an ℓ-*hole* is a hole of length ℓ. A *triangle* is a cycle of length 3 and a *square* is a 4-hole.

Trotignon and Vušković [23] studied the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord. The main motivation to investigate this class was to find a structure theorem for it, a kind of result which is not very frequent in the literature. Basically, this structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set of basic graphs and applying a series of known "gluing" operations. Another interesting property of this class is that it belongs to the family of the χ-bounded graphs, introduced by Gyárfás [12] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is χ-*bounded* with χ-*binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$, where $\chi(G')$ denotes the chromatic number of G' and $\omega(G')$ denotes the size of a maximum clique in G'. The research in this area is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded, see [20] for a survey. Note that perfect graphs are a χ-bounded family of graphs with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements. Also, by Vizing's theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [26]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [23]. Also in [23] the following results are obtained for graphs in $\mathcal{C}$: an $\mathcal{O}(nm)$ algorithm for optimal vertex-colouring, an $\mathcal{O}(n + m)$ algorithm for maximum clique, an $\mathcal{O}(nm)$ recognition algorithm, and the NP-completeness of the maximum stable set problem.

In the present paper we consider the complexity of determining the chromatic index of graphs in $\mathcal{C}$. In particular, we investigate how structure results can be used to solve the edge-colouring problem. We also investigate the subclasses obtained from $\mathcal{C}$ by forbidding 4-holes and/or 6-holes. Tables 1 and 2 summarize the main results achieved in the present work.

The results of Tables 1 and 2 show that, even for graph classes with strong structure and powerful decompositions, the edge-colouring problem may be difficult.

The class initially investigated in this work is the class $\mathcal{C}$ of graphs with no cycle with a unique chord. Each non-basic graph in this class can be decomposed [23] by special cutsets: 1-cutsets, proper 2-cutsets or proper 1-joins. We prove that edge-colouring is NP-complete for graphs in $\mathcal{C}$. We consider, then, a subclass $\mathcal{C}' \subset \mathcal{C}$ whose graphs are the graphs in $\mathcal{C}$ that do not have a 4-hole. By forbidding 4-holes we avoid decompositions by joins, which are difficult to deal with in edge-colouring [1,10,11]. That is, each non-basic graph in $\mathcal{C}'$ can be decomposed of 1-cutsets and proper 2-cutsets. For this class $\mathcal{C}'$ we establish a dichotomy: edge-colouring is NP-complete for graphs in $\mathcal{C}'$ with maximum degree 3 and polynomial for graphs in $\mathcal{C}'$ with maximum degree **not** 3. We determine also a necessary condition for a graph $G \in \mathcal{C}'$ of maximum degree 3 to be Class 2. This condition is having graph P^* – a subgraph of the Petersen graph – as a basic block in the decomposition tree. As a consequence, if both 4-holes and 6-holes are forbidden, the chromatic index of graphs with no cycle with a unique chord can be determined in polynomial time. The results achieved in this work have connections with other areas of research in edge-colouring, as we describe in the following three observations.

The first observation refers to the complexity dichotomy result found for class $\mathcal{C}'$. This dichotomy presents great interest since, to the best of our knowledge, this is the first class for which edge-colouring is NP-complete for graphs with a given fixed maximum degree Δ and is polynomial for graphs with maximum degree $\Delta' > \Delta$, as the reader may verify in the NP-completeness results reviewed in the beginning of the present section. It is interesting to observe that the only regular graphs in $\mathcal{C}'$ are the Petersen graph, the Heawood graph, the complete graphs, and the holes. As a consequence, edge-colouring is NP-complete when restricted to $\mathcal{C}'$, but polynomial when restricted to **regular** graphs in $\mathcal{C}'$.

The second observation is related to the study of *snarks* [22]. A *snark* is a cubic bridgeless graph with chromatic index 4. In order to avoid trivial (easy) cases, snarks are commonly restricted to have girth 5 or more and not to contain three edges whose deletion results in a disconnected graph, each of whose components is non-trivial. The study of snarks is closely related to the Four Colour Theorem. By the result of Lemma 8, the only **non-trivial** snark which has **no** cycle with a unique chord is the Petersen graph.

Finally, the third observation refers to the problem of determining the chromatic index of a k-partite graph, that is, a graph whose vertices can be partitioned into k stable sets. The problem is known to be polynomial [14,16] for $k = 2$ and for complete multipartite graphs. However, there is no explicit result in the literature regarding the complexity of determining the chromatic index of a k-partite graph for $k \geq 3$. From the proof of Theorem 2 we can observe that edge-colouring is NP-complete for k-partite r-regular graphs, for each $k \geq 3, r \geq 3$.

The remainder of the paper is organized as follows. In Section 2, we prove NP-completeness results regarding edge-colouring in the classes $\mathcal{C}$ and $\mathcal{C}'$. In Section 3, we review known results on the structure of graphs in $\mathcal{C}$ and obtain stronger structure results for graphs in $\mathcal{C}'$. In Section 4 we show how to determine in polynomial time the chromatic index of a graph in $\mathcal{C}'$ with maximum degree $\Delta \geq 4$. In Section 5 we further investigate graphs in $\mathcal{C}'$ with maximum degree 3: we show that edge-colouring can be solved in polynomial time if the inputs are restricted to regular graphs of $\mathcal{C}'$ and to 6-hole-free graphs of $\mathcal{C}'$.

2. NP-completeness results

In this section, we state NP-completeness results on the edge-colouring problem restricted to the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord and to the class $\mathcal{C}'$ composed of the graphs in $\mathcal{C}$ that do not contain a 4-hole. First, we prove that edge-colouring is NP-complete for regular graphs of $\mathcal{C}$ with fixed degree $\Delta \geq 3$. We observe that it can be shown that the construction of Cai and Ellis [5] which proves the NP-completeness of r-regular k-hole-free graphs, for $r \geq 3$ and $k \neq 4$, creates a graph with no cycle with a unique chord. Nevertheless, in the present section, we give a simpler construction. Second, we prove that edge-colouring is NP-complete for graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$. For the proof of this second result, we construct a replacement graph which is not present in any edge-colouring NP-completeness proof we could find in the literature.

We use the term CHRIND(P) to denote the problem of determining the chromatic index restricted to graph inputs with property P. For example, CHRIND(graph of $\mathcal{C}$) denotes the following problem:

INSTANCE: a graph G of $\mathcal{C}$.

QUESTION: is $\chi'(G) = \Delta(G)$?

The following theorem [13,17] establishes the NP-completeness of determining the chromatic index of Δ-regular graphs of fixed degree Δ at least 3:

Theorem 1 (*[13,17]*). *For each $\Delta \geq 3$, CHRIND(Δ-regular graph) is NP-complete.*

Please refer to Fig. 1. Graph Q_n, for $n \geq 3$, is obtained from the complete bipartite graph $K_{n,n}$ by removing an edge xy, by adding new pendant vertices u and v, and by adding pendant edges ux and vy. Graph Q'_n is obtained from Q_n by identifying vertices u and v into a vertex w. Observe that Q'_n is a graph of maximum degree n, and has $2n + 1$ vertices and $n^2 + 1$ edges. So, Q'_n is overfull and, hence [9], Class 2. Lemma 1 investigates the properties of graph Q_n, which is used as "gadget" in the NP-completeness proof of Theorem 2.

Lemma 1. *Graph Q_n is n-edge-colourable, and in any n-edge-colouring of Q_n, edges ux and vy receive the same colour.*

Proof. We use the notation from Fig. 1. First, we exhibit an n-edge-colouring of Q_n. Denote by $x_0, \ldots, x_{n-1}$ (resp. $y_0, \ldots, y_{n-1}$) the vertices of Q_n which belong to the same partition as x (resp. y), where $x = x_0$ (resp. $y = y_0$). An n-edge-colouring of Q_n is constructed as follows: just let the colour of edge x_iy_j be $(i + j \bmod n) + 1$ and let the colour of edges x_0u and y_0v be 1.

Now we prove that, in any n-edge-colouring of Q_n, edges ux and vy have the same colour. Suppose there is an n-edge-colouring π of Q_n where ux and vy have different colours. Consider the graph $Q'_n = (V', E')$ obtained from Q_n, by contracting vertices u and v into vertex w. Then we can construct an n-edge-colouring π' of Q'_n by setting $\pi'(e) = \pi(e)$ if $e \in E' \setminus \{ux, vy\}$, $\pi'(wx) = \pi(ux)$ and $\pi'(wy) = \pi(vy)$, which is a contradiction to the fact that Q'_n is Class 2. □

We prove in Theorem 2 the NP-completeness of edge-colouring regular graphs that do not contain a cycle with a unique chord for each fixed degree $\Delta \geq 3$.

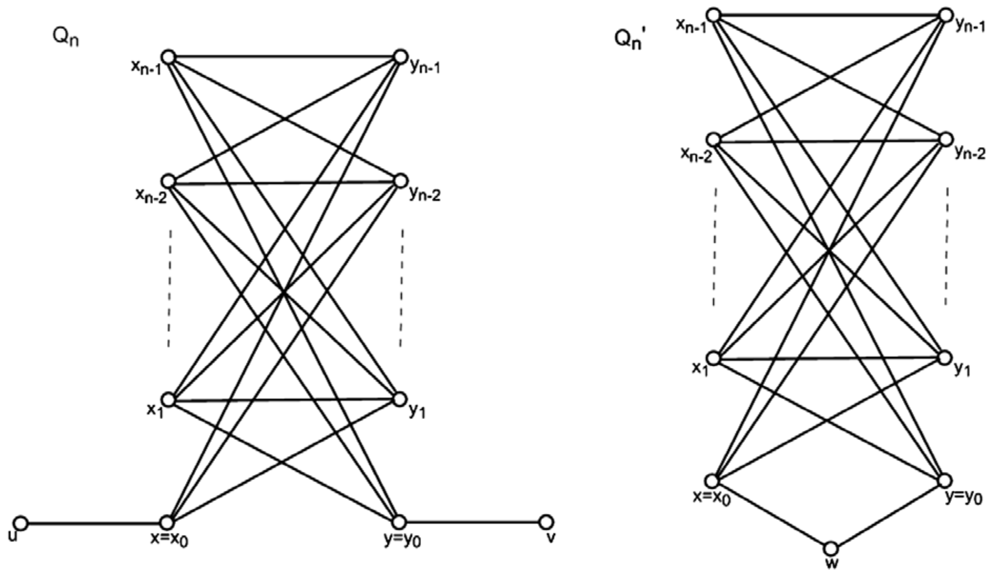

Fig. 1. NP-complete gadget Q_n and graph Q'_n.

Theorem 2. *For each $\Delta \geq 3$, CHRIND(Δ-regular graph in $\mathcal{C}$) is NP-complete.*

Proof. Let $G = (V, E)$ be an input of the NP-complete problem CHRIND(Δ-regular graph). Now, let G' be the graph obtained from G by removing each edge $pq \in E$ and adding a copy of Q_Δ, identifying vertices u and v of Q_Δ with vertices p and q of G. For each edge pq of G, denote H_{pq} the subgraph of G' isomorphic to Q_Δ whose pendant vertices are p and q. Observe that G' is also Δ-regular.

Claim 1: G' can be constructed in polynomial time from G. In fact, we make one substitution – by a copy of Q_Δ – for each edge of G, so that the construction time is linear on the number of edges of G.

Claim 2: if G is Δ-edge-colourable, then so is G'. Let π be a Δ-edge-colouring of G. We construct a Δ-edge-colouring π' of G' in the following way: for each edge pq of G, let the edges of H_{pq} in G' be coloured in such a way that the pendant edges have the colour $\pi(pq)$ — this colouring exists and is described by Lemma 1.

Claim 3: if G' is Δ-edge-colourable, then so is G. Let π' be a Δ-edge-colouring of G'. We construct a Δ-edge-colouring π of G as follows: let the colour in π of each edge pq of G be equal to the colour in π' of the pendant edges of H_{pq} (by Lemma 1, these two pendant edges must receive the same colour).

Claim 4: $G' \in \mathcal{C}$. Suppose G' has a cycle C with a unique chord $\alpha\beta$. Observe that, by construction, every edge of G' – and, in particular, chord $\alpha\beta$ – has both endvertices in the same copy of Q_Δ. Denote by $H_{p'q'}$ this copy and observe that cycle C, when restricted to $H_{p'q'}$, is a path between p' and q', and that $\alpha\beta$ is a unique chord of this path. But there is no path with a unique chord between the pendant vertices of Q_Δ, so that we have a contradiction. □

Observe that graph G' in the proof of Theorem 2 is tripartite with vertex tripartition (P_1, P_2, P_3) determined as follows:

- P_1 is the set whose elements are the original vertices of G and the vertices denoted by $y_1, \ldots, y_\Delta$ in each copy of Q_Δ;
- P_2 is the set whose elements are the vertices denoted by x_0 and y_0 in each copy of Q_Δ;
- P_3 is the set whose elements are the vertices denoted by $x_1, \ldots, x_\Delta$ in each copy of Q_Δ.

So, the following result holds:

Theorem 3. *For each $k \geq 3$, $\Delta \geq 3$, CHRIND(Δ-regular k-partite graph) is NP-complete.*

We emphasize that $\mathcal{C}$ is a class with strong structure [23], yet, it is NP-complete for edge-colouring. We manage in Section 4 to define a subclass of $\mathcal{C}$ where edge-colouring is solvable in polynomial time. Consider the class $\mathcal{C}'$ as the subset of the graphs of $\mathcal{C}$ that do not contain a square. The structure of graphs in $\mathcal{C}'$ is stronger than that of graphs in $\mathcal{C}$, and is described in detail in Section 3. Yet, the edge-colouring problem is still NP-complete for inputs in $\mathcal{C}'$, as we prove next in Theorem 4. We recall that the proof of Cai and Ellis [5] for the NP-completeness of edge-colouring cubic square-free graphs generates a graph which **has** a cycle with a unique chord. In addition, remark that the gadget Q_Δ used in the proof of the NP-completeness of edge-colouring graphs with no cycle with a unique chord **has** a square. So, we need an alternative construction, which is based on the gadget $\tilde{P}$ shown in Fig. 2. Graph $\tilde{P}$ is constructed in such a way that the identification of its pendant vertices generates a graph isomorphic to P^*, the graph obtained from the Petersen graph by removing one vertex. Graph P^* is a non-overfull Class 2 graph [15,6]. The properties of $\tilde{P}$ with respect to edge-colouring are described in Lemma 2.

Lemma 2. *Graph $\tilde{P}$ is 3-edge-colourable, and in any 3-edge-colouring of $\tilde{P}$, the edges ux and vy receive the same colour.*

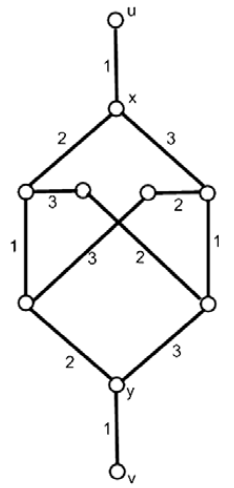

Fig. 2. 3-edge-colouring of gadget graph $\tilde{P}$.

Proof. Fig. 2 shows a 3-edge-colouring of $\tilde{P}$ — observe that edges ux and vy receive the same colour.

The fact that edges ux and vy always receive the same colour is a consequence of P^* being Class 2. The proof is similar to that of Lemma 1, except that gadget $\tilde{P}$ is used instead of Q_Δ. □

Theorem 4. *CHRIND(graph in $\mathcal{C}'$ with maximum degree 3) is NP-complete.*

Proof. The proof is similar to that of Theorem 2, except that $\Delta = 3$ and gadget $\tilde{P}$ is used instead of Q_Δ. □

Observe that the graph G' constructed in the proof of Theorem 4 is not regular. In fact, as we prove in Section 5.1, the edge-colouring problem can be solved in polynomial time if the input is restricted to cubic graphs of $\mathcal{C}'$.

3. Structure of graphs in $\mathcal{C}$ and $\mathcal{C}'$

The goal of the present section is to review structure results for the graphs in $\mathcal{C}$ and obtain stronger results for the subclass $\mathcal{C}'$. These results are used in Section 4 to edge-colour the graphs in $\mathcal{C}'$ with maximum degree at least 4. In the present section we review the results of Trotignon and Vušković [23] on the structure of graphs in $\mathcal{C}$ and obtain stronger results for graphs in $\mathcal{C}'$.

Let $\mathcal{C}$ be the class of the graphs that do not contain a cycle with a unique chord and let $\mathcal{C}'$ be the class of the graphs of $\mathcal{C}$ that do not contain a square. Trotignon and Vušković give a decomposition result [23] for graphs in $\mathcal{C}$ and graphs in $\mathcal{C}'$ in the following form: every graph in $\mathcal{C}$ or in $\mathcal{C}'$ either belongs to a basic class or has a cutset. Before we can state these decomposition theorems, we define the basic graphs and the cutsets used in the decomposition.

The *Petersen graph* is the graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cycles, and such that the only edges between some a_i and some b_i are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. We denote by P the Petersen graph and by P^* the graph obtained from P by removal of one vertex. Observe that $P \in \mathcal{C}$.

The *Heawood graph* is a cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$ so that $a_1a_2 \ldots a_{14}a_1$ is a cycle, and such that the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. We denote by H the Heawood graph and by H^* the graph obtained from H by removal of one vertex. Observe that $H \in \mathcal{C}$.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A strongly 2-bipartite graph is in $\mathcal{C}$ because any chord of a cycle is an edge between two vertices of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

For the purposes of this work, a graph G is called *basic*[1] if

1. G is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph; and
2. G has no 1-cutset, proper 2-cutset or proper 1-join (all defined next).

We denote by $\mathcal{C}_B$ the set of the basic graphs. Observe that $\mathcal{C}_B \subseteq \mathcal{C}$.

A *cutset* S of a connected graph G is a set of elements, vertices and/or edges, whose removal disconnects G. A decomposition of a graph is the removal of a cutset to obtain smaller graphs, called the *blocks* of the decompositions, by possibly adding some nodes and edges to connected components of $G \setminus S$. The goal of decomposing a graph is trying to solve a problem on the whole graph by combining the solutions on the blocks. For a graph $G = (V, E)$ and $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. The following cutsets are used in the known decomposition theorems of the class $\mathcal{C}$ [23]:

1 By the definition of [23], a basic graph is not, in general, indecomposable. However, our slightly different definition helps simplifying some of our proofs.

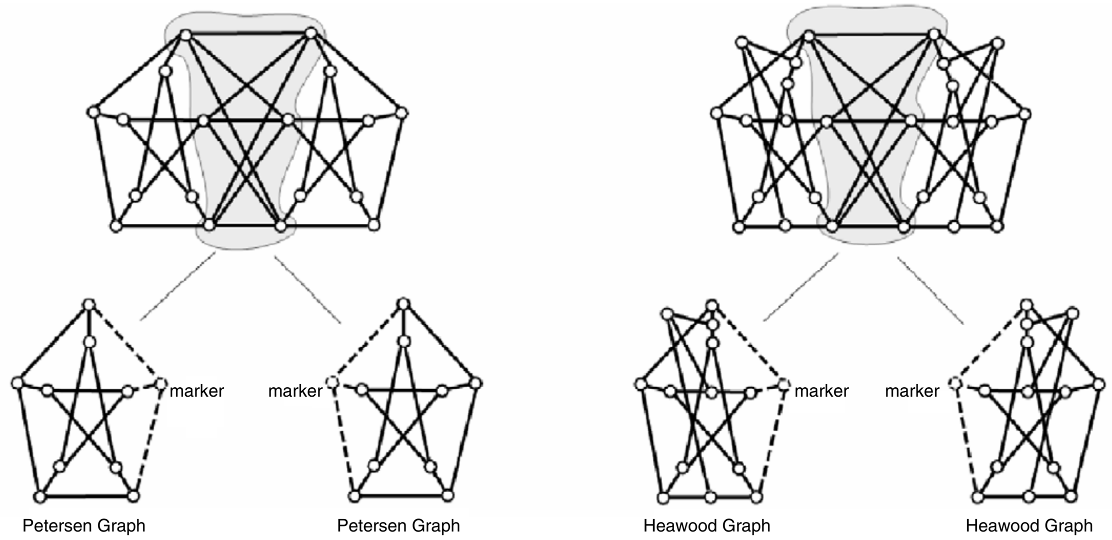

Fig. 3. Decomposition trees with respect to proper 1-joins. In the graph on the left, the basic blocks of decomposition are two copies of the Petersen graph. In the graph on the right, the basic blocks of decomposition are two copies of the Heawood graph.

- A *1-cutset* of a connected graph $G = (V, E)$ is a node v such that V can be partitioned into sets X, Y and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.
- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes a, b, both of degree at least three, such that V can be partitioned into sets X, Y and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.
- A *1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that there exist sets A, B satisfying:
  - $\emptyset \neq A \subseteq X, \emptyset \neq B \subseteq Y$;
  - $|X| \geq 2$ and $|Y| \geq 2$;
  - there are all possible edges between A and B;
  - there is no other edge between X and Y.

  We say that (X, Y, A, B) is a *split* of this 1-join.
  A *proper 1-join* is a 1-join such that A and B are stable sets of G of size at least two.

We can now state a decomposition result for graphs in $\mathcal{C}$:

Theorem 5 (*Trotignon and Vušković [23]*). *If $G \in \mathcal{C}$ is connected then either $G \in \mathcal{C}_B$ or G has a 1-cutset, or a proper 2-cutset, or a proper 1-join.*

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).
The *block* G_X (resp. G_Y) of a graph G with respect to a 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node y complete to A (resp. x complete to B). Nodes x, y are called *markers* of their respective blocks.
The *blocks* G_X and G_Y of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) are defined as follows. If there exists a node c of G such that $N_G(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block G_X (resp. G_Y) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new node c adjacent to a, b. Node c is called the *marker* of the block G_X (resp. G_Y).
The blocks with respect to 1-cutsets, proper 2-cutsets and proper 1-joins are constructed in such a way that they remain in $\mathcal{C}$, as shown by Lemma 3.

Lemma 3 (*Trotignon and Vušković [23]*). *Let G_X and G_Y be the blocks of decomposition of G with respect to a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

Observe that the Petersen graph and the Heawood graph may appear as a block of decomposition with respect to a proper 1-join, as shown in Fig. 3. However, these graphs cannot appear as a block of decomposition with respect to a proper 2-cutset, because they have no vertex with degree 2 to play the role of a marker.
Despite the fact that the Petersen graph and the Heawood graph do not appear as a block of decomposition with respect to a proper 2-cutset, they must be listed as basic blocks, because these graphs, themselves, are in $\mathcal{C}'$. So, the Petersen graph (resp. the Heawood graph) appears as a leaf of exactly one decomposition tree, namely, the decomposition tree of the Petersen graph (resp. the Heawood graph), itself — which is, actually, a trivial decomposition tree. Observe that graphs P^* (Petersen graph minus one vertex) and H^* (Heawood graph minus one vertex) may appear as a block with respect to a proper 2-cutset decomposition, as shown in Fig. 4.
We reviewed results that show how to decompose a graph of $\mathcal{C}$ into basic blocks: Theorem 5 states that each graph in $\mathcal{C}$ has a 1-cutset, a proper 2-cutset or a proper 1-join, while Lemma 3 states that the blocks generated with respect to any

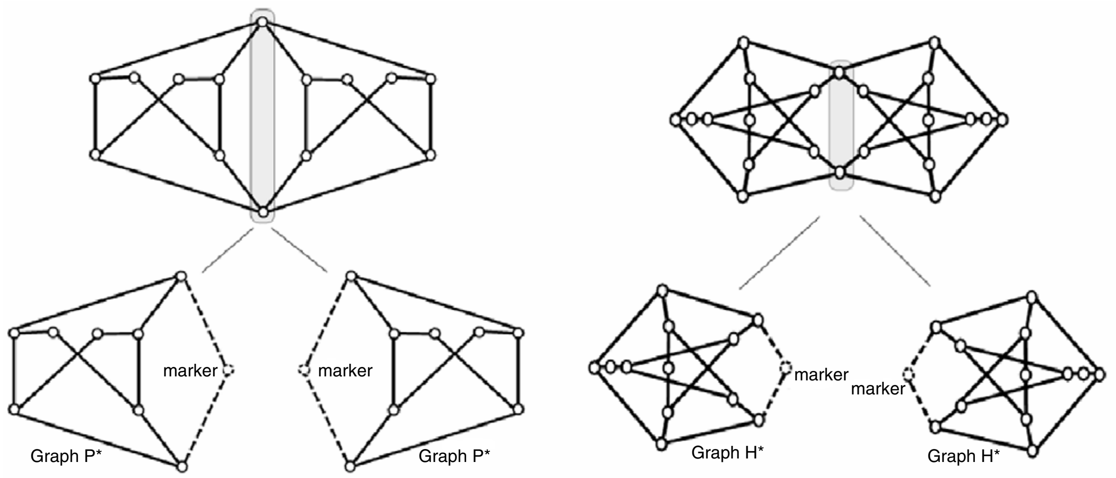

Fig. 4. Decomposition trees with respect to proper 2-cutsets. In the graph on the left, the basic blocks of decomposition are two copies of P^*. In the graph on the right, the basic blocks of decomposition are two copies of H^*.

of these cutsets are still in $\mathcal{C}$. We now obtain similar results for $\mathcal{C}'$. These results are not explicit in [23], but they can be obtained as consequences of results in [23] and by making minor modifications in its proofs. As we discuss in the following observation [4], for the goal of edge-colouring, we only need to consider the **biconnected** graphs of $\mathcal{C}'$.

Observation 1. *Let G be a connected graph with a 1-cutset with split (X, Y, v). The chromatic index of G is $\chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)\}$.*

By Observation 1, if both blocks G_X and G_Y are $\Delta(G)$-edge-colourable, then so is G. That is, once we know the chromatic index of the biconnected components of a graph, it is easy to determine the chromatic index of the whole graph. So, we may focus our investigation on the biconnected graphs of $\mathcal{C}'$.

Theorem 6 (*Trotignon and Vušković [23]*). *If $G \in \mathcal{C}'$ is biconnected, then either $G \in \mathcal{C}_B$ or G has a proper 2-cutset.*

Theorem 6 is an immediate consequence of Theorem 5: since G has no 4-hole, G cannot have a proper 1-join, and since G is biconnected, G cannot have a 1-cutset.

Next, in Lemma 4, we show that the blocks of decomposition of a biconnected graph of $\mathcal{C}'$ with respect to a proper 2-cutset, are also biconnected graphs of $\mathcal{C}'$. The proof of Lemma 4 is similar to that of Lemma 5.2 of [23]. For the sake of completeness, the proof, which uses the result of Theorem 7 below, is included here.

Theorem 7 (*Trotignon and Vušković [23]*). *Let $G \in \mathcal{C}$ be a connected graph. If G contains a triangle then either G is a complete graph, or some vertex of the maximal clique that contains this triangle is a 1-cutset of G.*

Lemma 4. *Let $G \in \mathcal{C}'$ be a biconnected graph and let (X, Y, a, b) be a split of a proper 2-cutset of G. Then both G_X and G_Y are biconnected graphs of $\mathcal{C}'$.*

Proof. We first prove that G is triangle-free. Suppose G contains a triangle. Then, by Theorem 7, either G is a complete graph, which contradicts the assumption that G has a proper 2-cutset, or G has a 1-cutset, which contradicts the assumption that G is biconnected. So G is triangle-free, and hence by construction, both of the blocks G_X and G_Y are triangle-free.

Now we show that G_X and G_Y are square-free. Suppose w.l.o.g. that G_X contains a square C. Since G is square-free, C contains the marker node M, which is not a real node of G, and $C = MazbM$, for some node $z \in X$. Since M is not a real node of G, we have $deg_G(z) > 2$, otherwise, z would be a marker of G_X. Let z' be a neighbor of z distinct of a and b. Since G is triangle-free, z' is not adjacent to a nor b. Since z is not a 1-cutset, there exists a path P in $G[X \cup \{a, b\}]$ from z' to $\{a, b\}$. We choose z' and P subject to the minimality of P. So, w.l.o.g., $z'Pa$ is a chordless path. Note that b is not adjacent to the neighbor of a along P because G is triangle-free and square-free, so that z is the unique common neighbor of a and b in G. So, by the minimality of P, vertex b does not have a neighbor in P. Now let Q be a chordless path from a to b whose interior is in Y. So, $bzz'PaQb$ is a cycle of G with a unique chord (namely az), contradicting the assumption that $G \in \mathcal{C}$.

By Lemma 3, G_X and G_Y both belong to $\mathcal{C}$, and since G_X and G_Y are both square-free, it follows that G_X and G_Y both belong to $\mathcal{C}'$.

Finally we show that G_X and G_Y are biconnected. Suppose w.l.o.g. that G_X has a 1-cutset with split (A, B, v). Since G is biconnected and $G[X \cup \{a, b\}]$ contains an ab-path, we have that $v \neq M$, where M is the marker of G_X. Suppose $v = a$. Then, w.l.o.g., $b \in B$, and $(A, B \cup Y, a)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$, if M is not a real node of G, contradicting the assumption that G is biconnected. So $v \neq a$ and by symmetry $v \neq b$. So $v \in X \setminus \{M\}$. W.l.o.g. $\{a, b, M\} \subset B$. Then $(A, B \cup Y, v)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$ if M is not a real node of G, contradicting the assumption that G is biconnected. $\square$

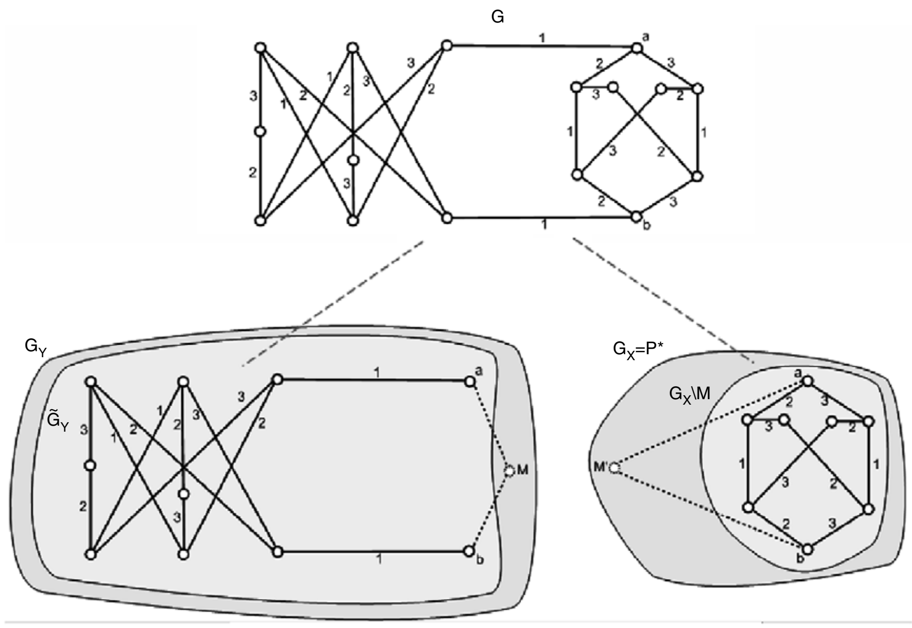

Fig. 5. Example of decomposition with respect to a proper 2-cutset $\{a, b\}$. Observe that the marker vertices and their incident edges – identified by dashed lines – do not belong to the original graph.

Observe that Lemma 3 is somehow stronger than Lemma 4. While Lemma 3 states that a graph is in $\mathcal{C}$ **if and only if** the blocks with respect to any cutset are also in $\mathcal{C}$, Lemma 4 establishes only one direction: **if** a graph is a biconnected graph of $\mathcal{C}'$, **then** the blocks with respect to any cutset are also biconnected graph of $\mathcal{C}'$. For our goal of edge-colouring, there is no need of establishing the "only if" part. Anyway, it is possible to verify that, if both blocks G_X and G_Y generated with respect to a proper 2-cutset of a graph G are biconnected graphs of $\mathcal{C}'$, then G itself is a biconnected graph of $\mathcal{C}'$.

Next lemma shows that every non-basic biconnected graph in $\mathcal{C}'$ has a decomposition such that one of the blocks is basic.

Lemma 5. *Every biconnected graph $G \in \mathcal{C}' \setminus \mathcal{C}_B$ has a proper 2-cutset such that one of the blocks of decomposition is basic.*

Proof. By Theorem 6 G has a proper 2-cutset. Consider all possible 2-cutset decompositions of G and pick a proper 2-cutset S that has a block of decomposition B whose size is smallest possible. By Lemma 4, $B \in \mathcal{C}'$ and is biconnected. So by Theorem 6, either B has a proper 2-cutset or it is basic. We now show that in fact B must be basic.

Let (X, Y, a, b) be a split with respect to S. Let M be the marker node of G_X, and assume w.l.o.g. that $B = G_X$. Suppose G_X has a proper 2-cutset with split (X_1, X_2, u, v). By minimality of $B = G_X$, $\{a, b\} \neq \{u, v\}$. Assume w.l.o.g. $b \notin \{u, v\}$. Note that since $deg_{G_X}(u) \geq 3$ and $deg_{G_X}(v) \geq 3$, it follows that $M \notin \{u, v\}$. Suppose $a \notin \{u, v\}$. Then w.l.o.g. $\{a, b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore $a \in \{u, v\}$. Then w.l.o.g. $\{b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore G_X does not have a proper 2-cutset, and hence it is basic. □

4. Chromatic index of graphs in $\mathcal{C}'$ with maximum degree at least 4

The first NP-completeness result of Section 2 proves that edge-colouring is difficult for the graphs in $\mathcal{C}$. We consider, further, the subclass $\mathcal{C}'$ and verify that the edge-colouring problem is still NP-complete when restricted to $\mathcal{C}'$. In the present section we apply the structure results of Section 3 to show that edge-colouring graphs in $\mathcal{C}'$ of maximum degree $\Delta \geq 4$ is polynomial by establishing that the only Class 2 graphs in $\mathcal{C}'$ are the odd order complete graphs. Remark that the NP-completeness holds only for 3-edge-colouring restricted to graphs in $\mathcal{C}'$ with maximum degree 3.

We describe, next, the technique applied to edge-colour a graph in $\mathcal{C}'$ by combining edge-colourings of its blocks with respect to a proper 2-cutset. Observe that the fact that a graph F is isomorphic to a block B obtained from a proper 2-cutset decomposition of G does **not** imply that G contains F: possibly B is constructed by the addition of a marker vertex. This is illustrated in the example of Fig. 5, where G is P^*-free, yet, graph P^* appears as a block with respect to a proper 2-cutset of G.

The reader will also observe that it is not necessary that a block of decomposition of G is $\Delta(G)$-edge-colourable in order that G itself is $\Delta(G)$-edge-colourable: graph G in Fig. 5 is 3-edge-colourable, while block P^* is not. This is an important observation: possibly, the edges adjacent to a marker vertex of a block of decomposition are not real edges of the original graph, or are already coloured by an edge-colouring of another block, so that these edges do not need to be coloured.

Observation 2. *Consider a graph $G \in \mathcal{C}'$ with the following properties:*

- *(X, Y, a, b) is a split of a proper 2-cutset of G;*
- *$\tilde{G}_Y$ is obtained from G_Y by removing its marker if this marker is not a real vertex of G;*

- $\tilde{\pi_Y}$ *is a* $\Delta(G)$*-edge-colouring of* $\tilde{G_Y}$*;*
- F_a *(resp.* F_b*) is the set of the colours in* $\{1, 2, \ldots, \Delta\}$ *not used by* $\tilde{\pi_Y}$ *in any edge of* $\tilde{G_Y}$ *incident to a (resp. b).*

If there exists a $\Delta(G)$*-edge-colouring* π_X *of* $G_X \setminus M$*, where M is the marker vertex of* G_X*, such that each colour used in an edge incident to a (resp. b) is in* F_a *(resp.* F_b*), then G is* Δ*-edge-colourable.*

The above observation shows that, in order to extend a $\Delta(G)$-edge-colouring of $\tilde{G_Y}$ to a $\Delta(G)$-edge-colouring of G, one must colour the edges of $G_X \setminus M$ in such a way that the colours of the edges incident to a (resp. b) are not used at the edges of $\tilde{G_Y}$ incident to a (resp. b). This guarantees that we create no conflicts. Moreover, there is no need to colour the edges incident to the marker M of G_X: if this marker is a vertex of G, its incident edges are already coloured by $\tilde{\pi}$, otherwise, these edges are not real edges of G. In the example of Fig. 5, we exhibit a 3-edge-colouring $\tilde{\pi_Y}$ of $\tilde{G_Y}$. In the notation of Observation 2, $F_a = \{2, 3\}$ and $F_b = \{2, 3\}$. We exhibit, also, a 3-edge-colouring of $G_X \setminus M$ such that the colours of the edges incident to a are $\{2, 3\} \subset F_a$ and the colours of the edges incident to b are $\{2, 3\} \subset F_b$. So, by Observation 2, we can combine colourings $\tilde{\pi_Y}$ and π_X in a 3-edge-colouring of G, as it is done in Fig. 5.

Before we proceed and show how to edge-colour graphs in $\mathcal{C}'$ with maximum degree $\Delta \geq 4$, we need to introduce some additional tools and concepts. A *partial k-edge-colouring* of a graph $G = (V, E)$ is a colouring of a subset E' of E, that is, a function $\pi : E' \to \{1, 2, \ldots, k\}$ such that no two adjacent edges of E' receive the same colour.

The *set of free colours* at vertex u with respect to a partial-edge-colouring $\pi : E' \to \mathbf{C}$ is the set $\mathbf{C} \setminus \pi(\{uv|uv \in E'\})$. The *list-edge-colouring* problem is described next. Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_e\}_{e\in E}$ be a collection which associates to each edge $e \in E$ a set of colours L_e called the *list* relative to e. It is asked whether there is an edge-colouring π of G such that $\pi(e) \in L_e$ for each edge $e \in E$. Theorem 8 is a result on list-edge-colouring which is applied, in this work, to edge-colour some of our basic graphs: strongly 2-bipartite graphs, Heawood graph and its subgraphs, and holes.

Theorem 8 (*Borodin, Kostochka, and Woodall [4]*). *Let* $G = (V, E)$ *be a bipartite graph and* $\mathcal{L} = \{L_e\}_{e\in E}$ *be a collection of lists of colours which associates to each edge* $uv \in E$ *a list* L_{uv} *of colours. If, for each edge* $uv \in E$, $|L_{uv}| \geq \max\{deg_G(u), deg_G(v)\}$*, then there is an edge-colouring* π *of G such that, for each edge* $uv \in E$, $\pi(uv) \in L_{uv}$.

We investigate, now, how to $\Delta(G)$-edge-colour a graph $G \in \mathcal{C}'$ by combining $\Delta(G)$-edge-colourings of its blocks with respect to a proper 2-cutset. More precisely, Lemma 6 shows how this can be done if one of the blocks is basic. Subsequently, we obtain, in Theorem 9 and Corollary 1, a characterization for graphs in $\mathcal{C}'$ of maximum degree at least 4 of its Class 2 graphs which establishes that edge-colouring is polynomial for these graphs.

Lemma 6. *Let* $G \in \mathcal{C}'$ *be a graph of maximum degree* $\Delta \geq 4$ *and let* (X, Y, a, b) *be a split of proper 2-cutset, in such a way that* G_X *is basic. If* G_Y *is* Δ*-edge-colourable, then G is* Δ*-edge-colourable.*

Proof. Denote by M the marker vertex of G_X and let $\tilde{G_Y}$ be obtained from G_Y by removing its marker if this marker is not a real vertex of G. Since $\tilde{G_Y}$ is a subgraph of G_Y, graph $\tilde{G_Y}$ is Δ-edge-colourable. Let π_Y be a Δ-edge-colouring of $\tilde{G_Y}$, i.e. a partial-edge-colouring of G, and let F_a and F_b be the sets of the free colours of a and b, respectively, with respect to the partial-edge-colouring π_Y. We show how to extend the partial-edge-colouring π_Y to G, as described in Observation 2, that is, by colouring the edges of $G_X \setminus M$. Since a and b are not adjacent, G_X is not a complete graph. Moreover, the block G_X cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and G_X has a marker vertex M of degree 2. So, G_X is isomorphic to an induced subgraph of P^*, or to an induced subgraph of H^*, or to a strongly 2-bipartite graph, or is a hole.

Case 1. G_X is a strongly 2-bipartite graph.

Since $deg_{G_X}(M) = 2$, vertex M belongs to the bipartition of G_X whose vertices have degree 2. So, vertices a and b belong to the bipartition of G_X whose vertices have degree larger than 2, and $|F_a| \geq deg_{G_X \setminus M}(a) \geq 2$ and $|F_b| \geq deg_{G_X \setminus M}(b) \geq 2$. Associate to each edge of $G_X \setminus M$ incident to a (resp. b) a list of colours equal to F_a (resp. F_b). To each of the other edges of $G_X \setminus M$, associate list $\{1, \ldots, \Delta\}$. Now, to each edge uv of $G_X \setminus M$, a list of colours is associated whose size is not smaller than $\max\{deg_{G_X \setminus M}(u), deg_{G_X \setminus M}(v)\}$ and, by Theorem 8, there is an edge-colouring π_1 of $G_X \setminus M$ from these lists. Finally, set $\pi := \pi_1$ for the edges of $G_X \setminus M$.

Case 2. G_X is a hole.

In this case, $G_X \setminus M$ is a path. Denote the vertices of $G_X \setminus M$ by $a = x_1, x_2, \ldots, x_k = b$, in such a way that $x_1x_2 \ldots x_k$ is a path. We now show that $k \geq 4$. Since a and b are not adjacent, $k \geq 3$. Suppose that $k = 3$. If M is a real node of G, then G_X is a square and it is an induced subgraph of G, contradicting the assumption that G is square-free. So M is not a real node of G, and hence $G_X \setminus M = X$. But, then, $|X| = 1$, contradicting the definition of a proper 2-cutset. Therefore, $k \geq 4$.

Observe that there is at least one colour c_α in F_a and one colour c_β in F_b. We construct a 3-edge-colouring π of $C_X \setminus M$ by setting $\pi(x_1x_2) := c_\alpha$ and $\pi(x_{k-1}x_k) := c_\beta$, and by colouring the other edges of $G_X \setminus M$ as follows. If $k = 4$, let $\pi(x_2x_3)$ be some colour in $\{1, 2, 3\} \setminus \{c_\alpha, c_\beta\}$, which is clearly a non-empty set. If $k \geq 5$, let $\mathcal{L}_2 = \{L_2, L_3, \ldots, L_{k-2}\}$ be a collection which associates to each edge x_ix_{i+1} a list of colours L_i such that:

- $L_i = \{1, 2, 3\} \setminus \{c_\alpha\}$, for $i = 2, 3, \ldots, k - 3$, and
- $L_{k-2} = \{1, 2, 3\} \setminus \{c_\beta\}$.

Observe that $G_X \setminus \{M, a, b\}$ is a path, hence bipartite of maximum degree 2, and that $|L_i| \geq 2$ for each $i = 2, \ldots, k - 2$, so that by, Theorem 8, there is an edge-colouring π_2 of $G_X \setminus \{M, a, b\}$ from the lists $\mathcal{L}_2$. Moreover, this colouring creates no conflicts with the colours c_α of x_1x_2 and c_β of $x_{k-1}x_k$, so that we can set $\pi := \pi_2$ for edges $x_2x_3, x_3x_4, \ldots, x_{k-2}x_{k-1}$.

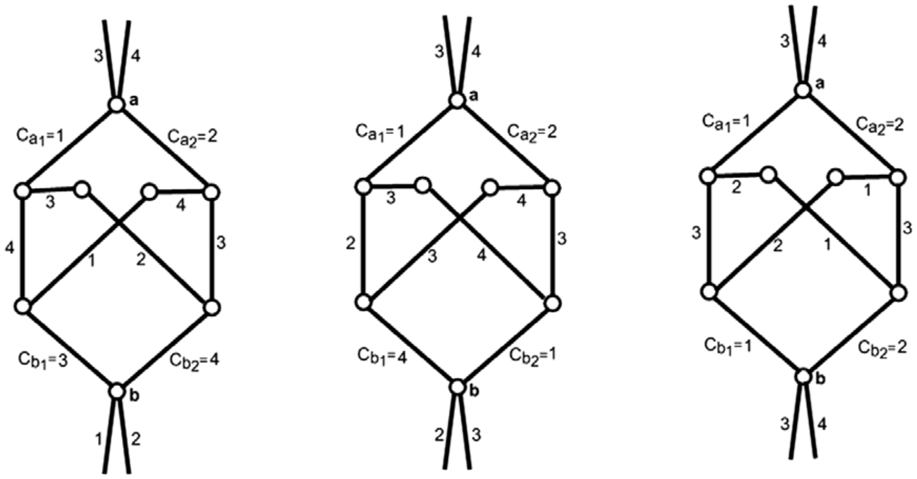

Fig. 6. Extending the colouring to the edges of G_X.

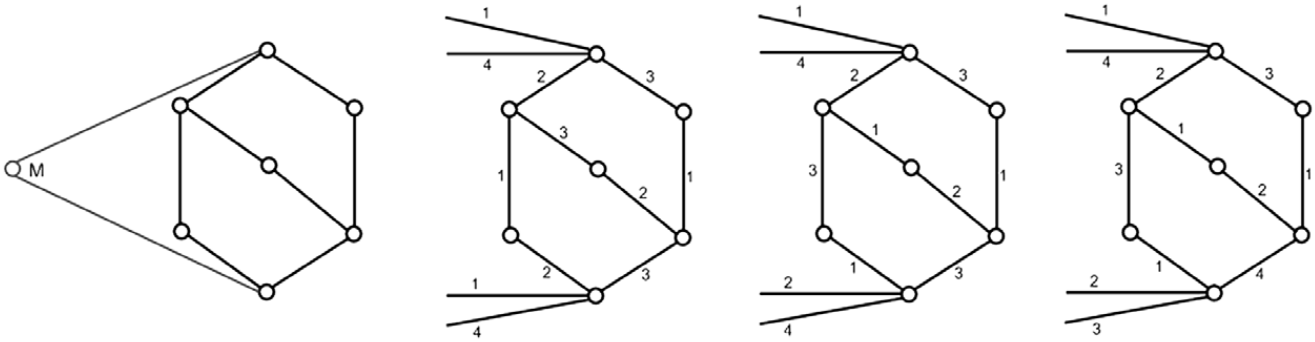

Fig. 7. Graph P^{**} and three edge-colourings of $P^{**} \setminus M$ subject to each possible free colour restriction.

Case 3. G_X is an induced subgraph of the Heawood graph.

Observe that a and b have only M as common neighbor in G_X, otherwise G_X has a square (recall that Heawood graph is square-free). We construct a 4-edge-colouring of $G_X \setminus M$. Denote the neighbors of a (resp. b) in $G_X \setminus M$ by $a_1, \ldots, a_x$ (resp. $b_1, \ldots, b_y$), where $x = deg_{G_X \setminus M}(a)$ (resp. $y = deg_{G_X \setminus M}(b)$). Note that $x, y \in \{1, 2\}$. Observe that F_a (resp. F_b) contains at least x (resp. y) colours, which we denote by $c_{a_1}, \ldots, c_{a_x}$ (resp. $c_{b_1}, c_{b_2}, \ldots, c_{b_y}$). Set the colour π of edge aa_i (resp. bb_j), for $i = 1, \ldots, x$ (resp. for $j = 1, \ldots, y$), to c_{a_i} (resp. c_{b_j}). Now, associate to each edge incident to a_i and different from aa_i a list of colours $\{1, 2, 3, 4\} \setminus \{c_{a_i}\}$. Similarly, associate to each edge incident to b_j and different of bb_j a list of colours $\{1, 2, 3, 4\} \setminus \{c_{b_j}\}$. Finally, associate to each of the other edges of $G_X \setminus \{M, a, b\}$ the list of colours $\{1, 2, 3, 4\}$. Observe that $G_X \setminus \{M, a, b\}$ is bipartite of maximum degree at most 3 and that each of the lists has 3 or 4 colours, so that, by Theorem 8, there is an edge-colouring π_3 of $G_X \setminus \{M, a, b\}$ from these lists, and we set $\pi := \pi_3$ for the edges of $G_Z \setminus M$.

Case 4.a: $G_X = P^*$.

Observe that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that exactly one of the following three possibilities holds:

- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0$;
- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$; or
- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 2$.

In the three cases, it is possible to extend the Δ-edge-colouring π_Y to G by colouring the edges of $G_X \setminus M$, as it is shown on Fig. 6.

Case 4.b: G_X is a proper induced subgraph of P^*.

We need to investigate which are the proper induced subgraphs of P^*. We invite the reader to verify that, except for graph P^{**} shown on the left of Fig. 7, each proper induced subgraph of P^* either has a 1-cutset or a proper 2-cutset, and we do not consider it because G_X is assumed basic, or is a hole, which is already considered in Case 2.

There is only one possible choice for the marker M of $G_X = P^{**}$, in the sense that, for any other choice of marker M', we have $G_X \setminus M' = G_X \setminus M$. As in Case 4.a, there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0$, 1 or 2. In Fig. 7 we exhibit three edge-colourings for $P^{**} \setminus M$, one for each possibility. □

Using Lemma 6 we can determine in polynomial time the chromatic index of the graphs of $\mathcal{C}'$, as we show in Theorem 9 and its Corollary 1.

Theorem 9. *If λ is an integer at least 4 and G is a connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \leq \lambda$, then G is λ-edge-colourable.*

Proof. We prove the theorem by induction. Let $G \in \mathcal{C}'$ be a connected graph with k vertices such that $\Delta(G) \leq \lambda$ and G is not a complete graph. By Theorem 6 either G is basic, or G has a 1-cutset, or G is biconnected and has a proper 2-cutset.

Suppose G is basic. If G is strongly 2-bipartite, then G is λ-edge-colourable because bipartite graphs are Class 1 and $\Delta(G) \leq \lambda$. If G is not strongly 2-bipartite, then G is a hole or a subgraph of the Petersen graph or of the Heawood graph, so that $\Delta(G) \leq 3 \leq \lambda - 1$ and G is λ-edge-colourable by Vizing's theorem. Assume, as induction hypothesis, that every connected non-complete graph $G' \in \mathcal{C}'$ with $k' < k$ vertices such that $\Delta(G') \leq \lambda$ is λ-edge-colourable.

Suppose G has a 1-cutset with split (X, Y, v). Note that blocks of decomposition G_X and G_Y are induced subgraphs of G and hence both belong to $\mathcal{C}'$. If G_X (resp. G_Y) is complete, then its maximum degree is at most $\lambda - 1$, so that G_X (resp. G_Y) is λ-edge-colourable by Vizing's theorem. If G_X (resp. G_Y) is not complete, G_X (resp. G_Y) is λ-edge-colourable by the induction hypothesis. In any case, both G_X and G_Y are λ-edge-colourable, and hence by Observation 1, graph G is λ-edge-colourable.

Finally, suppose G is biconnected and has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset such that block G_X is basic (note that such a cutset exists by Lemma 5). By Theorem 7, block G_X is not a complete graph. By Lemma 4, block G_Y is in $\mathcal{C}'$. By the induction hypothesis, block G_Y is λ-edge-colourable. By Lemma 6, graph G is λ-edge-colourable. □

Corollary 1. *A connected graph $G \in \mathcal{C}'$ of maximum degree $\Delta \geq 4$ is Class 2 if and only if it is an odd order complete graph.*

Proof. If G is complete, then the result clearly holds. So, we may assume G is not complete. Just choose $\lambda = \Delta$ in Theorem 9 to prove that every connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \geq 4$ is λ-edge-colourable, hence Class 1. □

5. Graphs of $\mathcal{C}'$ with maximum degree 3

Class $\mathcal{C}'$ has a stronger structure than $\mathcal{C}$, yet, edge-colouring problem is NP-complete for inputs in $\mathcal{C}'$. In fact, the problem is NP-complete for graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$. In this section, we further investigate graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$, providing two subclasses for which edge-colouring can be solved in polynomial time: cubic graphs of $\mathcal{C}'$ and 6-hole-free graphs of $\mathcal{C}'$.

5.1. Cubic graphs of $\mathcal{C}'$

In the present section, we prove the polynomiality of the edge-colouring problem restricted to cubic graphs of $\mathcal{C}'$. This is a direct consequence of Lemma 7, that states that every non-biconnected cubic graph is Class 2, and Lemma 8, that states that the Petersen graph is the only biconnected cubic Class 2 graph in $\mathcal{C}'$. We remark that the bipartite Heawood graph and the complete graph on four vertices are both cubic Class 1 graphs.

Lemma 7. *Let G be a connected cubic graph. If G has a 1-cutset, then G is Class 2.*

Proof. Denote by (X, Y, v) a split of a 1-cutset of G. Observe that v has degree 1 in exactly one of the blocks G_X and G_Y; assume, w.l.o.g. that this block is G_X. Let G'_X be the graph obtained from G_X by removing vertex v. Observe that G'_X has exactly one vertex of degree 2 and each of the other vertices has degree 3. Since the sum of the degrees of the vertices is even, G'_X has an even number of vertices of degree 3, say n. So, the number of edges in G'_X is $(3n + 2)/2$. Since $3\lfloor(n + 1)/2\rfloor = 3n/2 < (3n + 2)/2$, graph G'_X is overfull and hence Class 2. Since G'_X is a subgraph of G, and both G'_X and G have maximum degree 3, the graph G is itself Class 2. □

Lemma 8. *Let $G \in \mathcal{C}'$ be biconnected graph. If G is cubic, then G is isomorphic to the Petersen graph or to the Heawood graph or is a complete graph on four vertices.*

Proof. Suppose G is not basic. By Lemma 5, G has a proper 2-cutset such that one of the blocks is basic. Let (X, Y, a, b) be a split of such cutset, in such a way that G_X is basic, and denote by M the marker vertex of G_X. If $deg_{G_X}(a) = 1$, vertex M is the only neighbor of a and, clearly, is a 1-cutset of G_X. By Lemma 4, G_X is a biconnected graph of $\mathcal{C}'$. Since G_X is biconnected $deg_{G_X} \geq 2$. Let a' be a neighbor of a in G_X that is distinct from M. Since $\{M, a, b, a'\}$ cannot induce a square, b is not adjacent to a', and hence (since G is cubic) a' has two neighbors in $G_X \setminus \{a, b, M\}$. If $deg_{G_X}(a) = 2$ then $\{a', b\}$ is a proper 2-cutset of G, contradicting the assumption that G_X is basic. Hence $deg_{G_X}(a) \geq 3$, and by symmetry $deg_{G_X}(b) \geq 3$. Observe that each of the other vertices – different from a, b and M – has degree $\Delta(G)$. In other words, G_X is a graph with exactly one vertex of degree 2, and each of the other vertices has degree 3. But there is no graph in $\mathcal{C}_B$ with this property, and we have a contradiction to the fact that G_X is basic. So, G is basic and the statement of the lemma clearly holds. □

Theorem 10. *Let $G \in \mathcal{C}'$ be a connected cubic graph. Then G is Class 1 if and only if G is biconnected and is not isomorphic to the Petersen graph.*

Proof. If G is not biconnected, then, by Lemma 7, G is Class 2. If G is biconnected, then, by Lemma 8, G is isomorphic to the Petersen graph P or to the Heawood graph H or is a complete graph K_4 on four vertices. Remark that H is Class 1, because it is bipartite, and K_4 is Class 1, because it is a complete graph with even number of vertices. Hence, G is Class 2 if and only if it is isomorphic to the Petersen graph. □

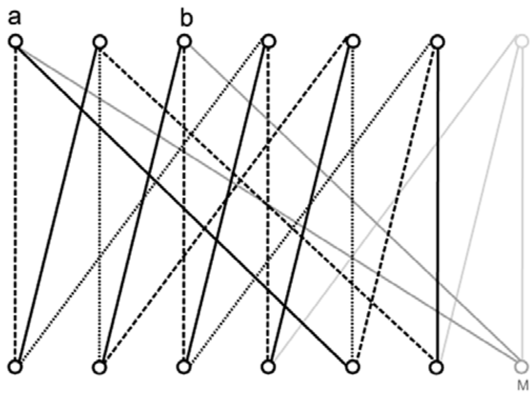

Fig. 8. A 3-edge-colouring of $H^* \setminus M$ such that the sets of the colours incident to vertex a and vertex b are the same.

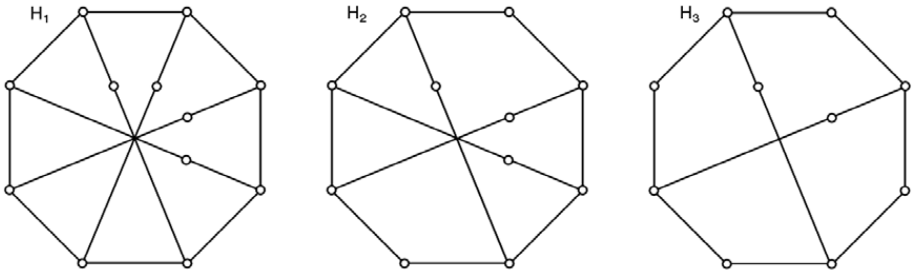

Fig. 9. Non-basic proper induced subgraphs of H^*.

5.2. 6-hole-free graphs of $\mathcal{C}'$

In the present section, we prove the polynomiality of the edge-colouring problem restricted to 6-hole-free graphs of $\mathcal{C}'$. This is a consequence of Lemma 9, a variation for 3-edge-colouring of Lemma 6.

Lemma 9. *Let $G \in \mathcal{C}'$ be a graph of maximum degree at most 3 and (X, Y, a, b) be a split of a proper 2-cutset, in such a way that G_X is basic but not isomorphic to P^*. If G_Y is 3-edge-colourable, then G is 3-edge-colourable.*

Proof. Assume G_Y is 3-edge-colourable. Denote by M the marker vertex of G_X and let $\tilde{G_Y}$ be obtained from G_Y by removing its marker if this marker is not a real vertex of G. Since $\tilde{G_Y}$ is a subgraph of G_Y, graph $\tilde{G_Y}$ is 3-edge-colourable. Let π_Y be a 3-edge-colouring of $\tilde{G_Y}$, i.e. a partial-edge-colouring of G, and let F_a and F_b be the sets of the free colours of a and b, respectively, with respect to the partial-edge-colouring π_Y. We show how to extend the partial-edge-colouring π_Y to G, as described in Observation 2, that is, by colouring the edges of $G_X \setminus M$. Since a and b are not adjacent, G_X is not a complete graph. Moreover, the block G_X cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and G_X has a marker vertex M of degree 2. Also, by assumption, block G_X is not isomorphic to P^*. So, G_X is isomorphic to a proper induced subgraph of P^*, or to an induced subgraph of H^*, or to a strongly 2-bipartite graph, or is a hole.

Case 1. G_X is a strongly 2-bipartite graph.

Similar to the Case 1 of the proof of Lemma 6, which also works for $\Delta = 3$.

Case 2. G_X is a hole.

Similar to the Case 2 of the proof of Lemma 6, where at most three colours are used in the edges of $G_X \setminus M$.

Case 3. G_X is an induced subgraph of H^*.

First, observe that $deg_{G_X \setminus M}(a) = 2$ and $deg_{G_X \setminus M}(b) = 2$, otherwise G_X has a decomposition by a 1-cutset or a proper 2-cutset and is not basic. Observe, also, that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$ or 2. We consider each case next.

If $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$, we must exhibit a 3-edge-colouring π of $G_X \setminus M$ such that the free colours at a and b are different. If M is a real node of G, then G_X is an induced subgraph of G, and hence $\Delta(G_X) \leq 3$. If M is not a real node of G, then by definition of proper 2-cutset both a and b have a neighbor in Y, and hence $\Delta(G_X) \leq 3$. So $\Delta(G_X) \leq 3$. Since G_X is bipartite, G_X has a 3-edge-colouring π'. So, let π be the restriction of π' to $G_X \setminus M$.

If $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 2$, we must exhibit a colouring of $G_X \setminus M$ such that the free colours at a and b are the same. We exhibit these colourings for each possible induced subgraph of the Heawood graph. First, consider the case $G_X = H^*$, whose colouring is given in Fig. 8.

Now, observe that each non-basic proper subgraph of H^* is a subgraph of the graph H_1 of Fig. 9, which is obtained from H^* by removing a vertex of degree 2. Graph H_2 of Fig. 9 is obtained from H_1 by removing one of the four vertices of degree 2 (any choice yields the same graph up to an isomorphism). Finally, the last non-basic proper subgraph of H^* is the graph H_3

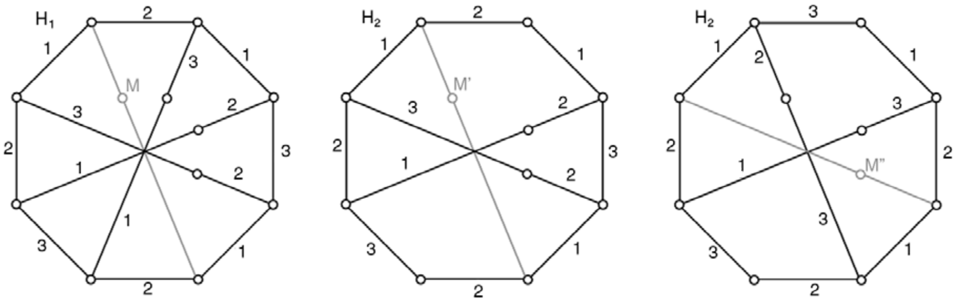

Fig. 10. 3-edge-colourings of H_1 and H_2, for each possible choice of marker.

of Fig. 9. Observe that there is only one possible choice M for the marker when $G_X = H_1$, in the sense that, for any other choice $\tilde{M}$, we have $G_X \setminus \tilde{M} = G_X \setminus M$. If $G_X = H_2$, there are two possible choices M' and M'' for the marker, in the sense that, for any other choice $\tilde{M}'$, we have $G_X \setminus \tilde{M}' = G_X \setminus M'$ or $G_X \setminus \tilde{M}' = G_X \setminus M''$. We show, in Fig. 10, one edge-colouring of $H_1 \setminus M$, and two edge-colourings of $H_2 \setminus M$, one for each possible choice of marker M. We do not consider here that case $G_X = H_3$ because H_3 is a strongly 2-bipartite graph, considered in Case 1.

Case 4. G_X is a proper subgraph of P^*.

As we already discussed in Case 4 of Lemma 6, except for graph P^{**} shown on the left of Fig. 7, each of the other proper induced subgraphs of P^* either has a 1-cutset or a proper 2-cutset, and we do not consider because G_X is basic, or is a hole, which are considered in Case 2. There is only one possible choice of marker M_1 for the case $G_X = P^{**}$, in the sense that for any other choice of marker M_1', we have $G_X \setminus M_1' = G_X \setminus M_1$. Observe, also, that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$ or 2. These two possibilities are considered in the first two colourings of Fig. 7. □

Remark that the NP-Complete gadget $\tilde{P}$ of Fig. 2 is constructed from P^*. The NP-completeness of edge-colouring graphs in $\mathcal{C}'$ is obtained as a consequence of $P^* \in \mathcal{C}'$. Using Lemma 9, we can prove that if the special graph P^* does not appear as a leaf in the decomposition tree, i.e., as a basic block when we recursively apply the proper 2-cutset decomposition to a biconnected graph $G \in \mathcal{C}'$ of maximum degree 3, then G is Class 1.

Theorem 11. *Let $G \in \mathcal{C}'$ be a connected graph of maximum degree 3. If G does not contain a 6-hole all of whose nodes are of degree 3, then G is Class 1.*

Proof. Assume the theorem does not hold and let G be a counterexample with fewest number of nodes. So G is a connected graph of $\mathcal{C}'$ of maximum degree 3, it does not contain a 6-hole all of whose nodes are of degree 3, and it is not 3-edge-colourable. By Theorem 6 either G is basic, or it has a 1-cutset, or it is biconnected and has a proper 2-cutset.

Suppose G is basic. G cannot be strongly 2-bipartite nor an induced subgraph of Heawood graph, since bipartite graphs are Class 1 [26]. Graph G cannot be a complete graph on four vertices, because such a graph is 3-edge-colourable. G cannot be a hole since it has maximum degree 3. So G must be an induced subgraph of the Petersen graph. G cannot be isomorphic to P nor P^*, because both of these graphs contain a 6-hole all of whose nodes are of degree 3. But all the other induced subgraphs of the Petersen graph are in fact 3-edge-colourable [6]. Therefore G cannot be basic.

Now suppose that G has a 1-cutset with split (X, Y, v). Note that blocks of decomposition are induced subgraphs of G, and hence both are connected graphs of $\mathcal{C}'$ that do not contain a 6-hole all of whose nodes are of degree 3. If $\Delta(G_X) = 3$ then since G is a minimum counterexample, G_X is 3-edge-colourable. If $\Delta(G_X) \leq 2$ then G_X is 3-edge-colourable by Vizing's theorem. So G_X is 3-edge-colourable, and similarly so is G_Y. But then by Observation 1, G is also 3-edge-colourable, a contradiction.

Therefore G is biconnected and has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset such that block G_X is basic (note that such a cutset exists by Lemma 5). By Lemma 4 both of the blocks G_X and G_Y are biconnected graphs of $\mathcal{C}'$. Since the marker node M is of degree 2 in both G_X and G_Y, and $G_X \setminus M$ and $G_Y \setminus M$ are both induced subgraphs of G, it follows that neither G_X nor G_Y can contain a 6-hole all of whose nodes are of degree 3. If M is a real node of G, then G_X and G_Y are both induced subgraphs of G, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. If M is not a real node of G, then by definition of proper 2-cutset both a and b have a neighbor in both X and Y, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. Since both G_X and G_Y have fewer nodes than G, it follows either from minimality of counterexample G or by Vizing's theorem that both G_X and G_Y are 3-edge-colourable. Since G_X does not contain a 6-hole all of whose nodes are of degree 3, G_X is not isomorphic to P^*, and hence by Lemma 9, G is 3-edge-colourable, a contradiction. □

Corollary 2. *Every connected 6-hole-free graph of $\mathcal{C}'$ with maximum degree 3 is Class 1.*

A natural question in connection with Theorem 12 is whether forbidding 6-holes would make it easier to edge-colour graphs of $\mathcal{C}$, and the answer is **no**. By observing graph G' of the proof of Theorem 2, one can easily verify that this graph has no 6-hole, so that the following theorem holds:

Theorem 12. *For each $\Delta \geq 3$, CHRIND(Δ-regular 6-hole-free graph in $\mathcal{C}$) is NP-complete.*

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