Pairwise Interaction Games

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Abstract. We study the complexity of computing Nash equilibria in games where players arranged as the vertices of a graph play a *symmetric 2-player game* against their neighbours. We call this a *pairwise interaction game*. We analyse this game for n players with a fixed number of actions and show that (1) a mixed Nash equilibrium can be computed in constant time for any game, (2) a pure Nash equilibrium can be computed through Nash dynamics in polynomial time for games with symmetrisable payoff matrix, (3) determining whether a pure Nash equilibrium exists for zero-sum games is NP-complete, and (4) counting pure Nash equilibria is #P-complete even for 2-strategy games. In proving (3), we define a new *defective* graph colouring problem called *Nash colouring*, which is of independent interest, and prove that its decision version is NP-complete. Finally, we show that pairwise interaction games form a proper subclass of the usual graphical games.

Key words: Nash equilibrium, graphical game, computational complexity, pairwise interaction

1 Introduction

1.1 Overview

Von Neumann and Morgenstern [30] proposed *game theory* as a mathematical tool for analysing the behaviour of rational players in *strategic games*. In such a game, each player has a set *pure strategies* (actions) and a player's payoff depends on the strategies chosen by him and his opponents. Players endeavour selfishly to maximise their own payoff. A player can play one of the pure strategies or a *mixed strategy* in which a pure strategy is selected at random. Strategic games are analysed so as to determine how the game may be played "optimally". Several solution concepts have been developed for this purpose; central among them is the *Nash equilibrium* due to Nash [31]. A set of strategies, with one strategy for each player, is in Nash equilibrium if no player can benefit by unilaterally changing his strategy. A Nash equilibrium is called *pure* if each player plays a *pure* strategy and *mixed* if players play mixed strategies. While mixed Nash equilibria are guaranteed to exist in any finite game by the celebrated theorem of Nash [31], pure Nash equilibria are not guaranteed to exist in general.

Plausibility of an equilibrium concept like the Nash equilibrium is partly determined by the complexity of computing equilibria [20]. As a result, many recent studies have focused on the complexity of finding Nash equilibria (see, for example, [1, 8, 15, 18, 20]). For the complexity problem to be meaningful, however, the game, particularly its payoffs, should allow a compact representation [37].

Many succinctly representable games have been studied in the literature (see [32, p.40] for a list) of which *graphical games* proposed by Kearns *et al.* [26] has received much attention (e.g. [7, 16, 17, 22]). In these games, players are arranged as the vertices of a graph and can play the game only with their immediate neighbours. In effect, a vertex k of degree d_k plays a $(d_k + 1)$-player game. If the number of pure strategies available to k is r, payoffs for k can be specified using r^{d_k+1} numbers. Thus, an n-player game requires at most $nr^{\Delta+1}$ numbers to describe the game where Δ is the maximum degree of the graph. This is a huge improvement over the representation size of normal-form games especially if the graph is sparse (i.e. when $\Delta \ll n$).

The representation can be further simplified using symmetries in games. A game is *symmetric* if every player has the same payoff matrix and a player's payoff depends only on the player's strategy and the number of other players playing each pure strategy available. Symmetric games have important applications in areas like automated-agent designs, game theoretic auction modelling [9] and evolutionary game theory. Indeed, the study of symmetric games started during the initial development of game theory (see, for instance, [30, 31]). In this paper we consider only symmetric games.

The idea of playing games on graphs predates the idea of the graphical games. Nearly a decade before the graphical games were introduced by [26], Nowak and May [33] empirically studied the impact that placing players at the vertices of a grid graph has in the emergence of cooperation. Their work stimulated research in this area, and a spate of new studies followed, studying the impact of many other types of graph, often with the conclusion that interaction graphs influence the evolution of cooperation (see [12, 34, 36, 39] and the references therein). In this setting, every vertex plays a symmetric 2-player game with all its immediate neighbours. This captures the natural tendency of players to treat each interaction as a separate 2-player game when the interactions are pairwise. This game, which we call a *pairwise interaction game* (formal definition is provided in the next section), is the subject of this paper. There is a sharp contrast between this game and the graphical games: in a single instance of this game, vertex k of degree d_k plays a 2-player game with each of its d_k neighbours and receives the accumulated payoff, whereas in a graphical game, k receives a payoff by engaging in a $(d_k + 1)$-player game once with all its neighbours.

Predictably, pairwise interaction games can be represented even more succinctly than symmetric graphical games. More precisely, the payoffs for an n-player game with r strategies can be simply described by an $r \times r$ matrix. It can be assumed that players know how to compute the payoff for a given strategy combination of his neighbours using this matrix. Clearly, this can always be computed in polynomial time. Despite its extremely compact representation, this model has many applications. It is often the case that strategic interactions in political, social and economic situations are pairwise [4]. As a result, this model is popular, for example, in Complex System Theory (e.g. [3]), in Artificial Intelligence (e.g. [27]) and in Evolutionary Game Theory (e.g. [39]). Furthermore, there are studies that use this model with 2×2 symmetric coordination games to investigate the emergence of contagion and Nash equilibria (e.g. [2, 4, 6]).

Surprisingly though, to the best of our knowledge, a systematic study of pairwise interaction games has not been carried out from the computational game theory perspective. That is the main purpose of this paper. Perhaps unsurprisingly, it turns out that the set of pairwise interaction games is in fact a *proper* subclass of the graphical games. What makes our study even more interesting, but surely disappointing from a game-theoretic point of view, is the fact that even for this simple subclass, the problem of deciding whether a pure Nash equilibrium exists is hard for zero-sum games with more than two strategies. On the other hand, we show that the problem of computing a mixed Nash equilibrium is trivial, so computing a pure Nash equilibrium in a pairwise interaction game is much harder than computing a mixed Nash equilibrium. Furthermore given recent negative results for the graphical games (e.g. [16, 17, 22]), our study is interesting since it identifies a large class of pairwise interaction games for which a pure Nash equilibrium always exists and is easy to compute.

1.2 Our results

In this paper, we study n-player pairwise interaction games with a fixed number of strategies r. Clearly, Nash's theorem [31] that there exists a mixed Nash equilibrium in all finite games holds for pairwise interaction games. Thus, we have the following easy theorem about mixed strategies. The proof of this theorem and all other omitted proofs are given in Appendix.

Theorem 1. *For any pairwise interaction game with a fixed number of pure strategies, a symmetric mixed Nash equilibrium can be computed in constant time. This strategy corresponds to all players playing the symmetric mixed Nash equilibrium strategy for the 2-player game.*

Although mixed Nash equilibria exist in any game, there is no convincing justification for players deliberately randomising their actions [35, p.37]. Hence, the pure Nash equilibrium is considered a better solution concept for games where one exists. This gives rise to two computational problems: (1) does a given game have any pure Nash equilibrium?, and (2) if one exists, can it be computed in polynomial time? We address these questions for pairwise interaction games. We first prove the following theorem that Nash dynamics converges for games with symmetric matrices. It is the simple dynamics in which, at every iteration, some player switches to the best response to the current strategies of their neighbours, and its convergence implies the existence of a pure Nash equilibrium.

Theorem 2. *For any pairwise interaction game with r strategies and a symmetric payoff matrix, the Nash dynamics converges in at most $n2^{K/2}(2\Delta + 1)^{K+1}$ steps, where $K = r(r+1)/2 - 2$ and Δ is the maximum vertex degree.*

We show further that adding a constant to any column of the payoff matrix does not affect the Nash equilibria. So, the above result applies to games with payoff matrices that can be symmetrised using this operation. This, in particular, means that the above result applies to all 2-strategy games.

Perhaps more significantly, in Section 4 we prove the following theorem for zero-sum games.

Theorem 3. *For all $r \geq 3$, and all antisymmetric $r \times r$ payoff matrices $\mathbf{A}$ such that the 2-player game has a unique mixed strategy which is not a pure strategy, deciding whether there is a pure Nash equilibrium in the pairwise-interaction game with payoff matrix $\mathbf{A}$ on a Δ-regular graph is NP-complete.*

That the mixed strategy cannot be a pure strategy is clear, since otherwise all players playing this strategy would give a pure Nash equilibrium by Theorem 1. The condition of having a unique mixed strategy is made for technical reasons, and we believe the theorem to be true without this assumption. However, we note that having a unique mixed strategy is generic, and we give a short proof of this in Lemma 5.

In Section 5 we show that even for some 2-strategy pairwise interaction games for which the problem of finding a pure Nash equilibrium is in P, the problem of exactly counting them is #P-hard. Surprisingly, it turns out that even approximately counting them in polynomial time is not possible unless NP=RP. Finally, we have the following theorem that pairwise interaction games form a (small) proper subset of symmetric graphical games. Thus our hardness results are stronger than those previously known for graphical games.

Theorem 4. *For given $r > 2$ and $\Delta = \Omega(r)$, pairwise interaction games form only a small fraction of symmetric graphical games on Δ-regular graphs.*

1.3 Related Work

Two-strategy games The *parity affiliation game* [18] with all edge weights -1's and the *cut game* [5, 10] with all edge weights $+1$'s correspond to the pairwise interaction game with payoff matrix $\mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Any pure Nash equilibrium in these games is a STABLE-CONFIGURATION and a MAX-CUT, in the sense of Schäffer and Yannakakis [40]. Thus, finding a pure Nash equilibrium for these games is P-complete [40]. On the other hand, if all edge weights are equal to $+1$ in the parity affiliation game, it is equivalent to the pairwise interaction game with $\mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is an easier game to solve as we will see later.

We also note that some 2-strategy games considered here are essentially equivalent to the *defective 2-colouring* problem [14]. Similar results to those we give are known for the defective 2-colouring problem by a result of Lovász [29].

Our convergence proof in Theorem 2 employs a potential function similar to that used in the convergence of Hopfield's neural networks [24] and other related problems. However, our proof applies to more than two strategies. In addition, due to the simplified nature of pairwise interaction games, we are able to show that a pure Nash equilibrium can be computed in polynomial time.

Complexity of games The problem of computing mixed Nash equilibria of n-player normal-form games is PPAD-complete for all $n \geq 2$ [8, 15]. When $n \geq 4$, this problem is equivalent to the problem of computing Nash equilibria in graphical games of maximum degree $\Delta \geq 3$, with two strategies per player [21]. Hence, the latter is also PPAD-complete [15]. Some positive results are known when these games are symmetric. A mixed Nash equilibrium of a symmetric n-player normal-form game with r strategies can be computed in polynomial time if $r = O(\log n / \log\log n)$ [37]. Using this result, it is shown in [7] that for the symmetric graphical games with degree Δ, the equilibrium can be computed in polynomial time if $r = O(\log \Delta / \log\log \Delta)$. Moreover, a mixed Nash equilibrium of graphical games on a path can be computed in polynomial time [17].

For symmetric n-player normal-form game with a two strategies, it is known that there always exists a pure Nash equilibrium [9]. Ryan *et al.* [38] consider *circuit symmetric game* [41], in which payoff functions are represented as circuits, and extend the results of [9] by showing that a pure Nash equilibrium can be computed in polynomial time with two strategies. They show further that determining the existence of a pure Nash equilibrium is NP-complete with more than two strategies. For graphical games, the problem of determining whether there exist a pure Nash equilibrium is NP-complete, in general, even if all players have only two strategies and neighbourhoods of size 2 [19]. Interestingly, a dichotomy result was proved in [16] for a class of symmetric graphical games on a d-dimensional torus, that the problem of determining whether the game has a pure Nash equilibrium is polynomial if $d = 1$ and NEXP-complete if $d > 1$.

Finally, the problem of counting Nash equilibria is generally hard. Counting the number of (mixed) Nash equilibria is #P-hard even for symmetric 2-player games [11]. For graphical games, counting the number of pure Nash equilibria is #P-hard even for symmetric games with neighbourhood size of 2 [7].

2 Preliminaries

2.1 Notations

If all elements of a matrix $\mathbf{A}$ or a vector $\mathbf{n}$ are positive, we write $\mathbf{A} \geq \mathbf{0}$ or $\mathbf{n} \geq \mathbf{0}$ respectively. Here and elsewhere, matrices and column vectors with all 1's and 0's are denoted by $\mathbf{1}$ and $\mathbf{0}$ respectively. By $\mathbf{A}^T$ and $\mathbf{n}^T$, we denote the transpose of $\mathbf{A}$ and $\mathbf{n}$ respectively. We write column vectors as row vectors with the transpose operation, e.g. $(n_0, \ldots, n_{r-1})^T$. An integer set $\{0, \ldots, n-1\}$ is denoted by $[n]$. Interchangeably, we refer to a participant in the game as a player or vertex, since each participant is represented by a graph vertex in pairwise interaction games.

2.2 Strategic games

Definition 1. *A* normal-form game *is given by a set of players $\mathcal{Q}$, and for each player $k \in \mathcal{Q}$ a finite set of pure strategies $\mathcal{S}_k$ and a payoff function u_k : $(\times_{k\in\mathcal{Q}}\mathcal{S}_k) \to \mathbb{R}$.*

A *pure strategy* of player k is an element of $\mathcal{S}_k$. A *mixed strategy* for player k is a probability distribution Σ_k over $\mathcal{S}_k$, so is a nonnegative vector of length $|\mathcal{S}_k|$. A set of strategies $s = (s_1, \ldots, s_k, \ldots, s_n)$ $(s_k \in \mathcal{S}_k)$ is called a *pure strategy profile*, and $\sigma = (\sigma_1, \ldots, \sigma_k, \ldots, \sigma_n)$ $(\sigma_k \in \Sigma_k)$ is called a *mixed strategy profile*.

A 2-player normal-form game can be conveniently represented by two real matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. The game is *symmetric* if $\mathbf{B} = \mathbf{A}^T$, and is *zero-sum* if $\mathbf{A} + \mathbf{B} = \mathbf{0}$. Hence for a symmetric zero-sum game, $\mathbf{A}$ is *antisymmetric*. Thus, one payoff matrix $\mathbf{A}$ is sufficient to describe any symmetric 2-player game.

Let $G = (V, E)$ be a graph. Let $\mathcal{N}(k) = \{v \in V \mid (k, v) \in E\}$, and $d_k = |\mathcal{N}(k)|$. By s_{-k} and σ_{-k} we denote the pure and mixed strategies of all neighbours of k respectively. Then, the pairwise interaction game is defined as follows.

Definition 2. *A* pairwise interaction game $\mathcal{G}$ *is defined by*

- *An undirected, connected graph $G = (V, E)$, where the vertices $V = \{0, ..., n-1\}$ represent players.*
- *A symmetric 2-player game $\langle \mathcal{S}, \mathbf{A} \rangle$, where $\mathcal{S} = \{0, \ldots, r-1\}$ is the set of pure strategies available to each vertex and $\mathbf{A} = (a_{ij})$ $(i, j \in [r])$ is the payoff matrix. We shall denote the set of mixed strategies over $\mathcal{S}$ by Σ. To avoid trivialities, we will always assume $r \geq 2$. If the 2-player game is zero-sum, we refer to the pairwise interaction game as zero-sum.*
- *The payoff for any player k $(k \in [n])$ is defined as*

$$u\big(\sigma_k; \sigma_{-k}\big) \;=\; \sum_{p \in \mathcal{N}(k)} \sigma_k^T \mathbf{A} \sigma_p \;. \tag{1}$$

Let $\mathcal{B}(\sigma_{-k})$ be the set of mixed strategy best responses of vertex k to the neighbour strategies σ_{-k}. Then we have

$$\mathcal{B}(\sigma_{-k}) \;=\; \{\sigma_k \in \Sigma \mid u(\sigma_k; \sigma_{-k}) \geq u(\sigma'_k; \sigma_{-k}) \;\forall \sigma'_k \in \Sigma \;\} \;. \tag{2}$$

Definition 3. *A strategy profile $\sigma^* = \big\{\sigma_0^*, \ldots, \sigma_k^*, \ldots, \sigma_{n-1}^*\big\}$ $(\sigma_k^* \in \Sigma)$ is a* mixed Nash equilibrium *if $\sigma_k^* \in \mathcal{B}(\sigma_{-k})$ $\forall k \in V$. We say σ^* is a* pure Nash equilibrium *if it is a pure strategy profile. We say σ^* is a* strict Nash equilibrium *if σ_k^* is the unique best response for all $k \in V$, and a* weak Nash equilibrium *if there is any vertex w for which σ_w^* is not the unique best response.*

Let $n_j^{(k)}$ denote the number of neighbours of k playing strategy $j \in \mathcal{S}$. We shall call a combination of neighbour strategies a *neighbourhood*. Then, instead of using s_{-k} to denote it, for symmetric games, it is convenient to use a column vector of $n_j^{(k)}$ with one entry for each $j \in [r]$, e.g. $\mathbf{n_k} = \big(n_0^{(k)}, \ldots, n_{r-1}^{(k)}\big)^T$ where $\sum_{j=0}^{r-1} n_j^{(k)} = d_k$. Using this notation, for pure strategies, (1) can be rewritten as

$$u\big(s_k; n_0^{(k)}, \ldots, n_{r-1}^{(k)}\big) \;=\; \sum_{j \in \mathcal{S}} n_j^{(k)} a_{s_k j}. \tag{3}$$

Similarly, (2) could be written as $\mathcal{B}(n_0^{(k)}, \ldots, n_{r-1}^{(k)})$. We will use these notations in the analysis of pure Nash equilibria, and (1) and (2) for mixed Nash equilibria.

Now the following proposition holds.

Proposition 1. *Adding an arbitrary constant to all entries of any column of* **A** *does not affect the Nash equilibria of pairwise interaction games.*

We next define Nash dynamics and provide a proposition that links its convergence and the existence of a pure Nash equilibrium (see, for example, [18]).

Definition 4. *Nash Dynamics or Best Response Dynamics: In this dynamics, at every step, some player playing a suboptimal strategy improves his payoff by switching to the best response.*

Proposition 2. *If the Nash dynamics converges, then there is a pure Nash equilibrium.*

We use the following notion of equivalence of games throughout the paper.

Definition 5. *Two games are* equivalent *if they have identical best responses to every combination of opponents' strategies.*

3 Symmetric payoff matrices

In this section we prove Theorem 2 about pairwise interaction games with symmetrisable payoff matrix **A**. The following lemma shows that there always exists a pure Nash equilibrium for these games.

Lemma 1. *An r-strategy pairwise interaction game with symmetric payoff matrix* **A** *has a pure Nash equilibrium.*

Proof (Sketch). We prove this using a potential function $\psi : \mathcal{S}^n \to \mathbb{R}$. Let $s = (s_1, \ldots, s_k, \ldots, s_n) \in \mathcal{S}^n$ be a pure strategy profile. Then $\psi(s)$ is defined as

$$\psi(s) \;=\; \sum_{k \in V} u\big(s_k; n_0^{(k)}, \ldots, n_{r-1}^{(k)}\big) \;=\; \sum_{k \in V} \sum_{j \in S} n_j^{(k)} a_{s_k j} \; .$$

Thus, the value of the potential function ψ is equal to the sum of the payoffs of all players. It is then easy to show that whenever a player improves his payoff by $\theta_k > 0$, the potential function increases in value by $2\theta_k$.

The Nash dynamics converges for these games, but how long does this take? Unfortunately, most of the results known about the convergence of the Nash dynamics are negative (e.g. [18]). The Hopfield's neural networks [24], for example, has a similar convergence and is known to take exponential time in the worst case [42]. On the contrary, as Theorem 2 states, the convergence is fast for the games considered in Lemma 1. To prove this, we need the following lemma.

Lemma 2. *Any $r \times r$ payoff matrix* **A** *can be rescaled such that the minimum difference between any two payoffs is one, and at least one payoff is zero. This can be done without affecting the Nash equilibria or the symmetry of* **A***, if* **A** *is symmetric. The rescaling requires only constant time for fixed r.*

Proof Sketch of Theorem 2: We first rescale the payoff matrix using Lemma 2. Then, there is at least one payoff 0 and another ± 1. We consider the remaining payoffs as variables. Let $\mathbf{v} = (v_1, \ldots, v_K)^T$ denote the vector containing these variables, where K is the total number of variables. As $\mathbf{A}$ is symmetric, we have $K = r(r+1)/2 - 2$. Consider a vertex k with degree d. Let $P(r,d) = \binom{d+r-1}{r-1}$ be the number different neighbourhood configurations for k. Let $\mathbf{a}_i$ $(i \in [r])$ denote the rows of the payoff matrix. For each configuration of the neighbour strategies $\mathbf{n}_p$ $(p \in [P])$, the best response is determined by the ordering of $\mathbf{a}_i \mathbf{n}_p$ $(i \in [r])$. Now, for each neighbourhood $\mathbf{n}_p$ $(p \in [P])$, and each pair of strategies i and j $(0 \le i < j \le r-1)$, we add the inequality $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p \ge 1$ if i yields a higher payoff than j, and we add $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p = 0$ if both strategies yield the same payoff. These inequalities form a convex nonempty polyhedron in K-dimensional. It is nonempty because the original payoffs satisfy all these inequalities. This polyhedron defines a class of games that are equivalent to $\mathbf{A}$ and have the property that every best response move improves the player's payoff by at least 1. Let $\mathbf{Nv} = \mathbf{b}$ be the set of K inequalities that are tight at a vertex of the polyhedron. Applying Cramer's rule on this, we can find the coordinates of this vertex in terms the elements of $\mathbf{n}_p$'s. Then, the Hadamard's inequality can be used to bound these coordinates in terms of Δ and r, which actually are the payoffs of an equivalent game. In this context, we may allow exponential dependence on r, since this is assumed to be constant. For the new, equivalent game, $\psi(s)$ is polynomially bounded and its value increases by at least 2 at every step. It can then be shown that the Nash dynamics converges as claimed. □

As mentioned before, the payoff matrix of any 2-strategy game can be symmetrised. Let $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ be the original payoff matrix. This can be symmetrised to give $\mathbf{P} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, where $\alpha = P - S$ and $\beta = R - T$ (see Proposition 1). So, Theorem 2 applies to these games. However, in the following theorem, we get tighter results than that of Theorem 2 and 6 for regular graphs by exploiting the unique properties of the game, albeit using essentially similar techniques.

Theorem 5. *For any 2-strategy pairwise interaction game on a Δ-regular graph $G = (V, E)$ with n vertices and m edges, starting from an arbitrary initial state, the Nash dynamics converges in at most $3n/2$ steps if $\alpha + \beta$ and β are of opposite signs and m steps if they are of the same sign.*

It might be possible to extend the above result to non-regular graphs. As an evidence, consider the unweighted *cut game* where $\alpha = \beta = -1$. In this game, we have $0 \ge \psi(s) \ge \sum_{k \in V} -d_k = -2m$, so it takes only m steps for convergence on any graph. However, we give an alternative proof for general graphs.

Theorem 6. *For any 2-strategy pairwise interaction game with payoff matrix $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ on a graph with n vertices and m edges, starting from an arbitrary initial state, the Nash dynamics converges in at most (i) $3m - n$ steps if $T > R$ and $S > P$, (ii) $3m$ steps if $T < R$ and $S < P$, (iii) n steps, otherwise.*

Proof (Sketch). The proof is similar to the proofs of Lemma 1 and Theorem 5, but uses an algorithm similar to that of the graph partitioning algorithm of [23].

4 Zero-sum games

In this section, we study pairwise interaction zero-sum games with $r \geq 3$ strategies and prove Theorem 3. The main tool of the proof is the following proposition.

Proposition 3. *In a zero-sum pairwise-interaction game, the best response to any neighbourhood configuration yields a nonnegative payoff. Furthermore, in Nash equilibrium, every player earns zero payoff.*

The neighbourhoods in a Nash equilibrium can be characterised using the proposition above.

Definition 6. *In a pairwise-interaction zero-sum game, a neighbourhood will be called a Nash Equilibrium neighbourhood (NE neighbourhood) if the best response to the neighbourhood yields zero payoff.*

Corollary 1. *If* $\mathbf{n} = (n_0, \ldots, n_{k-1})^T$ *is a NE neighbourhood, then* $\mathbf{An} \leq \mathbf{0}$.

We now show that a highly nontrivial elimination of strategies is possible for zero-sum games, which , in a sense, is much stronger than the usual iterated elimination of dominated strategies. That is, if a strategy earns a negative payoff in any NE neighbourhood, it can be eliminated, implying that *any* surviving strategy is a best response to *any* NE neighbourhood.

Lemma 3. *If a strategy earns a negative payoff when played against a NE neighbourhood, no player will play it in any pure Nash equilibrium.*

We now consider the question of the existence of NE neighbourhoods for rational payoff matrices. But, as we shall see, this does not imply that a pure Nash equilibrium exists in a d-regular graph.

Lemma 4. *If* $\mathbf{A}$ *has rational entries then, for some integer* d*, there exists a NE neighbourhood for a vertex of degree* d.

The proof, that is deferred to Appendix, reveals a remarkable connection between a NE neighbourhood in a zero-sum pairwise interaction game and the optimal mixed strategy of the 2-player game. This suggests a heuristic approach to pairwise interaction games: from each player's point of view, their neighbourhood can be viewed as a single opponent playing a mixed strategy. For a general pairwise interaction game, this approach has no real validity. This can simply be illustrated using the maxcut game with payoff matrix $\mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on a d-regular graph where d is an odd number. The unique symmetric mixed Nash equilibrium is $(1/2, 1/2)$, but the neighbourhood $(d/2, d/2)$ is impossible since d is an odd number. Nonetheless, there exists a pure Nash equilibrium for this game by Theorem 6 and 5. Thus, the above mentioned connection is not valid for general games. Since individual players are not able to play mixed strategies, the above view is asymmetric. But, surprisingly, for zero-sum games it is actually valid.

In this context, Lemma 4 can be linked to some well-known mixed strategy results. Consider games with a *unique mixed strategy*, by which we mean the games with a unique NE neighbourhood. These are games for which the surviving strategies are *completely mixed* [25]. (A completely mixed strategy is one for which every pure strategy has a positive probability.) Kaplansky [25] showed that a symmetric zero-sum game can be completely mixed only if the number of strategies is odd. Thus, for games with a unique NE neighbourhood, the number of surviving strategies must be odd. For the remainder of this section we consider only payoff matrices $\mathbf{A}$ which have unique mixed strategies. But we note that there is always a matrix arbitrarily close to $\mathbf{A}$ for which this is true.

Lemma 5. *Let $\mathbf{A} = (a_{ij})$ be an antisymmetric payoff matrix. Then, there always exists an antisymmetric payoff matrix $\mathbf{B} = (b_{ij})$ that has a unique mixed strategy and satisfies $|a_{ij} - b_{ij}| \leq 1/M$, for any $M > 0$.*

Next, let us define an interesting computation problem related to improper vertex colouring that will be used in the proof of Theorem 3.

Definition 7. *NASH-COLOURABLE is a decision problem whose instance is a graph $G = (V, E)$, a set of colours $\{1, \dots, r\}$ and, for each vertex degree d in G, a set of r nonnegative integers $\left\{c_1^d, \dots, c_r^d\right\}$ such that $d = \sum_{i=1}^r c_r^d$. The question is: is there an improper vertex colouring of G with r colours such that a vertex with degree d has exactly c_i^d neighbours with colour $i \in [r]$?*

If the answer is positive, the graph G will be said to be Nash colourable *and the particular assignment of colours will be called a* Nash colouring *of the graph.*

Definition 8. *$(c_1, \dots, c_r)_\Delta$-NASH-COLOURABLE will mean the Nash colouring problem for Δ-regular graphs. In this case we will write $(c_1^\Delta, \dots, c_r^\Delta) = (c_1, \dots, c_r)$, so $\Delta = \sum_{i=1}^r c_r$. Since there is only one vertex degree, we will specify $(c_1, \dots, c_r)$ in the prefix.*

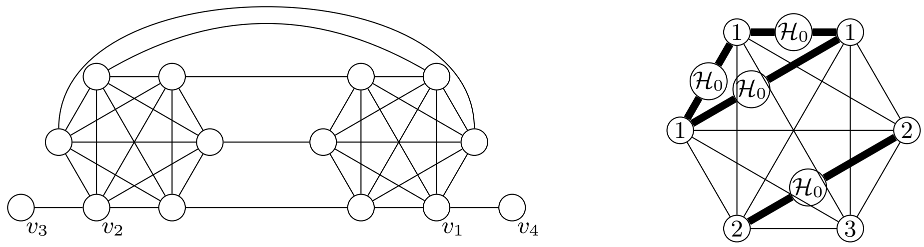
Proof Sketch of Theorem 3: A Nash equilibrium for the zero-sum pairwise interaction game is equivalent to finding a Nash colouring of the graph. The result then following from Theorem 7 that NASH-COLOURABLE is NP-complete. □

The following theorem shows that NASH-COLOURABLE is NP-complete.

Theorem 7. *If $r \geq 3$, and $c_1, \dots, c_r$ are any positive integers such that $\sum_{i=1}^r c_i = \Delta$, then the problem $(c_1, \dots, c_r)_\Delta$-NASH-COLOURABLE is NP-complete.*

Proof. The problem is clearly in NP. To prove that it is NP-hard, we reduce CHROMATIC-INDEX of r-regular graphs to $(c_1, \dots, c_r)_\Delta$-NASH-COLOURABLE. The hardness then follows from the result of [28] that CHROMATIC-INDEX is NP-complete for Δ-regular graphs with degree $\Delta \geq 3$.

Our reduction uses two Δ-cliques joined by a matching with one edge broken, as the basic gadget. We will call this gadget a *heavy edge* and denote it by $\mathcal{H}_0$. The heavy edge gadget for $\Delta = 6$ is shown in Fig. 1(a). In a valid $(c_1, \dots, c_r)_\Delta$-Nash colouring, each Δ-clique must contain c_i vertices of colour i and each vertex has one external edge which must be coloured (i, i). The matching means the two cliques have to be coloured identically. So the two external edges have to be coloured (x, x) for some $x \in [r]$.



(a) The heavy edge gadget ($\mathcal{H}_0$) for $\Delta = 6$. (b) $\mathcal{H}_1$ for $(3, 2, 1)_6$.

Fig. 1. $\mathcal{H}_0$ forces v_1, v_2, v_3, and v_4 to be coloured identically in any $(c_1, \ldots, c_r)_6$-Nash colouring, so in $\mathcal{H}_1$ we get subcliques of size $3, 2$ and 1 with colours $1, 2$ and 3 respectively.

Next we use $\mathcal{H}_0$ to construct another gadget $\mathcal{H}_1$ which basically is a Δ-clique in which r subcliques of size $c_1, \ldots, c_r$ are connected by heavy edges (see Fig. 1(b)). Then, we join two $\mathcal{H}_1$ gadgets with an r-edge matching, one for each subclique (see Fig. 2). Let us call the final gadget $\mathcal{H}$. This gadget then has $2(c_i - 1)$ external edges of each colour i.

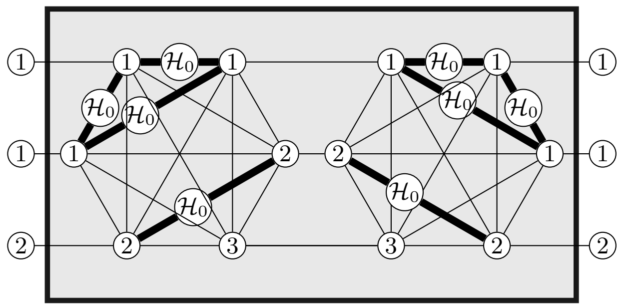


Fig. 2. Gadget $\mathcal{H}$ for $(3, 2, 1)_6$-NASH-COLOURABLE, i.e. $c_1 = 3, c_2 = 2, c_3 = 1$. The vertices are labelled with a possible colour assignment.

Now, take an instance of the CHROMATIC INDEX problem on a r-regular graph $G = (V, E)$. Take two copies of G. Replace each vertex by a Δ-clique. Now join these Δ-cliques corresponding to v in the two copies with the gadget $\mathcal{H}$ to get a Δ-regular graph. Let us call the resulting graph G^*.

It can now be shown that G^* has a $(c_1, \ldots, c_r)_\Delta$-Nash colouring if and only if the chromatic index of G is r. Finally, note that the new graph can be constructed in polynomial time, which completes the proof.

5 Some further results

We have already noted that the Nash equilibria related problems in pairwise interaction games have similarity to many other well-known interesting problems. Along this line, the following two theorems consider some games whose Nash equilibria correspond to maximal independent sets in the corresponding graphs. For these games, Theorem 8 shows that exactly counting the Nash equilibria is hard while Theorem 9 proves that even counting them approximately is hard.

Theorem 8. *Suppose a* 2*-strategy pairwise interaction game with payoff matrix* $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ *is played on a graph of maximum degree* 4*. Let* d_k *be the degree of vertex* k*. Then, if* $d_k\gamma$ *is not an integer for all* $k \in V$ *and the payoffs are such that* $T > R$*,* $S > P$ *and* $0 < \gamma \leq 1/4$ *or* $3/4 < \gamma \leq 1$*, where* $\gamma = (S - P)/(T + S - R - P)$*, the problem of counting the pure Nash equilibria is #P-complete.*

Theorem 9. *For the same game considered in Theorem 8 except that the game is played on a graph of maximum degree* 7 *and the payoffs are such that* $0 < \gamma \leq 1/7$ *or* $6/7 < \gamma \leq 1$*, there does not exist a fully polynomial time approximation scheme (FPTAS) to count the pure Nash equilibria unless RP=NP.*

Although we have obtained mainly positive results for the problem of computing a pure Nash equilibrium in 2-strategy pairwise interaction games, the computation is inherently sequential for some games and the following theorem holds.

Theorem 10. *The problem of computing pure Nash equilibria is P-complete for some* 2*-strategy pairwise interaction games.*

Thus finding a pure Nash equilibrium almost certainly cannot be done in constant time, in sharp contrast to Theorem 1 for mixed Nash equilibria.

Finally, we provide a proof sketch of Theorem 4.

Proof Sketch of Theorem 4: The comparison is made by identifying symmetric games with their equivalence class (see Definition 5). Let $P(r, \Delta) = \binom{\Delta+r-1}{r-1}$ be the number of ordered partitions of Δ into r parts. For symmetric graphical games, there are $\Gamma(r, \Delta) = r^P$ different games, since there are that number of best-response tables. For pairwise-interaction games, the number of possible games $\mathcal{G}(r, \Delta)$ can be bounded as follows. Recall from the proof of Theorem 2 that, for any $p \in [P]$, the best response is determined by the signs of $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p$ $(0 \leq i < j \leq r-1)$. Now $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p = 0$ determines a hyperplane through the origin in the space of the entries of the payoff matrix a_{ij} $(i, j \in [r])$. Each cell in the arrangement gives an equivalence class of payoffs, such that their ordering is the same for all neighbourhood configurations. Thus we can bound the number of games by the number of cells in this arrangement using [13, Theorem 2]. □

6 Open problems

We have initiated a systematic study of pairwise interaction games and presented results for the games with symmetric or antisymmetric payoff matrix. A natural extension of our work is to investigate the remaining case, i.e. the games with asymmetric payoff matrices that are not antisymmetric. We note that there are matrices of this kind for which it is easy to compute a pure Nash equilibrium. Hence, we believe that there are easy as well as hard cases left to study. But, we conjecture that there is a dichotomy in that the problem of deciding whether a pure Nash equilibrium exists is in P or NP-complete. Similarly, we believe that for the problem of counting Nash equilibria there is a dichotomy, thus the problem is in P or in #P-complete. We leave finding the dichotomy conditions as open problems. For the approximate counting problem, we showed that there does not exist an FPTAS even for some 2-strategy games. Here again, we believe that there is a dichotomy and leave the proof as an open problem.

There are many other ways in which our work could be extended. Recall that for the zero-sum pairwise interaction games, we proved that it is NP-complete to determine whether a pure Nash equilibrium exists, only for games with a unique mixed Nash equilibrium. It might be possible to extend our proof to games with more than one mixed Nash equilibria, but we have not done this. However, we conjecture that, even in this case, finding whether a pure Nash equilibrium exists is NP-complete unless the game can be reduced to a 1-strategy game.

Another interesting open problem is to find whether the Nash dynamics converges to any pure Nash equilibrium of the payoff matrix, i.e. of the 2-player game. Surprisingly, even for two-strategy games with more than one Nash equilibria this is not obvious. Finally, considering our hardness results, another topic of interest is to explore the approximate Nash equilibria for these games.

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Appendix

The missing proofs to be added here.....