Even-hole-free graphs that do not contain diamonds: a structure theorem and its consequences

Ton Kloks Haiko Müller Kristina Vušković*
School of Computing, University of Leeds, Leeds LS2 9JT, UK
{kloks|hm|vuskovi}@comp.leeds.ac.uk

July 11, 2007

Abstract

In this paper we consider the class of simple graphs defined by excluding, as induced subgraphs, even holes (*i.e.,* chordless cycles of even length) and diamonds (*i.e.,* a graph obtained from a clique of size 4 by removing an edge). We say that such graphs are (even-hole, diamond)-free. For this class of graphs we first obtain a decomposition theorem, using clique cutsets, bisimplicial cutsets (which is a special type of a star cutset) and 2-joins. This decomposition theorem is then used to prove that every graph that is (even-hole, diamond)-free contains a *simplicial extreme* (*i.e.,* a vertex that is either of degree 2 or whose neighborhood induces a clique). This characterization implies that for every (even-hole, diamond)-free graph G, $\chi(G) \leq \omega(G) + 1$ (where χ denotes the chromatic number and ω the size of a largest clique). In other words, the class of (even-hole, diamond)-free graphs is a χ-bounded family of graphs with the Vizing bound for the chromatic number.

The existence of simplicial extremes also shows that (even-hole, diamond)-free graphs are β-perfect, which implies a polynomial time coloring algorithm, by coloring greedily on a particular, easily constructable, ordering of vertices. Note that the class of (even-hole, diamond)-free graphs can also be recognized in polynomial time.

Keywords: Even-hole-free graphs; decomposition; χ-bounded families; β-perfect graphs; greedy coloring algorithm.

1 Introduction

All graphs in this paper are finite, simple and undirected. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is *F-free* if it does not contain F. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is *$\mathcal{F}$-free* if it is F-free, for every $F \in \mathcal{F}$.

Many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. Most famous such example is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ denotes the *chromatic number* of H, *i.e.,* the minimum number of colors needed to color the vertices of H so that no two vertices receive the same color, and $\omega(H)$ denotes the size of a largest clique in H (where a *clique* is a graph in which every pair of vertices are adjacent). The famous Strong Perfect

*This work was supported in part by EPSRC grant EP/C518225/1.

Graph Theorem (conjectured by Berge [3] and proved by Chudnovsky, Robertson, Seymour and Thomas [4]) states that a graph is perfect if and only if it does not contain an odd hole nor an odd antihole (where a *hole* is a chordless cycle of length at least four, it is *odd* or *even* if it contains an odd or even number of nodes, and an *antihole* is a complement of a hole).

In the last 15 years a number of other classes of graphs defined by excluding a family of induced subgraphs have been studied, perhaps originally motivated by the study of perfect graphs. The kinds of questions this line of research was focused on were whether excluding induced subgraphs affects the global structure of the particular class in a way that can be exploited for putting bounds on parameters such as χ and ω, constructing optimization algorithms (problems such as finding the size of a largest clique or a minimum coloring) and recognition algorithms. A number of these questions were answered by obtaining a structural characterization of a class through their decomposition (as was the case with the proof of the Strong Perfect Graph Theorem).

The structure of even-hole-free graphs was first studied by Conforti, Cornuéjols, Kapoor and Vušković in [7], where a decomposition theorem is obtained for this class, that was then used in [8] for constructing a polynomial time recognition algorithm. One can find a maximum weight clique of an even-hole-free graph in polynomial time, since as observed by Farber [11] 4-hole-free graphs (where a *4-hole* is a hole of length 4) have $\mathcal{O}(n^2)$ maximal cliques and hence one can list them all in polynomial time. In [18] da Silva and Vušković show that every even-hole-free graph contains a vertex whose neighborhood is *triangulated* (*i.e.,* does not contain a hole), and in fact they prove this result for a larger class of graphs that contains even-hole-free graphs (for the class of 4-hole-free odd-signable graphs, to be defined later). This characterization leads to a faster algorithm for computing a maximum weight clique in even-hole-free graphs (and in fact in 4-hole-free odd-signable graphs). More recently, Addario-Berry, Chudnovsky, Havet, Reed and Seymour [1], settle a conjecture of Reed, by proving that every even-hole-free graph contains a *bisimplicial vertex* (a vertex whose set of neighbors induces a graph that is the union of two cliques). This immediately implies that if G is a non-null even-hole-free graph, then $\chi(G) \leq 2\omega(G)-1$ (observe that if v is a bisimplicial vertex of G, then its degree is at most $2\omega(G)-2$, and hence G can be colored with at most $2\omega(G)-1$ colors).

The study of even-hole-free graphs is motivated by their connection to β-perfect graphs introduced by Markossian, Gasparian and Reed [16]. For a graph G, let $\delta(G)$ be the minimum degree of a vertex in G. Consider the following total order on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order gives the upper bound: $\chi(G) \leq \beta(G)$, where $\beta(G) = \max\{\delta(G')+1 : G'$ is an induced subgraph of $G\}$. A graph is *β-perfect* if for each induced subgraph H of G, $\chi(H) = \beta(H)$.

It is easy to see that β-perfect graphs belong to the class of even-hole-free graphs. A *diamond* is a cycle of length 4 that has exactly one chord. A *cap* is a cycle of length greater than four that has exactly one chord, and this chord forms a triangle with two edges of the cycle (*i.e.,* it is a short chord).

Markossian, Gasparian and Reed [16] show that (even-hole, diamond, cap)-free graphs are β-perfect. They show that a minimal β-imperfect graph that is not an even hole contains no simplicial extreme (where a vertex is *simplicial* if its neighborhood set induces a clique, and it is a *simplicial extreme* if it is either simplicial or of degree 2). They then prove that (even-hole, diamond, cap)-free graphs must always have a simplicial extreme.

This result was then generalized by de Figuiredo and Vušković [12], who show that (even-hole, diamond, cap-on-6-vertices)-free graphs contain a simplicial extreme, and hence are β-perfect. In the same paper they conjecture that in fact (even-hole, diamond)-free graphs are β-perfect, which we prove here.

In this paper we obtain a decomposition theorem for (even-hole, diamond)-free graphs that uses clique cutsets, bisimplicial cutsets (which is a special type of a star cutset) and 2-joins. This decomposition theorem is then used to prove that every graph that is (even-hole, diamond)-free contains a simplicial extreme, implying that they are β-perfect. We note that there are (even-hole, cap)-free graphs that are not β-perfect, see Figure 1. Total characterization of β-perfect graphs remains open, as well as their recognition. Clearly, since even-hole-free graphs can be recognized in polynomial time [8], so can (even-hole, diamond)-free graphs. Our result shows that (even-hole, diamond)-free graphs can be colored in polynomial time, by coloring greedily on a particular easily constructable ordering of vertices. (We note that for every graph G, there exists an ordering of its vertices on which the greedy coloring will give a $\chi(G)$-coloring of G, the difficulty being in finding this ordering). Whether even-hole-free graphs can be colored in polynomial time remains open.

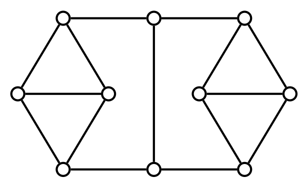


Figure 1: An (even-hole, cap)-free graph that is not β-perfect.

The fact that (even-hole, diamond)-free graphs have simplicial extremes implies that for such a graph G, $\chi(G) \leq \omega(G)+1$ (observe that if v is a simplicial extreme of G, then its degree is at most $\omega(G)$, and hence G can be colored with at most $\omega(G)+1$ colors). So this class of graphs belongs to the family of χ-bounded graphs, introduced by Gyárfás [13] as a natural extension of the family of perfect graphs: a family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$. Note that perfect graphs are a χ-bounded family of graphs with the χ-binding function $f(x) = x$. So a natural question to ask is: what choices of forbidden induced subgraphs guarantee that a family of graphs is χ-bounded. Much research has been done in this area, for a survey see [17]. We note that most of that research has been done on classes of graphs obtained by forbidding a finite number of graphs. Since there are graphs with arbitrarily large chromatic number and girth [10], in order for a family of graphs defined by forbidding a finite number of graphs (as induced subgraphs) to be χ-bounded, at least one of these forbidden graphs needs to be acyclic. Vizing's Theorem [21] states that for a simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ (where $\Delta(G)$ denotes the maximum vertex degree of G, and $\chi'(G)$ denotes the chromatic index of G, *i.e.*, the minimum number of colors needed to color the edges of G so that no two adjacent edges receive the same color). This implies that the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x+1$. This special upper bound for the chromatic number is called the *Vizing bound*. There is a list of nine forbidden induced subgraphs, called the Beineke graphs, that characterizes the class of line graphs of simple graphs [2]. It turns out that by excluding only two of the Beineke graphs, namely claws and $K_5 - e$'s (where a *claw* is a graph that has 4 nodes and 3 edges whose one vertex is adjacent to all the others, and $K_5 - e$ is the graph obtained from a clique on 5 nodes by removing an edge), one gets a family of graphs with the Vizing bound [15]. We obtain the Vizing bound for the chromatic number by forbidding a family of graphs none of which is acyclic.

The essence of even-hole-free graphs is actually captured by their generalization to signed graphs, called the odd-signable graphs, and in fact the decomposition theorem that we prove in this paper is for the class of graphs that generalizes (even-hole, diamond)-free graphs in this way. Odd-signable graphs are introduced in Section 1.1, and the decomposition theorem is described in Section 1.2. In Section 1.3 we give an idea why a very strong decomposition theorem was required to prove the existence of simplicial extremes in (even-hole, diamond)-free graphs. In Section 1.4, using a technique of Keijsper and Tewes [14], we extend the β-perfection of (even-hole, diamond)-free graphs to a class that now includes all of the previously known classes of β-perfect graphs. In Section 1.5 we introduce the terminology and notation that will be used throughout the paper.

1.1 Odd-signable graphs

We *sign* a graph by assigning $0, 1$ weights to its edges. A graph is *odd-signable* if there exists a signing that makes every triangle odd weight and every hole odd weight. We now characterize odd-signable graphs in terms of forbidden induced subgraphs, that are two types of *3-path configurations* (3PC's) and even wheels.

Let x, y be two distinct nodes of G. A 3PC(x, y) is a graph induced by three chordless x, y-paths, such that any two of them induce a hole. We say that a graph G contains a 3PC$(\cdot,\cdot)$ if it contains a 3PC(x, y) for some $x, y \in V(G)$. 3PC$(\cdot,\cdot)$'s are also known as *thetas* in [5].

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be six distinct nodes of G such that $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles. A 3PC$(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths $P_1 = x_1, \dots, y_1$, $P_2 = x_2, \dots, y_2$ and $P_3 = x_3, \dots, y_3$, such that any two of them induce a hole. We say that a graph G contains a 3PC$(\triangle, \triangle)$ if it contains a 3PC$(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. 3PC$(\triangle, \triangle)$'s are also known as *prisms* in [4] and *stretchers* in [9].

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \dots, x_n$. Node x is the *center* of the wheel. Edges xx_i, for $i \in \{1, \dots, n\}$, are called *spokes* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length greater than 1. Wheel (H, x) is *even* if it has an even number of sectors. If a wheel (H, x) has n spokes, the it is also referred to as an *n-wheel*.

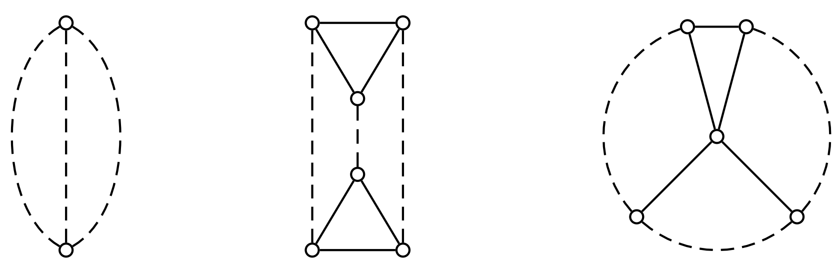


Figure 2: 3PC$(\cdot,\cdot)$, 3PC$(\triangle, \triangle)$ and an even wheel.

Figure 2 depicts a 3PC$(\cdot,\cdot)$, 3PC$(\triangle, \triangle)$ and an even wheel. In this and other figures throughout the paper, solid lines represent edges and dotted lines represent paths of length at least one.

It is easy to see that 3PC$(\cdot,\cdot)$'s, 3PC$(\triangle,\triangle)$'s and even wheels cannot be contained in even-hole-free graphs. In fact they cannot be contained in odd-signable graphs. The following characterization of odd-signable graphs states that the converse also holds, and it is an easy consequence of a theorem of Truemper [20].

Theorem 1.1 ([6])
A graph is odd-signable if and only if it is (even-wheel, 3PC$(\cdot,\cdot)$*,* 3PC$(\triangle,\triangle)$*)-free.*

This characterization of odd-signable graphs will be used throughout the paper.

1.2 Decomposition theorem

For $x \in V(G)$, $N(x)$ denotes the set of nodes of G that are adjacent to x, and $N[x] = N(x) \cup \{x\}$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph of G induced by V'. For $x \in V(G)$, the graph $G[N(x)]$ is called the *neighborhood* of x. For $S \subseteq V(G)$, $N[S]$ is defined to be S together with the set of all nodes of $V(G) \setminus S$ that have a neighbor in S. For an induced subgraph H of G, $N[H] = N[V(H)]$.

Let G be a connected graph. We first introduce three types of cutsets that will be used in our decomposition theorem.

A node set $S \subseteq V(G)$ is a *clique cutset* of G if S induces a clique and $G \setminus S$ is disconnected.

A node set S is a *bisimplicial cutset* of G with *center* x if for some wheel (H, x) of G and for some long sector S_1 of (H, x) with endnodes x_1 and x_2, the following hold.

(i) $S = X_1 \cup X_2 \cup \{x\}$, where $X_1 = N[x_1] \cap N(x)$ and $X_2 = N[x_2] \cap N(x)$.

(ii) $G \setminus S$ contains connected components C_1 and C_2 such that $V(S_1) \setminus \{x_1, x_2\} \subseteq V(C_1)$ and $V(H) \setminus (V(S_1) \cup S) \subseteq V(C_2)$.

Note that in a diamond-free graph the following hold:

(i) $X_1 \cap X_2 = \varnothing$,

(ii) both X_1 and X_2 induce cliques, and

(iii) for every $u \in X_1$ (resp. $u \in X_2$) $X_1 = N[u] \cap N(x)$ (resp. $X_2 = N[u] \cap N(x)$).

We say that S is a bisimplicial cutset that separates S_1 from $H \setminus S_1$.

G has a *2-join*, denoted by $V_1|V_2$, with special sets (A_1, A_2, B_1, B_2) that are nonempty and disjoint, if the nodes of G can be partitioned into sets V_1 and V_2 so that the following hold.

(i) For $i = 1, 2$, $A_i \cup B_i \subseteq V_i$.

(ii) Every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, and these are the only adjacencies between V_1 and V_2.

(iii) For $i = 1, 2$, the graph induced by V_i, $G[V_i]$, contains a path with one endnode in A_i and the other in B_i. Furthermore, if $|A_i| = |B_i| = 1$, then $G[V_i]$ is not a chordless path.

We now introduce two classes of graphs that have no clique cutset, bisimplicial cutset nor a 2-join.

Let x_1, x_2, x_3, y be four distinct nodes of G such that x_1, x_2, x_3 induce a triangle. A 3PC$(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P_{x_1y} = x_1, \ldots, y$, $P_{x_2y} = x_2, \ldots, y$ and $P_{x_3y} = x_3, \ldots, y$, such that any two of them induce a hole. We say that a graph G contains a 3PC$(\triangle, \cdot)$ if it contains a 3PC$(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$. Note that in a $\Sigma =$ 3PC$(\triangle, \cdot)$ at most one of the paths may be of length one. If one of the paths of Σ is of length 1, then Σ is also a wheel that is called a *bug*. If all of the paths of Σ are of length greater than 1, then Σ is a *long* 3PC$(\triangle, \cdot)$, see Figure 3. 3PC$(\triangle, \cdot)$'s are also known as *pyramids* in [4].

Figure 3: A long 3PC$(\triangle, \cdot)$ and a bug.

We now define nontrivial basic graphs. Let L be the line graph of a tree. Note that every edge of L belongs to exactly one maximal clique, and every node of L belongs to at most two maximal cliques. The nodes of L that belong to exactly one maximal clique are called *leaf nodes*. A clique of L is *big* if it is of size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an *internal segment* if it has its endnodes in distinct big cliques (when P is of length 0, it is called an internal segment when the node of P belongs to two big cliques). The other paths P are called *leaf segments*. Note that one of the endnodes of a leaf segment is a leaf node.

A *nontrivial basic graph* R is defined as follows: R contains two adjacent nodes x and y, called the *special* nodes. The graph L induced by $R \setminus \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R, each leaf node of L is adjacent to exactly one of the two special nodes, and no other node of L is adjacent to special nodes. The last condition for R is that no two leaf segments of L with leaf nodes adjacent to the same special node have their other endnode in the same big clique. The *internal segments* of R are the internal segments of L, and the *leaf segments* of R are the leaf segments of L together with the node in $\{x, y\}$ to which the leaf segment is adjacent to.

Let G be a graph that contains a nontrivial basic graph R with special nodes x and y. R^* is an *extended nontrivial basic graph* of G if R^* consists of R and all nodes $u \in V(G) \setminus V(R)$ such that for some big clique K of R and for some $z \in \{x, y\}$, $N(u) \cap V(R) = V(K) \cup \{z\}$. We also say that R^* is an *extension* of R. See Figure 4.

A graph is *basic* if it is one of the following graphs:

(1) a clique,
(2) a hole,
(3) a long 3PC$(\triangle, \cdot)$, or
(4) an extended nontrivial basic graph.

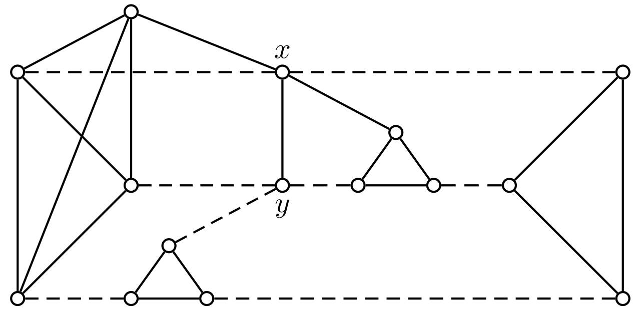


Figure 4: An extended nontrivial basic graph.

Theorem 1.2 *A connected (diamond, 4-hole)-free odd-signable graph is either basic, or it has a clique cutset, a bisimplicial cutset or a 2-join.*

The two key structures in the proof of this decomposition theorem are wheels and $3PC(\triangle, \cdot)$'s. A *proper wheel* is a wheel that is not a bug. Proper wheels are decomposed with bisimplicial cutsets in Section 3. Once the proper wheels are decomposed for the rest of the proof we assume that the graph does not contain a proper wheel. In fact, the proof of the decomposition theorem consists of a sequence of structures that are decomposed (when present in the graph) in that particular order. Once one structure is decomposed for the rest of the proof it is assumed that the graph does not contain that structure. Finding this sequence is the key to any decomposition theorem, and is the most difficult part of it.

The rest of the structures that are decomposed will arise from $3PC(\triangle, \cdot)$. The key is to use either bisimplicial cutsets or 2-joins to separate different paths of a $3PC(\triangle, \cdot)$. But this will not be possible if there exist paths P, as in Figure 5, called the crosspaths (to be defined formally in Section 4). On the other hand, not all of the $3PC(\triangle, \cdot)$'s need to be decomposed because they could be a part of a basic graph.

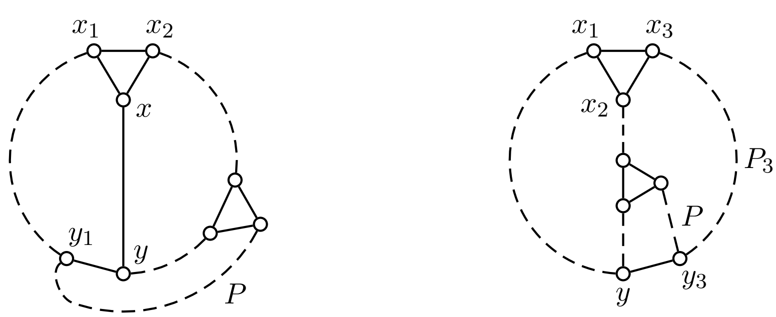


Figure 5: A crosspath P that prevents, in the first case, $N[x]$ from being a bisimplicial cutset separating different sectors of the bug, and in the second case, the existence of a 2-join that separates path P_3 from the other two path of the $3PC(x_1x_2x_3, y)$.

The proof of Theorem 1.2 follows from the following three lemmas, proved in Sections 3, 8 and 10 respectively.

Lemma 1.3 *Let G be a connected (diamond, 4-hole)-free odd-signable graph. If G does not contain a $3PC(\triangle, \cdot)$, then G is either a clique or a hole, or it has a clique cutset or a bisimplicial cutset.*

Lemma 1.4 *Let G be a connected (diamond, 4-hole)-free odd-signable graph. If G contains a* $3\text{PC}(\triangle, \cdot)$ *but does not contain a* $3\text{PC}(\triangle, \cdot)$ *with a crosspath, then either G is a long* $3\text{PC}(\triangle, \cdot)$ *or it has a clique cutset, a bisimplicial cutset or a 2-join.*

Lemma 1.5 *Let G be a connected (diamond, 4-hole)-free odd-signable graph. If G contains a* $3\text{PC}(\triangle, \cdot)$ *with a crosspath, then either G is an extended nontrivial basic graph or G has a clique cutset, a bisimplicial cutset or a 2-join.*

In a connected graph G, a node set S is a k-star cutset if $G \setminus S$ is disconnected, and for some clique C in S of size k, $S \setminus C \subseteq N[C]$. A 1-star cutset is also known as a *star cutset*, a 2-star cutset is also known as a *double star cutset*, and a 3-star cutset is also known as a *triple star cutset*. In [7] Conforti, Cornuéjols, Kapoor and Vušković decompose even-hole-free graphs (in fact 4-hole-free odd-signable graphs) using 2-joins and star, double star and triple star cutsets. This decomposition theorem was strong enough to obtain a decomposition based recognition algorithm for even-hole-free graphs [8], but even at the time it was clear that it was not the strongest possible decomposition theorem for even-hole-free graphs. In [19] da Silva and Vušković are working on obtaining a decomposition theorem for even-hole-free graphs (in fact for 4-hole-free odd-signable graphs) that uses just 2-joins and star cutsets. The approach is to first reduce the problem to the diamond-free case and then use Theorem 1.2.

1.3 Simplicial extremes

Recall that a vertex v is a *simplicial extreme* of a graph G, if it is either a simplicial vertex (*i.e.*, a vertex whose neighborhood induces a clique) or a vertex of degree 2. In Section 11 we use Theorem 1.2 to prove the following characterization of (even-hole, diamond)-free graphs.

Theorem 1.6 *Every (even-hole, diamond)-free graph contains a simplicial extreme.*

This characterization and the following characterization of minimal β-imperfect graphs, by Markossian, Gasparian and Reed [16], proves that (even-hole, diamond)-free graphs are β-perfect.

Lemma 1.7 ([16]) *A minimal β-imperfect graph that is not an even hole, contains no simplicial extreme.*

Theorem 1.8 *Every (even-hole, diamond)-free graph is β-perfect.*

Proof: Follows from Theorem 1.6 and Lemma 1.7. □

Theorems 1.6 and 1.8 were actually conjectured to be true by de Figueiredo and Vušković [12]. In [12] they prove that every (even-hole, diamond, cap-on-6-vertices)-free is β-perfect by showing the following characterization of this class of graphs.

Theorem 1.9 ([12]) *If G is an (even-hole, diamond, cap-on-6-vertices)-free graph, then one of the following holds.*

(1) G is triangulated.

(2) For every edge xy, G has a simplicial extreme in $G \setminus N[\{x, y\}]$.

Similar property was used in [1] to prove that every even-hole-free graph has a bisimplicial vertex.

Theorem 1.10 ([1]) *If G is even-hole-free then the following hold.*

(1) If K is a clique of G of size at most 2 such that $N[K] \neq V(G)$, then G has a bisimplicial vertex in $G \setminus N[K]$.

(2) If H is a hole of G such that $N[H] \neq V(G)$, then G has a bisimplicial vertex in $G \setminus N[H]$.

Such characterizations allowed for certain types of double star cutsets to be used in the inductive proofs of Theorem 1.9 and Theorem 1.10. For assume that Theorem 1.9 (resp. Theorem 1.10) holds for all graphs with fewer vertices than G, and suppose that for an edge xy, $N[\{x,y\}]$ is a double star cutset of G. Then we can conclude that for every connected component C of $G \setminus N[\{x,y\}]$, there exists a simplicial extreme (resp. bisimplicial vertex) of G in C.

For the class of (even-hole, diamond)-free graphs it is not even the case that for every vertex there is a simplicial extreme outside the neighborhood of that vertex. The graph in Figure 6 is (even-hole, diamond)-free, and its only simplicial extremes are in the neighborhood of vertex x. Note that this graph contains a cap on 6 vertices. Also, all the vertices of this graph, except x, are bisimplicial vertices, so for any edge there is a bisimplicial vertex outside of the neighborhood of that edge.

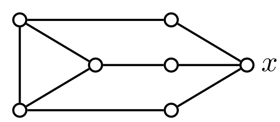


Figure 6: An (even-hole, diamond)-free graph whose only simplicial extremes are in the neighborhood of x.

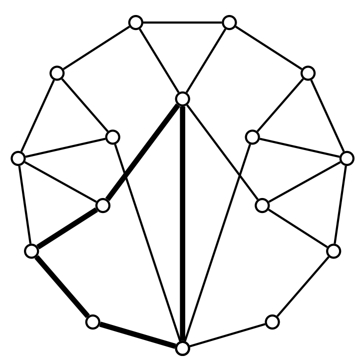


Figure 7: An (even-hole, diamond)-free graph G, bold edges denote a hole H such that no vertex of $G - N[H]$ is a simplicial extreme of G.

(2) of Theorem 1.10 is used to help prove (1). Figure 7 shows that an analogous property does not hold in our case: bold edges denote a hole H such that no vertex of $G \setminus N[H]$ is a simplicial extreme of G.

We prove Theorem 1.6 by proving the following characterization of (even-hole, diamond)-free graphs.

Theorem 1.11 *If G is an (even-hole, diamond)-free graph, then one of the following holds.*

(1) G is a clique.

(2) G contains two nonadjacent simplicial extremes.

This characterization does not allow us to use double star cutset decompositions in our proof, not even star cutset decompositions. We really had to strengthen our decomposition theorem as much as we could, in order to make it useful for proving Theorem 1.11.

1.4 Enlarging the class of β-perfect graphs obtained

All the β-perfect graphs obtained so far have simplicial extremes, and hence have the following special property: for every induced subgraph H of G, either $\chi(H) = \omega(H)$ or $\chi(H) = 3 > 2 = \omega(H)$. In [14] Keijsper and Tewes introduce a more general type of β-perfect graphs by proving the following extension of the result in [12].

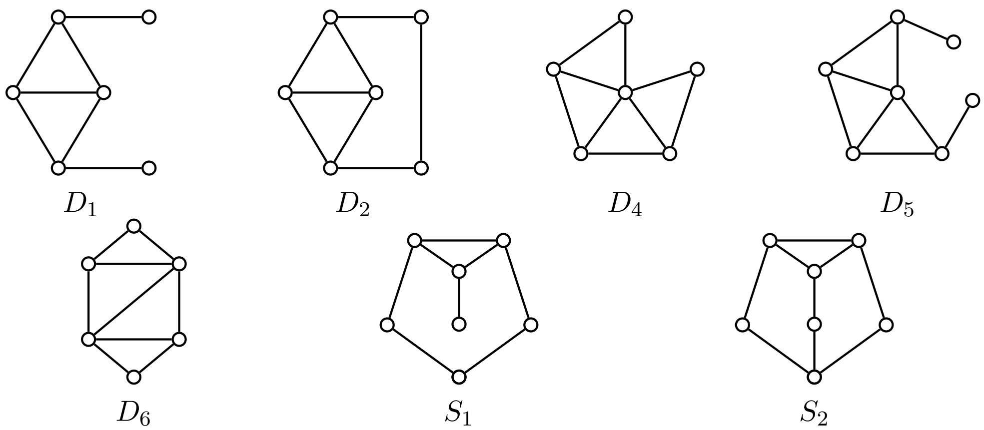
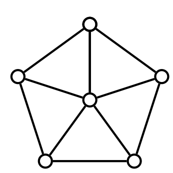


Figure 8: Forbidden subgraphs for β-perfect graphs.

Theorem 1.12 ([14]) *If G is an even-hole-free graph that contains none of the graphs in Figure 8, then G is β-perfect.*



D_3

Figure 9: The complete 5-wheel.

Note that, as evidenced by the graph in Figure 9, the graphs satisfying the condition of Theorem 1.12 need not have simplicial extremes and in general do not have the special property described above.

We now extend Theorem 1.8 using the technique used by Keijsper and Tewes to prove Theorem 1.12.

Lemma 1.13 ([14]) *Let H be a minimal induced subgraph of G that satisfies $\beta(G) = \delta(H) + 1$. If H is (4-hole, 6-hole)-free, then H contains a diamond if and only if H contains D_1, D_2, D_3, D_4, D_5 or D_6.*

Lemma 1.14 ([14]) *Let H be a minimal induced subgraph of G that satisfies $\beta(G) = \delta(H) + 1$, and assume that H is 4-hole-free. If H contains a D_3, then H contains D_1, D_2, D_4 or D_6 or $\chi(H) = \beta(H)$.*

Lemma 1.15 ([14]) *Let G be an even-hole-free graph and let H be an induced subgraph of G such that $\beta(G) = \delta(H) + 1$. If H contains a simplicial extreme then $\chi(G) = \beta(G)$.*

Corollary 1.16 *Every (even-hole, D_1, D_2, D_4, D_5, D_6)-free graph is β-perfect.*

Proof: Let G be an (even-hole, D_1, D_2, D_4, D_5, D_6)-free graph. It suffices to prove that $\chi(G) = \beta(G)$. Let H be a minimal induced subgraph of G satisfying $\beta(G) = \delta(H) + 1$. If H is diamond-free, then by Theorem 1.6, H contains a simplicial extreme, and hence by Lemma 1.15, $\chi(G) = \beta(G)$. If H contains a diamond, then by Lemma 1.13, H contains a D_3, and hence Lemma 1.14 gives $\chi(G) \leq \beta(G) = \delta(H) + 1 = \beta(H) = \chi(H) \leq \chi(G)$. □

This class of graphs now includes all of the previously known classes of β-perfect graphs.

1.5 Terminology and notation

A *path* P is a sequence of distinct nodes $x_1, \ldots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. These are called the *edges* of the path P. Nodes x_1 and x_n are the *endnodes* of the path. The nodes of $V(P)$ that are not endnodes are called the *intermediate* nodes of P. Let x_i and x_l be two nodes of P, such that $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the $x_i x_l$-subpath of P. Let Q be the $x_i x_l$-subpath of P. We write $P = x_1, \ldots, x_{i-1}, Q, x_{l+1}, \ldots, x_n$. A *cycle* C is a sequence of nodes $x_1, \ldots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, \ldots, x_n$ form a path and $x_1 x_n$ is an edge. The edges of the path $x_1, \ldots, x_n$ together with the edge $x_1 x_n$ are called the *edges* of cycle C. The *length* of a path P (resp. cycle C) is the number of edges in P (resp. C).

Given a path or a cycle Q in a graph G, any edge of G between nodes of Q that is not an edge of Q is called a *chord* of Q. Q is *chordless* if no edge of G is a chord of Q. As mentioned earlier a *hole* is a chordless cycle of length at least 4. It is called a k-hole if it has k edges. A k-hole is even if k is even, and it is odd otherwise.

Let A, B be two disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, \ldots, x_n$ *connects A and B* if either $n = 1$ and x_1 has a neighbor in A and B, or $n > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between A and B* if in $G[V(P) \cup A \cup B]$ no path connecting A and B is shorter than P. The direct connection P is said to be *from A to B* if x_1 is adjacent to a node in A and x_n is adjacent to a node in B.

A note on notation: For a graph G, let $V(G)$ denote its node set. For simplicity of notation we will sometimes write G instead of $V(G)$, when it is clear from the context that we want to refer to the node set of G. We will not distinguish between a node set and the graph induced by that node set. Also a singleton set $\{x\}$ will sometimes be denoted with just x. For example, instead of "$u \in V(G) \setminus \{x\}$", we will write "$u \in G \setminus x$". These simplifications of notation will take place in the proofs, whereas the statements of results will use proper notation.

2 Appendices to a hole

In our decomposition theorem we use bisimplicial cutsets and 2-joins to break apart holes of the graph. We begin by analyzing particular types of paths, called the appendices, that connect nodes of a hole. Throughout this section we assume that G is a (diamond, 4-hole)-free odd-signable graph.

Definition 2.1 *Let H be a hole. A chordless path $P = p_1, \ldots, p_k$ in $G \setminus H$ is an* appendix *of H if no node of $P \setminus \{p_1, p_k\}$ has a neighbor in H, and one of the following holds:*

(i) $k = 1$ and (H, p_1) is a bug ($N(p_1) \cap V(H) = \{u_1, u_2, u\}$, such that u_1u_2 is an edge), or

(ii) $k > 1$, p_1 has exactly two neighbors u_1 and u_2 in H, u_1u_2 is an edge, p_k has a single neighbor u in H, and $u \notin \{u_1, u_2\}$.

Nodes u_1, u_2, u are called the attachments *of appendix P to H. We say that u_1u_2 is the* edge-attachment *and u is the* node-attachment.

Let H'_P (resp. H''_P) be the u_1u-subpath (resp. u_2u-subpath) of H that does not contain u_2 (resp. u_1). H'_P and H''_P are called the sectors *of H w.r.t. P.*

Let Q be another appendix of H, with edge attachment v_1v_2 and node-attachment v. Appendices P and Q are said to be crossing *if one sector of H w.r.t. P contains v_1 and v_2, say H'_P does, and $v \in V(H''_P) \setminus \{u\}$, see Figure 10.*

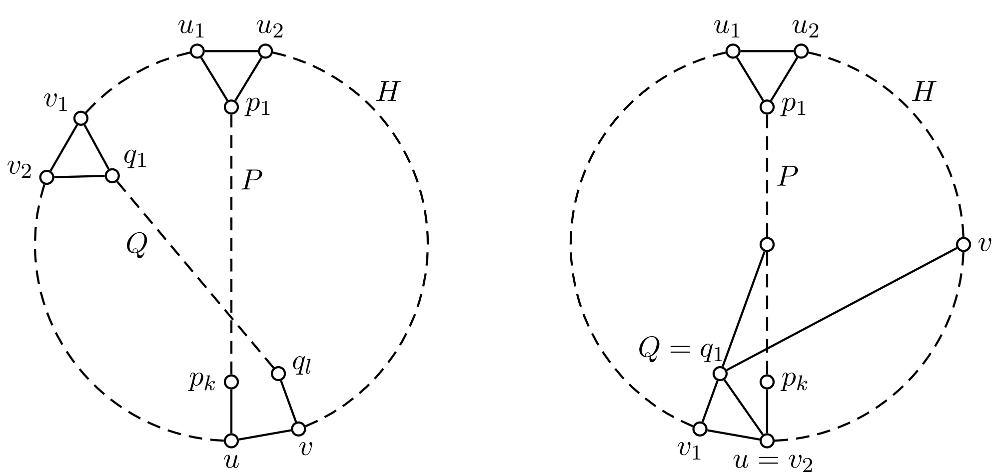


Figure 10: Crossing appendices P and Q of a hole H.

Lemma 2.2 *Let P be an appendix of a hole H, with edge-attachment u_1u_2 and node-attachment u. Let H'_P (resp. H''_P) be the sector of H w.r.t. P that contains u_1 (resp. u_2). Let $Q = q_1, \ldots, q_l$ be a chordless path in $G \setminus H$ such that q_1 has a neighbor in H'_P, q_l has a neighbor in H''_P, no node of $Q \setminus \{q_1, q_l\}$ is adjacent to a node of H and one of the following holds:*

(i) $l = 1$, q_1 is not adjacent to u, and if u_1 (resp. u_2) is the unique neighbor of q_1 in H'_P (resp. H''_P), then u_2 (resp. u_1) is not adjacent to q_1, or

(ii) $l > 1$, $N(q_1) \cap V(H) \subseteq V(H'_P) \setminus \{u\}$, $N(q_l) \cap V(H) \subseteq V(H''_P) \setminus \{u\}$, q_1 *has a neighbor in* $H'_P \setminus \{u_1\}$, *and* q_l *has a neighbor in* $H''_P \setminus \{u_2\}$.

Then Q *is also an appendix of* H *and its node-attachment is adjacent to* u. *Furthermore, no node of* P *is adjacent to or coincident with a node of* Q.

Proof: Let $P = p_1, \ldots, p_k$ and assume that p_1 is adjacent to u_1 and u_2. Let u'_1 (resp. u'_2) be the neighbor of q_1 in H'_P that is closest to u (resp. u_1). Let u''_1 (resp. u''_2) be the neighbor of q_l in H''_P that is closest to u (resp. u_2). Note that either $u'_1 \neq u_1$ or $u''_1 \neq u_2$. Let S'_1 (resp. S'_2) be the u'_1u-subpath (resp. u'_2u_1-subpath) of H'_P, and let S''_1 (resp. S''_2) be the u''_1u-subpath (resp. u''_2u_2-subpath) of H''_P. Let H' (resp. H'') be the hole induced by $H'_P \cup P$ (resp. $H''_P \cup P$).

First suppose that $l = 1$. Note that q_1 cannot be coincident with a node of P. Suppose q_1 has a neighbor in P. Note that if q_1 is adjacent to p_1, then since there is no diamond, $u'_1 \neq u_1$ and $u''_1 \neq u_2$. But then $P \cup S'_1 \cup S''_1 \cup q_1$ contains a 3PC(q_1, u). So q_1 has no neighbor in P. Since $H \cup q_1$ cannot induce a 3PC(u'_1, u''_1), q_1 has at least three neighbors in H. Since (H, q_1) cannot be an even wheel, without loss of generality q_1 has an odd number of neighbors in H'_P and an even number of neighbors in H''_P. Since $H'' \cup q_1$ cannot induce a 3PC(u''_1, u''_2) nor an even wheel with center q_1, $u''_1u''_2$ is an edge. If u'_2 is not adjacent to u, then $H'' \cup S'_2 \cup q_1$ induces either an even wheel with center u_2 (when $u_2 = u''_2$) or a 3PC$(p_1u_1u_2, q_1u''_1u''_2)$ (when $u_2 \neq u''_2$). So u'_2 is adjacent to u, and the lemma holds.

Now suppose that $l > 1$. So $u'_1 \neq u_1$ and $u''_1 \neq u_2$. Not both q_1 and q_l can have a single neighbor in H, since otherwise $H \cup Q$ induces a 3PC(u'_1, u''_1). Without loss of generality $u''_1 \neq u''_2$.

Suppose that $u''_1u''_2$ is not an edge. A node of P must be adjacent to or coincident with a node of Q, else $H'' \cup Q \cup S'_1$ induces a 3PC(q_l, u). Note that no node of $\{q_1, q_l\}$ is coincident with a node of $\{p_1, p_k\}$, and if a node of Q is coincident with a node of P, then a node of Q is also adjacent to a node of P. Let q_i be the node of Q with highest index that has a neighbor in P. (Note that q_i is not coincident with a node of P). Let p_j be the node of P with highest index adjacent to q_i. If $j > 1$ and $i > 1$, then $H \cup \{p_j, \ldots, p_k, q_i, \ldots, q_l\}$ contains a 3PC(q_l, u). If $i = 1$, then $S'_1 \cup S''_1 \cup Q \cup \{p_j, \ldots, p_k\}$ induces a 3PC(q_1, u). So $i > 1$, and hence $j = 1$. If $i < l$, then $S''_1 \cup S''_2 \cup P \cup \{q_i, \ldots, q_l\}$ induces a 3PC(p_1, q_l). So $i = l$. Since $H \cup q_l$ cannot induce a 3PC(u''_1, u''_2), (H, q_l) is a wheel. But then one of the wheels (H, q_l) or (H'', q_l) must be even.

Therefore $u''_1u''_2$ is an edge. Suppose that $u'_1 \neq u'_2$. Then by symmetry, $u'_1u'_2$ is an edge, and hence $H \cup Q$ induces a 3PC$(q_1u'_1u'_2, q_lu''_1u''_2)$. Therefore $u'_1 = u'_2$, *i.e.*, Q is an appendix of H.

Suppose that a node of P is adjacent to or coincident with a node of Q. Let q_i be the node of Q with highest index adjacent to a node of P, and let p_j be the node of P with lowest index adjacent to q_i. If $i > 1$ and $j < k$, then $H \cup \{p_1, \ldots, p_j, q_i, \ldots, q_l\}$ induces an even wheel with center u_2 or a 3PC$(p_1u_1u_2, q_lu''_1u''_2)$. If $i = 1$, then $P \cup Q \cup S'_1 \cup S''_1$ contains a 3PC(q_1, u). So $i > 1$, and hence $j = k$.

If p_k has a unique neighbor in Q, then $Q \cup S'_1 \cup S''_1 \cup p_k$ induces a 3PC(q_i, u). So p_k has more than one neighbor in Q.

Suppose that $k = 1$. Then either $S'_2 \cup S''_2 \cup Q \cup p_1$ or $S'_1 \cup S''_1 \cup Q \cup p_1$ induces an even wheel with center p_1. So $k > 1$.

Let T' (resp. T'') be the hole induced by $S'_1 \cup S''_1 \cup Q$ (resp. $S'_2 \cup S''_2 \cup Q$). If both (T', p_k) and (T'', p_k) are wheels, then one of them is even. So p_k has exactly two neighbors in Q.

Since $T'' \cup p_k$ cannot induce a 3PC$(\cdot,\cdot)$, $N(p_k) \cap Q = \{q_i, q_{i-1}\}$. (Note that q_{i-1} is not coincident with a node of P, since $j = k$). If no node of $P \setminus p_k$ has a neighbor in Q, then $(H \setminus ((S'_1 \cup S''_1) \setminus u'_1)) \cup P \cup Q$ induces a 3PC$(p_1u_1u_2, p_kq_iq_{i-1})$. So a node of $P \setminus p_k$ has a neighbor in Q. Let p_t be such a node with lowest index. Let q_s be the node of Q with highest index adjacent to p_t. Note that by the choice of i and j, $s \leq i-1$. If $t \neq k-1$ then $H''_P \cup \{p_1, \ldots, p_t, p_k, q_s, \ldots, q_l\}$ induces an even wheel with center q_l or a 3PC$(q_lu''_1u''_2, p_kq_iq_{i-1})$. So $t = k-1$, *i.e.*, p_k and p_{k-1} are the only nodes of P that have a neighbor in Q. If $s \neq 1$ then $(H \setminus S''_2) \cup P \cup \{q_s, \ldots, q_l\}$ induces an even wheel with center p_k. So $s = 1$. If $i - 1 = 1$ then $\{q_{i-1}, q_i, p_k, p_{k-1}\}$ induces a diamond. So $i - 1 > 1$, *i.e.*, p_k is not adjacent to q_1. But then $S'_1 \cup \{q_1, \ldots, q_{i-1}, p_{k-1}, p_k\}$ induces a 3PC(q_1, p_k).

Therefore, no node of P is adjacent to or coincident with a node of Q. If u'_1u is not an edge, then $(H \setminus S''_2) \cup P \cup Q$ induces a 3PC(u'_1, u). Therefore u'_1u is an edge. □

Lemma 2.3 *Let $P = p_1, \ldots, p_k$ be an appendix of a hole H, with edge-attachment u_1u_2 and node-attachment u, with p_1 adjacent to u_1, u_2. Let $Q = q_1, \ldots, q_l$ be another appendix of H, with edge-attachment v_1v_2 and node-attachment v, with q_1 adjacent to v_1, v_2. If P and Q are crossing, then one of the following holds:*

(i) uv is an edge,

(ii) $l = 1$, $u \in \{v_1, v_2\}$ and q_1 has a neighbor in $P \setminus \{p_k\}$, or

(iii) $k = 1$, $v \in \{u_1, u_2\}$ and p_1 has a neighbor in $Q \setminus \{q_l\}$.

Proof: Let H'_P (resp. H''_P) be the sector of H w.r.t. P that contains u_1 (resp. u_2). Without loss of generality $\{v_1, v_2\} \subseteq H'_P$ and v_1 is the neighbor of q_1 in H'_P that is closer to u_1. Assume uv is not an edge.

By Lemma 2.2 either $v_2 = u$ or $u_2 = v$. Without loss of generality assume that $v_2 = u$. Let S_1 (resp. S_2) be the uv-subpath (resp. u_2v-subpath) of H''_P. A node of P must be coincident with or adjacent to a node of Q, else $H'_P \cup S_2 \cup P \cup Q$ induces a 3PC$(p_1u_1u_2, q_1v_1u)$ or an even wheel with center u_1. Note that no node of $\{q_1, q_l\}$ is coincident with a node of $\{p_1, p_k\}$. Let q_i be the node of Q with lowest index adjacent to P. (So q_i is not coincident with a node of P). Let p_j be the node of P with lowest index adjacent to q_i.

If $j < k$ and $i < l$, then $H \cup \{p_1, \ldots, p_j, q_1, \ldots, q_i\}$ induces a 3PC$(p_1u_1u_2, q_1v_1u)$ or an even wheel with center u_1.

Suppose $j = k$. Note that since there is no diamond, p_k is not adjacent to q_1. If $N(p_k) \cap Q = q_i$, then $S_1 \cup Q \cup p_k$ induces a 3PC(u, q_i). So p_k has more than one neighbor in Q. Let T' (resp. T'') be the hole induced by $S_1 \cup Q$ (resp. $(H \setminus (S_1 \setminus v)) \cup Q$). Note that (T', p_k) is a wheel. If (T'', p_k) is also a wheel, then one of these two wheels must be even. So (T'', p_k) is not a wheel, and hence $k > 1$ and p_k has exactly two neighbors in Q. $N(p_k) \cap Q = \{q_i, q_{i+1}\}$, else $T'' \cup p_k$ induces a 3PC$(\cdot,\cdot)$. But then $H'_P \cup S_2 \cup Q \cup p_k$ induces a 3PC$(q_1v_1u, p_kq_iq_{i+1})$.

So $j < k$, and hence $i = l$. In particular, q_l is the only node of Q that has a neighbor in P. If $l > 1$ then $S_1 \cup Q \cup \{p_j, \ldots, p_k\}$ contains a 3PC(u, q_l). So $l = 1$. □

3 Proper wheels

Definition 3.1 *A* bug *is a wheel with three sectors, exactly one of which is short. A* proper wheel *is a wheel that is not a bug.*

In this section we prove the following theorem. Lemma 1.3 will follow from it.

Theorem 3.2 *Let G be a (diamond, 4-hole)-free odd-signable graph. If G contains a proper wheel (H,x), then for some two distinct long sectors S_i and S_j of (H,x), G has a bisimplicial cutset with center x that separates S_i from $H \setminus S_i$, and G has a bisimplicial cutset with center x that separates S_j from $H \setminus S_j$.*

Throughout this section we assume that G is a (diamond, 4-hole)-free odd-signable graph, and (H,x) is a proper wheel of G with fewest number of spokes, and among all proper wheels with the same number of spokes as (H,x), H has the shortest length. Let $x_1,\dots,x_n$ be the neighbors of x in H, appearing in this order when traversing H. For $i=1,\dots,n$, let X_i be the set of nodes comprised of x_i and all nodes of G that are adjacent to both x and x_i. Since G has no diamond, for every $i=1,\dots,n$, X_i induces a clique. Furthermore, for $i,j \in \{1,\dots,n\}$, $i \neq j$, if x_ix_j is an edge then $X_i = X_j$, and otherwise $X_i \cap X_j = \emptyset$. Let $X = X_1 \cup \ldots \cup X_n \cup \{x\}$. For $i=1,\dots,n$, let S_i be the sector of (H,x) whose endnodes are x_i and x_{i+1} (here and throughout this section we assume that indices are taken modulo n).

Lemma 3.3 *Let u be a node of $G \setminus (V(H) \cup \{x\})$ that has a neighbor in H. Then u is one of the following types.*

Type 1: *Node u is not adjacent to x and it has exactly one neighbor in H.*

Type 2: *Node u is not adjacent to x and it has exactly two neighbors in H. These two neighbors are furthermore adjacent and belong to a long sector of (H,x).*

Type b: *(H,x) is a 5-wheel, u is not adjacent to x, (H,u) is a bug, for some sector S_i, u has two adjacent neighbors in $V(S_i) \setminus \{x_i, x_{i+1}\}$, and its third neighbor in H is x_{i+3}.*

Type bx: *Node u is adjacent to x, for some sector S_i, $N(u) \cap V(H) \subseteq V(S_i) \setminus \{x_i, x_{i+1}\}$, and $V(S_i) \cup \{u,x\}$ induces a bug.*

Type x: *$N(u) \cap (V(H) \cup \{x\}) = \{x\}$.*

Type x1: *For some $i \in \{1,\dots,n\}$, $N(u) \cap (V(H) \cup \{x\}) = \{x, x_i\}$, and sectors S_i and S_{i-1} are long.*

Type x2: *For some $i \in \{1,\dots,n\}$, $N(u) \cap (V(H) \cup \{x\}) = \{x, x_i, x_{i+1}\}$, and x_ix_{i+1} is an edge.*

Type wx1: *For some $i \in \{1,\dots,n\}$, $N(u) \cap \{x, x_1,\dots,x_n\} = \{x, x_i\}$, sectors S_i and S_{i-1} are long, u has a neighbor in every long sector of (H,x), and either u has a neighbor in both $S_{i-1} \setminus \{x_i\}$ and $S_i \setminus \{x_i\}$, or in neither $S_{i-1} \setminus \{x_i\}$ nor $S_i \setminus \{x_i\}$.*

Type wx2: *For some $i \in \{1,\dots,n\}$, $N(u) \cap \{x, x_1,\dots,x_n\} = \{x, x_i, x_{i+1}\}$, x_ix_{i+1} is an edge, u has a neighbor in every long sector of (H,x), and u has a neighbor in exactly one of $S_{i-1} \setminus \{x_i\}$ or $S_{i+1} \setminus \{x_{i+1}\}$.*

Proof: We consider the following cases.

Case 1: u is not adjacent to x.

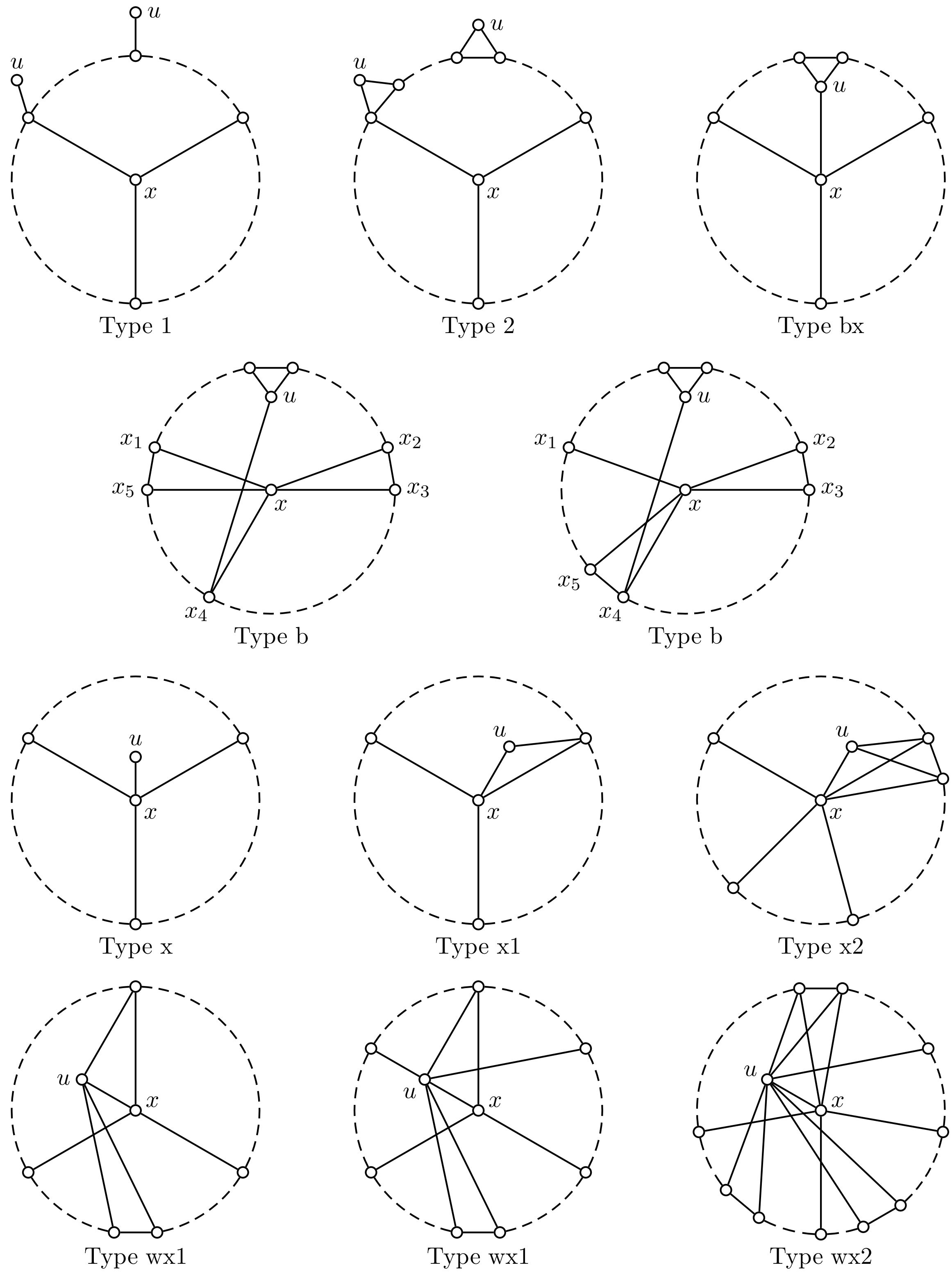


Figure 11: Different types of adjacencies between a vertex u and a proper wheel (H, x).

If u has a neighbor in $\{x_1, \dots, x_n\}$, then since G has no diamond nor a 4-hole, $|N(u) \cap \{x_1, \dots, x_n\}| \leq 1$. Therefore, if u has no neighbor in an interior of some long sector, then u is of type 1. So assume that u has a neighbor in the interior of without loss of generality S_1. Let v_1 (resp. v_2) be the neighbor of x_1 (resp. x_2) in $H \setminus S_1$.

Suppose that $N(u) \cap H \subseteq S_1$. Let u_1 (resp. u_2) be the neighbor of u in S_1 that is closest to x_1 (resp. x_2). If $u_1 = u_2$, then u is of type 1. If u_1u_2 is an edge, then u is of type 2. So assume that $u_1 \neq u_2$ and u_1u_2 is not an edge. Let S_1' be the u_1u_2-subpath of S_1. Since G has no 4-hole nor a diamond, S_1' is of length greater than two. But then $(H \setminus (S_1' \setminus \{u_1, u_2\})) \cup \{u, x\}$ induces a proper wheel with center x, that contradicts the choice of (H, x).

So assume that u has a neighbor in $H \setminus S_1$.

Case 1.1: u has a unique neighbor u' in S_1.

$N(u) \cap H \subseteq S_1 \cup \{v_1, v_2\}$, else $(H \setminus \{v_1, v_2\}) \cup \{u, x\}$ contains a 3PC(x, u'). Node u must be adjacent to both v_1 and v_2, else $H \cup u$ induces a 3PC$(u', \cdot)$.

Suppose that x is not adjacent to v_2. If $u'x_2$ is not an edge, then $S_1 \cup \{u, x, v_2\}$ induces a 3PC(x_2, u'). So $u'x_2$ is an edge. But then $\{u', x_2, v_2, u\}$ induces a 4-hole. So x is adjacent to v_2, and by symmetry it is also adjacent to v_1.

Let H' be the hole induced by $(H \setminus S_1) \cup u$. Since G has no diamond, (H', x) is a proper wheel, and it has fewer spokes than (H, x), a contradiction.

Case 1.2: u has two nonadjacent neighbors in S_1.

$N(u) \cap H \subseteq S_1 \cup \{v_1, v_2\}$, else $(H \setminus \{v_1, v_2\}) \cup \{u, x\}$ contains a 3PC(x, u). Node u must have an odd number of neighbors in S_1, since otherwise $S_1 \cup \{u, x\}$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center u. Since (H, u) cannot be an even wheel, u must be adjacent to both v_1 and v_2. Since $|N(u) \cap \{x_1, \dots x_n\}| \leq 1$, without loss of generality u is not adjacent to x_2. If x is not adjacent to v_2, then $S_1 \cup \{u, x, v_2\}$ contains a 3PC(u, x_2). So x is adjacent to v_2. But then u cannot be adjacent to x_1, and by symmetry x is adjacent to v_1, contradicting the fact that $|N(u) \cap \{x_1, \dots, x_n\}| \leq 1$.

Case 1.3: u has exactly two neighbors in S_1, and they are adjacent.

By Cases 1.1 and 1.2, for every sector S_j, $j \in \{1, \dots, n\}$, if u has a neighbor in the interior of S_j, u has exactly two neighbors in S_j, and these two neighbors are adjacent. Note that if u is adjacent to x_1 (resp. x_2), then since G has no diamond, u cannot have a neighbor in the interior of sector S_n (resp. S_2). Since (H, u) cannot be an even wheel, u must be adjacent to x_i, for some $2 < i \leq n$. Since $|N(u) \cap \{x_1, \dots, x_n\}| \leq 1$, node u is not adjacent to x_1 nor x_2.

Suppose that u has exactly three neighbors in H, *i.e.*, (H, u) is a bug. Let H' and H'' be the two holes, distinct from H, contained in $H \cup u$. Then by minimality of (H, x), both (H', x) and (H'', x) must be bugs, and hence (H, x) is a 5-wheel and u is of type b.

Now we may assume that u has more than three neighbors in H. In fact, since (H, u) cannot be an even wheel, u has at least five neighbors in H. If $x_i = v_2$ then $S_1 \cup \{u, x, v_2\}$ induces a 3PC$(\triangle, \triangle)$. So $x_i \neq v_2$, and by symmetry $x_i \neq v_1$. Let x_i' be the neighbor of x_i in S_i. Let H' be the subpath of H from x_1 to x_i', that contains x_2. By symmetry we may assume that u has a neighbor in $H \setminus H'$. Note that this implies that $i \neq n$. If $x_i' \neq x_n$ then $(H \setminus H') \cup (S_1 \setminus x_1) \cup \{u, x, x_i\}$ contains a 3PC(x, u). So $x_i' = x_n$. Then u has a neighbor in the interior of S_n, and hence it has exactly two neighbors in S_n, say u_1 and u_2, and u_1u_2 is an edge. But then $S_n \cup \{u, x, x_i\}$ induces a 3PC(xx_ix_i', uu_1u_2).

Case 2: u is adjacent to x.

Case 2.1: $N(u) \cap \{x_1, \ldots, x_n\} = \varnothing$.

Assume u is not of type x. Then without loss of generality u has a neighbor in the interior of long sector S_1. If u has a unique neighbor u' in S_1, then $S_1 \cup \{u, x\}$ induces a 3PC(x, u'). Since $S_1 \cup \{u, x\}$ cannot induce an even wheel with center u, node u must have an even number of neighbors in S_1. So if u has a neighbor in a sector of (H, x), it has an even number of neighbors in that sector, and hence u has an even number of neighbors in H. Since $H \cup u$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel, u has exactly two neighbors in H, and these two neighbors are adjacent. Therefore, u is of type bx.

Case 2.2: $N(u) \cap \{x_1, \ldots, x_n\} \neq \varnothing$.

Since G has no diamond, $|N(u) \cap \{x_1, \ldots, x_n\}| \leq 2$. Then without loss of generality either $N(u) \cap \{x_1, \ldots, x_n\} = \{x_2\}$ and sectors S_1 and S_2 are long, or $N(u) \cap \{x_1, \ldots, x_n\} = \{x_2, x_3\}$ and x_2x_3 is an edge. So if u has no neighbor in the interior of some long sector of (H, x), then u is of type x1 or x2. Assume that u does have a neighbor in the interior of some long sector of (H, x).

Case 2.2.1: $N(u) \cap \{x_1, \ldots, x_n\} = \{x_2\}$.

Suppose that u has no neighbor in sectors $S_3, \ldots, S_n$. Then without loss of generality u has a neighbor in $S_1 \setminus x_2$. Let u_1 be such a neighbor that is closest to x_1, and let S_1' be the u_1x_2-subpath of S_1. Since G has no 4-hole nor a diamond, S_1' is of length greater than two. If u has no neighbor in $S_2 \setminus x_2$, then $(H \setminus (S_1' \setminus \{u_1, x_2\})) \cup \{u, x\}$ induces an even wheel with center x. So u has a neighbor in $S_2 \setminus x_2$. Let u_2 be such a neighbor that is closest to x_3, and let S_2' be the u_2x_2-subpath of S_2. Let H' be the hole induced by $(H \setminus ((S_1' \cup S_2') \setminus \{u_1, u_2\})) \cup u$. Then (H', x) is a proper wheel with the same number of spokes as (H, x), but H' is shorter than H, contradicting our choice of (H, x). Therefore u must have a neighbor in $S_3 \cup \ldots \cup S_n$.

If u has exactly two neighbors in H, then $H \cup u$ induces a 3PC$(\cdot, \cdot)$. So (H, u) must be a wheel. Suppose that for some long sector S_i, $3 \leq i \leq n$, u has no neighbor in S_i. Let S be a sector of (H, u) that contains S_i. Then $S \cup \{u, x\}$ induces a wheel with center x that has at least three long sectors, and hence it is a proper wheel with fewer spokes than (H, x), a contradiction. So u has a neighbor in every long sector of (H, x).

Suppose u is not of type wx1. Then without loss of generality u has a neighbor in $S_1 \setminus x_2$ and no neighbor in $S_2 \setminus x_2$. Let S be a sector of (H, u) that does not contain x_2. Since $S \cup \{u, x\}$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel with center x, node x has an even number of neighbors in S. Now let S be a sector of (H, u) that contains x_2 and x_3. Then x has an even number of neighbors in S, else $S \cup \{u, x\}$ induces an even wheel. But then x has an even number of neighbors in H, a contradiction.

Case 2.2.2: $N(u) \cap \{x_1, \ldots, x_n\} = \{x_2, x_3\}$.

Suppose that u has no neighbors in sectors $S_4, \ldots, S_n$. Then without loss of generality u has a neighbor in $S_1 \setminus x_2$. Let u_1 be such a neighbor that is closest to x_1, and let S_1' be the u_1x_2-subpath of S_1. Since G has no 4-hole nor a diamond, S_1' is of length greater than two. If u has no neighbor in $S_3 \setminus x_3$, then $(H \setminus (S_1' \setminus \{u_1, x_2\})) \cup \{u, x\}$ induces a proper wheel with center x, that contradicts our choice of (H, x). So u has a neighbor in $S_3 \setminus x_3$. Let u_2 be such a neighbor that is closest to x_4. Let H' be the hole induced by u and the u_1u_2-subpath of H that does not contain x_2. Then (H', x) is an even wheel. Therefore, u must have a neighbor in $S_4 \cup \ldots \cup S_n$.

Note that (H, u) is a wheel. Suppose that for some long sector S_i, $4 \leq i \leq n$, u has no neighbor in S_i. Let S be a sector of (H, u) that contains S_i. Then $S \cup \{u, x\}$ induces a wheel with center x that has at least three long sectors, and hence it is a proper wheel with fewer spokes than (H, x), a contradiction. So u has a neighbor in every long sector of (H, x).

Let S be a sector of (H, u) that does not contain x_2, x_3. Since $S \cup \{u, x\}$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel with center x, node x has an even number of neighbors in S. This implies that if u has no neighbor in $(S_1 \cup S_3) \setminus \{x_2, x_3\}$, then x has an even number of neighbors in H, a contradiction. So without loss of generality u has a neighbor in $S_1 \setminus x_2$. Suppose u is not of type wx2. Then u has a neighbor in $S_3 \setminus x_3$. But then x has an even number of neighbors in H, a contradiction. □

Lemma 3.4 *If u and v are type wx1 or wx2 nodes w.r.t. (H, x) such that for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, $u \in X_i$ and $v \in X_j$, then $x_i x_j$ is an edge.*

In particular, (H, x) cannot have both a type wx1 and a type wx2 node; if there is a type wx1 node, then all type wx1 nodes are adjacent to the same node of $\{x_1, \ldots, x_n\}$; if there is a type wx2 node, then all type wx2 nodes are adjacent to the same two node of $\{x_1, \ldots, x_n\}$.

Proof: Assume $x_i x_j$ is not an edge. Then u and v are not adjacent, else the graph induced by the node set $\{x, x_i, x_j, u, v\}$ contains a diamond. Suppose there exist two distinct sectors S_l and S_k of (H, x), such that both u and v have a neighbor in both $S_l \setminus \{x_l, x_{l+1}\}$ and $S_k \setminus \{x_k, x_{k+1}\}$. Then $(S_l \setminus \{x_l, x_{l+1}\}) \cup (S_k \setminus \{x_k, x_{k+1}\}) \cup \{u, v, x\}$ contains a 3PC(u, v). So

(1) there cannot exist two distinct long sectors such that both u and v have a neighbor in the interior of both of the sectors.

We now consider the following two cases.

Case 1: u is of type wx1.

Without loss of generality $i = 2$. Then by Lemma 3.3, S_1 and S_2 are long sectors, and u either has neighbors in both $S_1 \setminus x_2$ and $S_2 \setminus x_2$, or in none of them. If v is not adjacent to x_1 nor x_3, then by Lemma 3.3, v has neighbors in the interior of both S_1 and S_2, and hence $((S_1 \cup S_2) \setminus \{x_1, x_3\}) \cup \{v, x\}$ contains a 3PC(x_2, v). Therefore, without loss of generality v is adjacent to x_3.

Suppose v is of type wx1. Then S_3 is also a long sector, and v has a neighbor in the interior of S_1. If v has a neighbor in $S_2 \setminus x_3$, then $S_1 \cup (S_2 \setminus x_3) \cup \{v, x\}$ contains a 3PC(x_2, v). So v has no neighbor in $S_2 \setminus x_3$, and by Lemma 3.3, it has no neighbor in $S_3 \setminus x_3$. By symmetry, u has no neighbor in $(S_1 \cup S_2) \setminus x_2$. But then $((S_1 \cup S_3) \setminus x_1) \cup \{u, v, x\}$ contains an even wheel with center x.

So v must be if type wx2. Then S_3 is a short sector and S_4 is long, and v has a neighbor in either the interior of S_2 or the interior of S_4 (but not both). Suppose that v has a neighbor in the interior of S_4. By (1), u cannot have a neighbor in $(S_1 \cup S_2) \setminus x_2$. But then $S_2 \cup (S_4 \setminus \{x_4, x_5\}) \cup \{u, v, x\}$ contains an even wheel with center x. So v has no neighbor in the interior of S_4, and hence it has a neighbor in the interior of S_2. By (1), u cannot have a neighbor in $(S_1 \cup S_2) \setminus x_2$. But then $(S_2 \setminus x_3) \cup (S_4 \setminus x_5) \cup \{u, v, x\}$ contains an even wheel with center x.

Case 2: u is of type wx2.

By Case 1 and symmetry, v is also of type wx2. We may assume without loss of generality that u is adjacent to x_2 and x_3, and that u has no neighbor in $S_3 \setminus x_3$. By the choice of u and v, v is not adjacent to x_2 nor x_3. Suppose that v is not adjacent to x_1 nor x_4. Without loss of generality $N(v) \cap \{x_1, \ldots, x_n\} = \{x_j, x_{j+1}\}$ and v has a neighbor in $S_{j+1} \setminus x_{j+1}$. But then both u and v have a neighbor in the interior of S_{j+1}, contradicting (1). Therefore v must be adjacent to either x_1 or x_4.

Suppose that v is adjacent to x_1. Then sectors S_1 and S_{n-1} are long, and by Lemma 3.3 at least one of them contains neighbors of both u and v in its interior. So by (1), $n = 5$. If v has a neighbor in interior of S_1, then $(H \setminus \{x_1, x_2, x_4\}) \cup \{u, v\}$ contains a 3PC(u, v). Otherwise, v has a neighbor in interior of S_4, and hence $(H \setminus \{x_2, x_4, x_5\}) \cup \{u, v\}$ contains a 3PC(u, v).

So v is adjacent to x_4. By (1), $n = 5$ and v has a neighbor in the interior of S_3. But then $(H \setminus \{x_1, x_2, x_4\}) \cup \{u, v\}$ contains a 3PC(u, v). □

Definition 3.5 *Let S_i be a long sector of (H, x). A chordless path $P = p_1, \dots, p_k$ in $G \setminus (V(H) \cup \{x\})$ is an* appendix *of the wheel (H, x), if no node of $P \setminus \{p_1, p_k\}$ has a neighbor in H, and one of the following holds:*

(i) $k = 1$ and p_1 is of type b with two neighbors in sector S_i and the third neighbor being x_l, or

(ii) $k > 1$, p_1 is of type 2 or bx with neighbors in sector S_i, p_k is adjacent to x_l, $l \in \{1, \dots, n\} \setminus \{i, i+1\}$, and it is of type 1 or x1, and $x_l x_i$ and $x_l x_{i+1}$ are not edges.

We say that P is an appendix of S_i to x_l. Note that if P is an appendix of (H, x), then it is also an appendix of H, so all the terminology introduced in Definition 2.1 applies.

Lemma 3.6 *If S_1 is a long sector of (H, x) such that there exists a type wx1 or wx2 node adjacent to an endnode of S_1, then S_1 has no appendix.*

Proof: Let u be a type wx1 or wx2 node adjacent to say x_1. If u is of type wx2, then it is adjacent to x_n and $x_1 x_n$ is an edge. Suppose that $P = p_1, \dots, p_k$ is an appendix of S_1 to x_l.

If u has a neighbor in P, let p_i be such a neighbor with highest index. Let S_1' be the subpath of S_1 whose one endnode is adjacent to u, the other endnode is adjacent to p_1, and no intermediate node of S_1' has a neighbor in $\{u, p_1\}$. If u has no neighbor in the interior of S_1, then $x_1 \in S_1'$, and if it does, then we assume that $x_1 \notin S_1'$. Let S be the sector of (H, u) that contains x_l. Note that S is a long sector of (H, u), and hence, since G does not contain a diamond, x may be adjacent to at most one endnode of S.

We first show that x is adjacent to an endnode of S. Assume it is not. If x_1 is adjacent to an endnode of S, say s, then since x is not adjacent to s, $\{s, x_1, x, u\}$ induces a diamond, a contradiction. So x_1 is not adjacent to any endnode of S. Since $S \cup \{u, x\}$ cannot induce a 3PC$(\cdot, \cdot)$, it must induce a wheel with center x. Since this wheel cannot be a proper wheel that has fewer spokes than (H, x), it must be a bug. Without loss of generality $x_l x_{l+1}$ is an edge. So $N(x) \cap S = \{x_l, x_{l+1}\}$.

Let S' (resp. S'') be the component of $S \setminus x_l$ (resp. $S \setminus x_{l+1}$) that contains x_{l+1} (resp. x_l). If u has a neighbor in P, then let H' (resp. H'') be the hole induced by $S' \cup \{u, x_l, p_i, \dots, p_k\}$ (resp. $S'' \cup \{u, p_i, \dots, p_k\}$). Otherwise let H' (resp. H'') be the hole induced by $S' \cup S_1' \cup P \cup \{u, x_l\}$ (resp. $S'' \cup S_1' \cup P \cup u$).

Since $H'' \cup x$ cannot induce a 3PC(x_l, u), (H'', x) is a wheel. But then (H', x) or (H'', x) is an even wheel.

Therefore, x must be adjacent to an endnode of S. We now consider the following two cases.

Case 1: u is of type wx1.

Since x must be adjacent to an endnode of S, G does not contain a diamond, and S_1 and S_n are long sectors, it follows that x_1 is an endnode of S. Since u is of type wx1 w.r.t. (H, x), either u has neighbors in both $S_1 \setminus x_1$ and $S_n \setminus x_1$, or in neither of them. Since x_l belongs to S, and x_1 is an endnode of S, it follows that u has no neighbor in $(S_1 \cup S_n) \setminus x_1$. So S contains x_2 or x_n. Since u is of type wx1 w.r.t. (H, x), it has a neighbor in every long sector of (H, x). Since, by definition of an appendix of a wheel (Definition 3.5), $l \neq 2$ and $x_l x_2$ is not an edge, it follows that u has a neighbor distinct from x_1 in the $x_1, \dots, x_l$ subpath of H that contains x_2. So S cannot contain x_2, and hence it contains x_n. Since $S \cup \{u, x\}$ cannot induce a proper wheel with center x that has fewer spokes than (H, x), it induces a bug. In particular, $N(x) \cap S = \{x_1, x_n\}$ and $l = n$.

If u has a neighbor in P, then let H' (resp. H'') be the hole induced by $S_n \cup \{u, p_i, \dots, p_k\}$ (resp. $(S \setminus S_n) \cup \{x_n, u, p_i, \dots, p_k\}$). Otherwise, let H' (resp. H'') be the hole induced by $S_n \cup S'_1 \cup P$ (resp. $(S \setminus S_n) \cup S'_1 \cup P \cup \{x_n, u\}$). Since neither $H' \cup x$ nor $H'' \cup x$ can induce a 3PC$(\cdot, \cdot)$, both (H', x) and (H'', x) are wheels, and hence one of them must be even.

Case 2: u is of type wx2.

Then by Lemma 3.3, x_n is an endnode of S, and u has a neighbor in the interior of S_1. Since $S \cup \{u, x\}$ cannot induce an even wheel, $N(x) \cap S = \{x_n, x_l\}$, *i.e.*, $l = n - 1$. Note that by definition of S'_1, $x_1 \notin S'_1$.

If u has a neighbor in P, then let H' (resp. H'') be the hole induced by $S_{n-1} \cup \{u, p_i, \dots, p_k\}$ (resp. $(S \setminus S_{n-1}) \cup \{x_{n-1}, u, p_i, \dots, p_k\}$). Otherwise, let H' (resp. H'') be the hole induced by $S_{n-1} \cup S'_1 \cup P \cup u$ (resp. $(S \setminus S_{n-1}) \cup S'_1 \cup P \cup \{x_{n-1}, u\}$). Since $H'' \cup x$ cannot induce a 3PC$(\cdot, \cdot)$, both (H', x) and (H'', x) are wheels, and hence one of them must be even. □

Lemma 3.7 *There exist at least two long sectors of* (H, x) *that have no appendix.*

Proof: Since there is no diamond, (H, x) has at least two long sectors. So we may assume that some long sector S_t has an appendix R. Note that R is also an appendix w.r.t. H. Let H'_R and H''_R be the two sectors of H w.r.t. R (Definition 2.1). Note that both H'_R and H''_R contain a long sector of (H, x). We now show that each of them must in fact contain a long sector of (H, x) that has no appendix.

Consider all long sectors S_i and their appendices P that have an associated sector H_P of H w.r.t. P, such that $H_P \subseteq H'_R$. Choose such a sector S_i and its appendix $P = p_1, \dots, p_k$ so that H_P is shortest possible. Let $u_1 u_2$ be the edge-attachment of P, and x_m its node-attachment. Without loss of generality assume that H_P contains sector S_m. Note that H_P contains at least one long sector of (H, x). Let S_j be such a long sector with lowest index (*i.e.*, it is such a long sector that is closest to x_m on H_P). Suppose that S_j contains an appendix $Q = q_1, \dots, q_l$ with node-attachment $x_{m'}$ and edge-attachment $v_1 v_2$. If P and Q are not crossing, then the choice of P is contradicted. So P and Q are crossing.

Suppose that $x_m x_{m'}$ is an edge. Then, since there is no diamond, S_m is a long sector, but by definition of an appendix of a wheel (Definition 3.5), $x_{m'}$ cannot be adjacent to x_j, and hence $j > m$, contradicting our choice of j. So $x_m x_{m'}$ is not an edge. Then by Lemma 2.3, without loss of generality $l = 1$ and $x_m \in \{v_1, v_2\}$. Since $l = 1$, q_1 is of type b w.r.t. (H, x). But then by Lemma 3.3, neither v_1 nor v_2 coincides with an endnode of S_j, and hence it is not possible that $x_m \in \{v_1, v_2\}$.

Therefore S_j cannot have an appendix. So some long sector of (H, x) contained in H'_R

does not have an appendix, and by symmetry some long sector of (H, x) contained in H''_R does not have an appendix. □

Lemma 3.8 *The intermediate nodes of the long sectors of* (H, x) *are contained in different connected components of* $G \setminus X$*.*

Proof: Assume not and let $P = p_1, \ldots, p_k$ be a direct connection in $G \setminus X$ from one long sector to another long sector of (H, x). We may assume that (H, x) and P are chosen so that $|P|$ is minimized. By definition of P, no node of P is of type x1, x2, wx1 nor wx2 w.r.t. (H, x). Also, the only nodes of (H, x) that may have a neighbor in the interior of P are the nodes of $\{x, x_1, \ldots, x_n\}$. By Lemma 3.3 and the definition of P, if some p_i, $1 < i < k$, has a neighbor in (H, x), then it has a unique neighbor in (H, x).

Claim 1: *At most two nodes of* $\{x_1, \ldots, x_n\}$ *may have a neighbor in* $P \setminus \{p_1, p_k\}$*, and if* x_i *and* x_j*,* $i \neq j$*, both have a neighbor in* $P \setminus \{p_1, p_k\}$*, then* $x_i x_j$ *is an edge.*

Proof of Claim 1: Let P' be a subpath of $P \setminus \{p_1, p_k\}$ whose one endnode is adjacent to x_i, the other to x_j, $i \neq j$, and no intermediate node of P' is adjacent to a node of $\{x_1, \ldots, x_n\}$. Then $x_i x_j$ is an edge, else $H \cup P'$ induces a $3PC(x_i, x_j)$.

If at least three nodes of $\{x_1, \ldots, x_n\}$ have a neighbor in $P \setminus \{p_1, p_k\}$, then since G has no diamond (and hence (H, x) has no consecutive short sectors), there would exist a subpath P' of $P \setminus \{p_1, p_k\}$ whose one endnode is adjacent to x_i, the other to x_j, $i \neq j$, no intermediate node of P' is adjacent to a node of $\{x_1, \ldots, x_n\}$, and $x_i x_j$ is not an edge. This completes the proof of Claim 1.

Without loss of generality p_1 has a neighbor in the interior of a long sector S_1 and p_k has a neighbor in the interior of a long sector S_l. Let u_1 (resp. u_2) be the neighbor of p_1 in S_1 that is closest to x_1 (resp. x_2). Let S'_1 (resp. S''_1) be the $x_1 u_1$-subpath (resp. $x_2 u_2$-subpath) of S_1. Let v_1 (resp. v_2) be the neighbor of p_k in S_l that is closest to x_l (resp. x_{l+1}). Let S'_l (resp. S''_l) be the $x_l v_1$-subpath (resp. $x_{l+1} v_2$-subpath) of S_l. If x has a neighbor in P, then let p_i (resp. p_j) be the node of P with lowest (resp. highest) index adjacent to x.

Claim 2: *If* x *has a neighbor in* P*, then* x_m*, for every* $m \in \{1, \ldots, n\}$*, has an even number of neighbors in* $p_i, \ldots, p_j$*.*

Proof of Claim 2: Let P' be a subpath of P such that the endnodes of P' are adjacent to x, and no intermediate node of P' is adjacent to x. If node x_m has an odd number of neighbors in P', then $P' \cup \{x, x_m\}$ induces a $3PC(\cdot, \cdot)$ or an even wheel with center x_m. So x_m has an even number of neighbors in P', and hence it has an even number of neighbors in $p_i, \ldots, p_j$. This completes the proof of Claim 2.

By Lemma 3.3 it suffices to consider the following three cases.

Case 1: p_1 is of type b.

Then (H, x) is a 5-wheel, p_1 is adjacent to x_4, and it has two adjacent neighbors in $S_1 \setminus \{x_1, x_2\}$. Let H' (resp. H'') be the hole induced by $S''_1 \cup S_2 \cup S_3 \cup p_1$ (resp. $S'_1 \cup S_4 \cup S_5 \cup p_1$). Since (H, x) is chosen to be a proper wheel in G with fewest number of spokes, both (H', x) and (H'', x) must be bugs. Since G does not contain a diamond, not both $x_3 x_4$ and $x_4 x_5$ can be edges. So without loss of generality we may assume that S_2 is a short sector and S_3 is a long sector.

We first show that p_k cannot be of type b. Assume it is. Note that p_k has a neighbor in interior of some long sector of (H, x), distinct from S_1. If S_4 is a long sector, then by symmetry we may assume that p_k has a neighbor in interior of S_3, and hence is adjacent to x_1. If S_5 is a long sector and p_k has neighbors in interior of S_5, then p_k is adjacent to x_3. Hence p_k is adjacent to either x_1 or x_3. If p_k is adjacent to x_1 and $k > 2$, then $(H \setminus \{x_2, x_3, x_5\}) \cup \{x, p_1, p_k\}$ contains a 3PC(x_1, x_4). If p_k is adjacent to x_1 and $k = 2$, then $S_1' \cup \{x, x_1, x_4, p_1, p_2\}$ induces a 3PC(x_1, p_1). So p_k is adjacent to x_3 and S_5 is a long sector. If $k > 2$, then $S_1'' \cup S_5' \cup \{x, x_3, x_4, p_1, p_k\}$ induces an even wheel with center x. If $k = 2$, then $S_3 \cup \{x, p_1, p_2\}$ induces a 3PC(x_3, x_4). Therefore, p_k cannot be of type b.

Case 1.1: x has a neighbor in P.

Suppose that x_4 does not have a neighbor in $p_2, \ldots, p_{i-1}$. By Claim 1, x_1 or x_2 does not have a neighbor in the interior of P. Without loss of generality x_1 does not. Then $S_1' \cup \{x, x_4, p_1, \ldots, p_i\}$ induces a 3PC(x, p_1). So x_4 has a neighbor in $p_2, \ldots, p_{i-1}$. Then by Claim 1, x_1, x_2 and x_3 do not have neighbors in $P \setminus \{p_1, p_k\}$. Node x_4 must have an even number of neighbors in $p_1, \ldots p_i$, since otherwise $S_1'' \cup \{x, x_4, p_1, \ldots, p_i\}$ induces an even wheel with center x_4. So by Claim 2, x_4 has an even number of neighbors in $p_1, \ldots, p_j$.

Suppose that p_k is of type bx. Then $j = k$. Suppose $l = 3$. If $N(x_4) \cap P \neq \{p_1, p_2\}$, then $S_1'' \cup S_3' \cup P \cup x_4$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x_4. So $N(x_4) \cap P = \{p_1, p_2\}$, and hence $S_1'' \cup S_3 \cup P$ induces a 3PC$(x_4 p_1 p_2, p_k v_1 v_2)$. Therefore $l = 5$. If $N(x_4) \cap P \neq \{p_1, p_2\}$, then $S_1' \cup S_5'' \cup P \cup x_4$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x_4. So $N(x_4) \cap P = \{p_1, p_2\}$, and hence $(H \setminus S_5') \cup P$ induces an even wheel with center p_1. Therefore p_k is not of type bx. So by Lemma 3.3, p_k is of type 1 or 2 w.r.t. (H, x).

Suppose $l = 3$. Let H' (resp. H'') be the hole induced by $S_1'' \cup S_3' \cup P$ (resp. $S_3' \cup \{x, p_j, \ldots, p_k\}$). Suppose that x_4 has a neighbor in $p_j, \ldots, p_k$. Recall that x_4 has an even number of neighbors in $p_1, \ldots, p_j$, and that x_4 cannot be adjacent to p_j (since x is adjacent to p_j). So (H', x_4) is a wheel. If (H'', x_4) is also a wheel, then one of (H', x_4) or (H'', x_4) is an even wheel. So (H'', x_4) is not a wheel. In particular, x_4 has a unique neighbor in $p_j, \ldots, p_k$ and $v_1 x_4$ is not an edge. But then $H'' \cup x_4$ induces a 3PC$(\cdot, \cdot)$. So x_4 has no neighbor in $p_j, \ldots, p_k$, and hence it has an even number of neighbors in P. If $v_1 x_4$ is an edge, then $H'' \cup x_4$ induces a 3PC(x, v_1). So $v_1 x_4$ is not an edge. Since $H' \cup x_4$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel with center x_4, $N(x_4) \cap P = \{p_1, p_2\}$. Since (H', x) cannot be an even wheel, x has an odd number of neighbors in P. But then $S_3'' \cup (P \setminus p_1) \cup x$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x. Therefore $l \neq 3$.

By symmetry we may assume that $l = 5$. Let H' (resp. H'') be the hole induced by $S_1' \cup S_5'' \cup P$ (resp. $S_5'' \cup \{x, p_j, \ldots, p_k\}$). If x_4 has a neighbor in $p_j, \ldots, p_k$, then either (H', x_4) is an even wheel, or $H'' \cup x_4$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x_4. So x_4 has no neighbor in $p_j, \ldots, p_k$, and hence it has an even number of neighbors in P. If $N(x_4) \cap P \neq \{p_1, p_2\}$, then $H' \cup x_4$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x_4. So $N(x_4) \cap P = \{p_1, p_2\}$, and hence $(H \setminus x_5) \cup P$ contains an even wheel with center p_1.

Case 1.2: x has no neighbor in P.

In particular, p_k is not of type bx, and hence it must be of type 1 or 2.

If x_1 has a neighbor in $P \setminus \{p_1, p_k\}$, then by Claim 1, x_4 does not, and hence $S_1' \cup (P \setminus p_k) \cup \{x, x_4\}$ contains a 3PC(x_1, p_1). So x_1 does not have a neighbor in $P \setminus \{p_1, p_k\}$.

If x_2 has a neighbor in $P \setminus \{p_1, p_k\}$, then by Claim 1, x_1 and x_4 do not, and hence $S_1' \cup (P \setminus p_k) \cup \{x, x_2, x_4\}$ contains a 3PC(x, p_1). So x_2 does not have a neighbor in $P \setminus \{p_1, p_k\}$, and by analogous argument neither does x_3.

If $l=3$, then $S_1 \cup S_3' \cup P \cup x$ induces a 3PC($p_1u_1u_2, xx_2x_3$). So $l \neq 3$, and by symmetry we may assume that $l=5$.

If x_4 has no neighbor in $P \setminus \{p_1, p_k\}$, then $S_1'' \cup S_5'' \cup P \cup \{x, x_4\}$ induces a 3PC(x, p_1). So x_4 has a neighbor in $P \setminus \{p_1, p_k\}$. If x_4 has a neighbor in $P \setminus \{p_1, p_2\}$, then $S_1' \cup S_5'' \cup (P \setminus p_2) \cup \{x, x_4\}$ contains a 3PC(x_1, x_4). So $N(x_4) \cap P = \{p_1, p_2\}$. But then $(H \setminus x_5) \cup P$ contains an even wheel with center p_1.

Case 2: p_1 is of type 1.

By Case 1 and symmetry, p_k is not of type b.

Case 2.1: x has a neighbor in P.

If neither x_1 nor x_2 has a neighbor in $p_2, \dots, p_{i-1}$, then $S_1 \cup \{x, p_1, \dots, p_i\}$ induces a 3PC(u_1, x). So without loss of generality x_1 has a neighbor in $p_2, \dots, p_{i-1}$. By Claim 1, $x_2, \dots x_l$ do not have neighbors in $P \setminus \{p_1, p_k\}$. Then x_1u_1 is an edge, else $S_1 \cup \{x, p_1, \dots, p_{i-1}\}$ contains a 3PC(u_1, x_1). So u_1x_2 cannot be an edge, else there is a 4-hole. Node x_1 has an odd number of neighbors in $p_1, \dots, p_i$, else $S_1 \cup \{x, p_1, \dots, p_i\}$ induces an even wheel. By Claim 2, x_1 has an odd number of neighbors in $p_1, \dots, p_j$.

Suppose that x_1v_1 is an edge. Then $l=n$, and p_k is either of type 1 or 2 (adjacent to x_1 and v_1). Since $S_n \cup \{x, p_j, \dots, p_k\}$ cannot induce a 3PC($\cdot,\cdot$) nor an even wheel with center x_1, node x_1 has an odd number of neighbors in $p_j, \dots, p_k$. Recall that no node of $P \setminus \{p_1, p_k\}$ can be adjacent to more than one node of $\{x, x_1, \dots, x_n\}$. In particular, p_j is not adjacent to x_1. So x_1 has an even number of neighbors in P, and hence $H \cup P$ induces an even wheel with center x_1. Therefore x_1v_1 is not an edge.

Since $S_l' \cup \{x, x_1, p_j, \dots, p_k\}$ cannot induce a 3PC($\cdot,\cdot$) nor an even wheel with center x_1, node x_1 has an even number of neighbors in $p_j, \dots, p_k$. So x_1 has an odd number of neighbors in P, and hence $S_1 \cup \dots \cup S_{l-1} \cup S_l' \cup P$ induces a 3PC($\cdot,\cdot$) or an even wheel with center x_1.

Case 2.2: x does not have a neighbor in P.

In particular, p_k is not of type bx.

We first show that neither x_1 nor x_2 has a neighbor in $P \setminus \{p_1, p_k\}$. Assume x_1 does. Then by Claim 1, $x_2, \dots, x_l$ do not have neighbors in P. If x_1u_1 is not an edge, then $S_1 \cup (P \setminus p_k) \cup x$ contains a 3PC(x_1, u_1). So x_1u_1 is an edge, and hence x_2u_1 is not. If $l=2$, then $(P \setminus p_1) \cup S_1 \cup S_2' \cup x$ contains a 3PC(x_1, x_2). So $l>2$. If $l=n$, then $(H \setminus x_1) \cup P \cup x$ contains an even wheel with center x. So $2<l<n$. Since $S_1 \cup \dots \cup S_{l-1} \cup S_l' \cup P$ cannot induce a 3PC($\cdot,\cdot$) nor an even wheel with center x_1, node x_1 has an even number of neighbors in P. But then $S_1 \cup S_l'' \cup P \cup x$ induces an even wheel with center x_1. Therefore x_1 does not have a neighbor in $P \setminus \{p_1, p_k\}$, and by symmetry neither does x_2.

If some x_t, $t \in \{3, \dots, n\}$ has a neighbor in $P \setminus \{p_1, p_k\}$, then $H \cup (P \setminus p_k)$ contains a 3PC(u_1, x_t). So no node of $x_1, \dots, x_n$ has a neighbor in $P \setminus \{p_1, p_k\}$. If $2<l<n$, then $S_1 \cup S_l' \cup P \cup x$ or $S_1 \cup S_l'' \cup P \cup x$ induces a 3PC(u_1, x). So without loss of generality $l=n$. But then $(H \setminus x_1) \cup P \cup x$ contains an even wheel with center x or a 3PC($\cdot,\cdot$).

Case 3: p_1 is of type 2 or bx.

By Lemma 3.3, Cases 1 and 2, and the symmetry, p_k is also of type 2 or bx. A node of $x_1, \dots, x_n$ must have a neighbor in $P \setminus \{p_1, p_k\}$, since otherwise $H \cup P$ induces a 3PC($\triangle,\triangle$) or an even wheel. Let x_t be the node of $\{x_1, \dots, x_n\}$ with smallest index that has a neighbor in $P \setminus \{p_1, p_k\}$.

Case 3.1: x has a neighbor in P.

First suppose that x_2 has a neighbor in $P \setminus \{p_1, p_k\}$. Recall that no node of $P \setminus \{p_1, p_k\}$ can be adjacent to more than one node of $\{x, x_1, \ldots, x_n\}$. In particular, x_2 is not adjacent to p_i. By Claim 1, $x_4, \ldots, x_n, x_1$ cannot have neighbors in $P \setminus \{p_1, p_k\}$. If $i > 1$ then x_2 must have an even number of neighbors in $p_2, \ldots, p_i$, else $S_1' \cup \{x, x_2, p_1, \ldots, p_i\}$ contains a 3PC$(\cdot,\cdot)$ or an even wheel with center x_2. Similarly, x_2 has an even number of neighbors in $p_j, \ldots, p_k$. So by Claim 2, x_2 has an even number of neighbors in $p_2, \ldots, p_k$. But then since $S_1' \cup S_l'' \cup S_{l+1} \cup \ldots \cup S_n \cup P \cup \{x, x_2\}$ cannot induce a 3PC$(\cdot,\cdot)$ nor an even wheel with center x_2, $u_2 \neq x_2$ and x_2 has exactly two neighbors in P, that are furthermore adjacent. But then $S_1 \cup S_l'' \cup S_{l+1} \cup \ldots \cup S_n \cup P$ induces a 3PC$(p_1 u_1 u_2, \triangle)$. So x_2 does not have a neighbor in $P \setminus \{p_1, p_k\}$, and by symmetry neither do x_1, x_l, x_{l+1}.

Node x_t must have an even number of neighbors in $p_2, \ldots, p_i$, else either $S_1' \cup \{x, x_t, p_1, \ldots, p_i\}$ (if $t = 3$) or $S_1'' \cup \{x, x_t, p_1, \ldots, p_i\}$ (otherwise) induces a 3PC$(\cdot,\cdot)$ or an even wheel with center x_t. By symmetry x_t has an even number of neighbors in $p_{j+1}, \ldots, p_k$. So by Claim 2, x_t has an even number of neighbors in P.

By symmetry we may assume that $t > l+1$. Then since $S_1'' \cup S_2 \cup \ldots \cup S_l' \cup P \cup x_t$ cannot induce a 3PC$(\cdot,\cdot)$ nor an even wheel with center x_t, node x_t has exactly two neighbors in P that are furthermore adjacent. But then $S_1'' \cup S_2 \cup \ldots S_{t-1} \cup P$ contains a 3PC$(p_k v_1 v_2, \triangle)$.

Case 3.2: x does not have a neighbor in P.

Then p_1 and p_k are both of type 2.

Suppose that x_2 has a neighbor in $P \setminus \{p_1, p_k\}$. By Claim 1, nodes $x_4, \ldots, x_n, x_1$ cannot have neighbors in $P \setminus \{p_1, p_k\}$. If x_3 has a neighbor in $P \setminus \{p_1, p_k\}$, then $(H \setminus x_2) \cup (P \setminus p_k) \cup x$ contains a 3PC$(\cdot,\cdot)$ or an even wheel with center x. So x_3 does not have a neighbor in $P \setminus \{p_1, p_k\}$.

If $l = 2$ then $(H \setminus x_2) \cup P \cup x$ contains a 3PC$(\cdot,\cdot)$ or an even wheel with center x. If $l = n$ then $(H \setminus x_1) \cup (P \setminus p_1) \cup x$ contains a 3PC$(\cdot,\cdot)$ or an even wheel with center x. So $2 < l < n$.

Suppose that p_1 is not adjacent to x_2. Let H' be the hole of $(H \setminus u_2) \cup (P \setminus p_k)$ that contains $x_1, \ldots, x_n$. Then (H', x) and a proper subpath of P contradict our choice of (H, x) and P. So p_1 must be adjacent to x_2.

Let H' be the hole contained in $(H \setminus (S_l' \cup x_2)) \cup P$. Since (H', x_2) cannot be an even wheel, x_2 has an even number of neighbors in P. If $x_1 x_{l+1}$ is not an edge, then $S_1' \cup S_l'' \cup P \cup \{x, x_2\}$ induces an even wheel with center x_2. Otherwise, $x_2 x_l$ cannot be an edge and hence $S_1' \cup S_l' \cup P \cup \{x, x_2\}$ induces an even wheel with center x_2.

Therefore x_2 cannot have a neighbor in $P \setminus \{p_1, p_k\}$. By symmetry neither can x_1, x_l, x_{l+1}.

Suppose that $l = 2$. Let H' be the hole induced by $S_1'' \cup S_2' \cup P$. If x_t has a unique neighbor in P, then $H' \cup P \cup \{x, x_t\}$ induces a 3PC$(x_2, \cdot)$. If x_t has two nonadjacent neighbors in P, then $H' \cup P \cup \{x, x_t\}$ contains a 3PC(x_2, x_t). So x_t has exactly two neighbors in P, that are furthermore adjacent. But then $H' \cup S_2 \cup S_3 \cup \ldots \cup S_{t-1}$ induces a 3PC$(\triangle, \triangle)$. Therefore $l > 2$, and by symmetry $l < n$.

Let x_t be a node of $\{x_1, \ldots, x_n\}$ that has a neighbor in $P \setminus \{p_1, p_k\}$. Without loss of generality $2 < t < l$. Since G has no diamond, x_t cannot be adjacent to both x_2 and x_l. Without loss of generality x_t is not adjacent to x_l. Let H' (resp. H'') be the hole induced by $S_1' \cup S_l' \cup P \cup x$ (resp. $S_1' \cup S_l'' \cup S_{l+1} \cup \ldots \cup S_n \cup P$). Since $H' \cup x_t$ cannot induce a 3PC$(\cdot,\cdot)$ nor an even wheel with center x_t, node x_t has an even number of neighbors in P. If x_t has more than two neighbors in P, then (H'', x_t) is an even wheel. So x_t has exactly two neighbors in P. If these two neighbors are not adjacent, then $H'' \cup x_t$ induces a 3PC$(\cdot,\cdot)$. So the two neighbors of x_t in P are adjacent. By Claim 1 and the choice of x_t, the only other node of $\{x_1, \ldots, x_n\}$ that may have a neighbor in $P \setminus \{p_1, p_k\}$ is x_{t+1}. But then $(H \setminus x_l) \cup P$ contains a 3PC$(p_1u_1u_2, \triangle)$. □

Lemma 3.9 *Suppose that S_1 is a long sector of (H, x) such that the following hold.*

(i) If there exists a type wx1 or wx2 node w.r.t. (H, x), then such a node is adjacent to x_1 or x_2.

(ii) S_1 has no appendix.

Then $S = X_1 \cup X_2 \cup \{x\}$ is a bisimplicial cutset separating S_1 from $H \setminus S_1$.

Proof: Note that if x_1x_n is an edge then $x_n \in X_1$, and if x_2x_3 is an edge then $x_3 \in X_2$. Assume S is not a cutset, and let $P = p_1, \ldots, p_k$ be a direct connection from S_1 to $H \setminus S_1$ in $G \setminus S$. By (i) and Lemma 3.4, no node of P is of type wx1 or wx2. By (ii), $k > 1$, *i.e.*, p_1 is not of type b. By Lemma 3.3 and the definition of P, p_1 has a neighbor in the interior of S_1 and it is of type 1, 2 or bx w.r.t. (H, x). By Lemma 3.8, either $p_k \in X \setminus S$ or p_k has a unique neighbor in $H \cup x$ that is a node of $\{x_3, \ldots, x_n\}$ that is not adjacent to neither x_1 nor x_2. So p_k is adjacent to some x_l, $l \neq 1, 2$ and x_lx_1 and x_lx_2 are not edges. So p_k is of type 1, x1 or x2. Without loss of generality we assume that if p_k is of type x2, then it is adjacent to x_{l+1}. Note that if p_k is of type x2 then, since G has no diamond, $x_{l+1}x_1$ is not an edge. By definition of P and Lemma 3.3, no node of $H \setminus S$ has a neighbor in $P \setminus \{p_1, p_k\}$, and no node of $P \setminus \{p_1, p_k\}$ is adjacent to more than one node of $H \cup x$.

Claim 1: *If a node of $X_1 \cap H$ (resp. $X_2 \cap H$) has a neighbor in $P \setminus \{p_1, p_k\}$, then no node of $X_2 \cap H$ (resp. $X_1 \cap H$) has a neighbor in $P \setminus \{p_1, p_k\}$.*

Proof of Claim 1: Assume not. Then there is a subpath P' of P whose one endnode is adjacent to a node of $X_1 \cap H$, the other endnode is adjacent to a node of $X_2 \cap H$ and no intermediate node of P' has a neighbor in H. But then $H \cup P'$ induces a 3PC$(\cdot, \cdot)$. This completes the proof of Claim 1.

Let u_1 (resp. u_2) be the neighbor of p_1 in S_1 that is closest to x_1 (resp. x_2). Let S_1' (resp. S_1'') be the x_1u_1-subpath (resp. u_2x_2-subpath) of S_1.

First suppose that x_1 has a neighbor in $P \setminus \{p_1, p_k\}$. Then by Claim 1, no node of $X_2 \cap H$ has a neighbor in $P \setminus \{p_1, p_k\}$. Let p_i (resp. p_j) be the neighbor of x_1 in $P \setminus \{p_1, p_k\}$ with lowest (resp. highest) index.

We now show that x has an even number of neighbors in $p_1, \ldots, p_j$. Let P' be a subpath of $p_1, \ldots, p_j$ whose endnodes are both adjacent to x_1 and no intermediate node is adjacent to x_1. Since $P' \cup \{x_1, x\}$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel with center x, node x has an even number of neighbors in P'. If p_1 is not adjacent to x_1, then x has an even number of neighbors in $p_1, \ldots, p_i$, else $S_1' \cup \{x, p_1, \ldots, p_i\}$ induces a 3PC$(\cdot, \cdot)$ or an even wheel with center x. So x has an even number of neighbors in $p_1, \ldots, p_j$.

This implies that the parity of the number of neighbors of x in P and in $p_j, \ldots, p_k$ are the same. Let H' (resp. H'') be the hole induced by $S_1'' \cup S_2 \cup \ldots \cup S_{l-1} \cup P$ (resp. $S_1 \cup S_2 \cup \ldots \cup S_{l-1} \cup \{p_j, \ldots, p_k\}$). Then either $H' \cup x$ induces a 3PC$(\cdot, \cdot)$ or one of $(H'x)$, (H'', x) is an even wheel.

Therefore x_1 does not have a neighbor in $P \setminus \{p_1, p_k\}$, and by symmetry neither does x_2.

Node p_1 must be of type 2 or bx, else $S_1 \cup P \cup \{x, x_l\}$ contains a 3PC(u_1, x).

Now suppose that x_n has a neighbor in $P \setminus \{p_1, p_k\}$. Then $x_n \in S$ and hence $x_n x_1$ is an edge. By Claim 1, x_n is the only node of H that has a neighbor in $P \setminus \{p_1, p_k\}$. Let p_i (resp. p_j) be the node of $P \setminus \{p_1, p_k\}$ with lowest (resp. highest) index adjacent to x_n.

We now show that x has an odd number of neighbors in $p_1, \ldots, p_j$. Let P' be a subpath of $p_1, \ldots, p_j$ whose endnodes are both adjacent to x_n, and no intermediate node of P' is adjacent to x_n. Since $P' \cup \{x_n, x\}$ cannot induce a 3PC$(\cdot, \cdot)$ nor an even wheel with center x, node x has an even number of neighbors in P'. Node x must have a neighbor in $p_1, \ldots, p_i$, else $S_1 \cup \{x_n, x, p_1, \ldots, p_i\}$ induces a 3PC$(p_1 u_1 u_2, x_1 x_n x)$ or an even wheel with center x_1. Since $S'_1 \cup \{x_n, x, p_1, \ldots, p_i\}$ cannot induce an even wheel, node x has an odd number of neighbors in $p_1, \ldots, p_i$. Therefore, x has an odd number of neighbors in $p_1, \ldots, p_j$.

This implies that the parities of the number of neighbors of x in P and in $p_j, \ldots, p_k$ are different. Let H' (resp H'') be the hole induced by $S''_1 \cup S_2 \cup \ldots \cup S_{l-1} \cup P$ (resp. $S_1 \cup S_2 \cup \ldots \cup S_{l-1} \cup \{x_n, p_j, \ldots, p_k\}$). Then either (H', x) or (H'', x) is an even wheel.

Therefore, x_n has no neighbor in $P \setminus \{p_1, p_k\}$, and by symmetry neither does x_3, *i.e.*, no node of H has a neighbor in $P \setminus \{p_1, p_k\}$.

Since P is not an appendix of (H, x), p_k is of type x2. But then $P \cup H$ induces a 3PC$(p_1 u_1 u_2, p_k x_l x_{l+1})$. □

Proof of Theorem 3.2: Let (H, x) be a proper wheel of G. If there is no node of type wx1 or wx2 w.r.t. (H, x), then the result follows from Lemma 3.7 and Lemma 3.9.

Suppose there exists a node u that is of type wx1 w.r.t. (H, x). Then for some $i \in \{1, \ldots, n\}$, $N(u) \cap \{x, x_1, \ldots, x_n\} = \{x, x_i\}$, and sectors S_i and S_{i-1} are both long. By Lemma 3.6, S_i and S_{i-1} have no appendices, and hence the result follows from Lemma 3.9.

Finally suppose there exists a node u that is of type wx2 w.r.t. (H, x). Then for some $i \in \{1, \ldots, n\}$, $N(u) \cap \{x, x_1, \ldots, x_n\} = \{x, x_i, x_{i+1}\}$, So sectors S_{i-1} and S_{i+1} are both long, and u is adjacent to an endnode of both of them. By Lemma 3.6, S_{i-1} and S_{i+1} have no appendices, and hence the result follows from Lemma 3.9. □

Proof of Lemma 1.3: Assume G does not have a 3PC$(\triangle, \cdot)$. If G does not contain a hole, then G is triangulated, and it is a well known result that a triangulated graph is either a clique or has a clique cutset. So assume G contains a hole H, but does not contain a clique cutset nor a bisimplicial cutset. Then by Theorem 3.2 G does not contain a proper wheel, and by our first assumption G does not contain a bug. So G does not contain a wheel.

Note that a node $x \notin V(H)$ cannot have two nonadjacent neighbors in H, else there is a 3PC$(\cdot, \cdot)$ or a wheel. Let C be a connected component of $G \setminus H$. Then $N(C) \cap H$ contains two nonadjacent nodes, else there is a clique cutset. Then there exists a path P in C such that the endnodes of P have nonadjacent neighbors in H. Let P be shortest such path. Then for some two nonadjacent nodes u and v of H, one endnode of P is adjacent to u, and the other one is adjacent to v. If no node of H has a neighbor in the interior of P, then $P \cup H$ induces an even wheel, a 3PC$(\triangle, \triangle)$, a 3PC$(\cdot, \cdot)$ or a 3PC$(\triangle, \cdot)$, contradicting our assumptions. So a node w of H has a neighbor in the interior of P. By the choice of P, w must be adjacent to both u and v. In fact $N(P) \cap H = \{u, v, w\}$. But then $H \cup P$ induces a wheel, a contradiction. □

4 Nodes adjacent to a 3PC($\triangle, \cdot$) and crossings

In light of Lemma 1.3, for the rest of the decomposition we focus on the case when the graph has a 3PC($\triangle, \cdot$). In this section we examine paths that connect different paths of a 3PC($\triangle, \cdot$). Throughout this section Σ denotes a 3PC($x_1x_2x_3, y$). The three paths of Σ are denoted P_{x_1y}, P_{x_2y} and P_{x_3y} (where P_{x_iy} is the path that contains x_i). Note that at most one of the paths of Σ is of length 1, and if one of the paths of Σ is of length 1, then Σ is a bug. For $i = 1, 2, 3$, we denote the neighbor of y in P_{x_iy} by y_i.

Lemma 4.1 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not contain a proper wheel. If $u \in V(G) \setminus V(\Sigma)$ has a neighbor in Σ, then u is one of the following types.*

Type p1: $|N(u) \cap V(\Sigma)| = 1$.

Type p2: $|N(u) \cap V(\Sigma)| = 2$ *and the two neighbors of u in Σ form an edge of one of the paths of Σ.*

Type pb: *For some $i \in \{1, 2, 3\}$, $N(u) \cap V(\Sigma) \subseteq V(P_{x_iy})$, and for $j \in \{1, 2, 3\} \setminus \{i\}$, $V(P_{x_iy}) \cup V(P_{x_jy}) \cup \{u\}$ induces a bug with center u.*

Type t3: $N(u) \cap V(\Sigma) = \{x_1, x_2, x_3\}$.

Type t3b: *Node u is adjacent to x_1, x_2 and x_3, and it has one more neighbor in Σ, say in $P_{x_iy} \setminus \{x_i\}$. Furthermore, for some $j \in \{1, 2, 3\} \setminus \{i\}$, $V(P_{x_iy}) \cup V(P_{x_jy}) \cup \{u\}$ induces a bug with center u.*

Type b: *Node u has exactly three neighbors in Σ. For some $i \in \{1, 2, 3\}$, u is adjacent to y_i, and the other two neighbors of u in Σ are contained in say in P_{x_jy}, for some $j \in \{1, 2, 3\} \setminus \{i\}$. Furthermore, $V(P_{x_iy}) \cup V(P_{x_jy}) \cup \{u\}$ induces a bug with center u.*

Proof: If for some $i \in \{1, 2, 3\}$, $N(u) \cap \Sigma \subseteq P_{x_iy}$, then u is of type p1, p2 or pb, else there is a diamond, 3PC($\cdot, \cdot$) or a proper wheel.

So assume without loss of generality that u has neighbors in both $P_{x_1y} \setminus y$ and $P_{x_2y} \setminus y$. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then P_{x_3y} is an appendix of H.

First suppose that u is not adjacent to all of x_1, x_2, x_3. Then, since there is no diamond, u is adjacent to at most one node of $\{x_1, x_2, x_3\}$. Suppose u is not adjacent to y. Then by Lemma 2.2 applied to H, P_{x_3y} and u, node u is also an appendix of H and its node-attachment is without loss of generality y_1. Furthermore, no node of P_{x_3y} is adjacent to u, and hence u is of type b. So u is adjacent to y. Then (H, u) must be a bug. Without loss of generality $N(u) \cap P_{x_1y} = \{y, y_1\}$ and $N(u) \cap P_{x_2y} = \{y, u_1\}$, where yu_1 is not an edge. If u has no neighbor in $P_{x_3y} \setminus y$, then $P_{x_2y} \cup P_{x_3y} \cup u$ induces a 3PC(y, u_1). So u has a neighbor in $P_{x_3y} \setminus y$. Node u cannot be adjacent to y_3, else $\{y_1, y, u, y_3\}$ induces a diamond. But then $P_{x_2y} \cup P_{x_3y} \cup u$ induces a proper wheel with center u.

Now assume that u is adjacent to all of x_1, x_2, x_3. Suppose u is not of type t3. Without loss of generality u has a neighbor in $P_{x_1y} \setminus x_1$. $P_{x_1y} \cup P_{x_2y} \cup u$ must induce a bug with center u, and similarly so must $P_{x_1y} \cup P_{x_3y} \cup u$. Hence u is of type t3b. □

Nodes adjacent to Σ are further classified as follows.

Type p: A node that is of type p1, p2 or pb w.r.t. Σ.

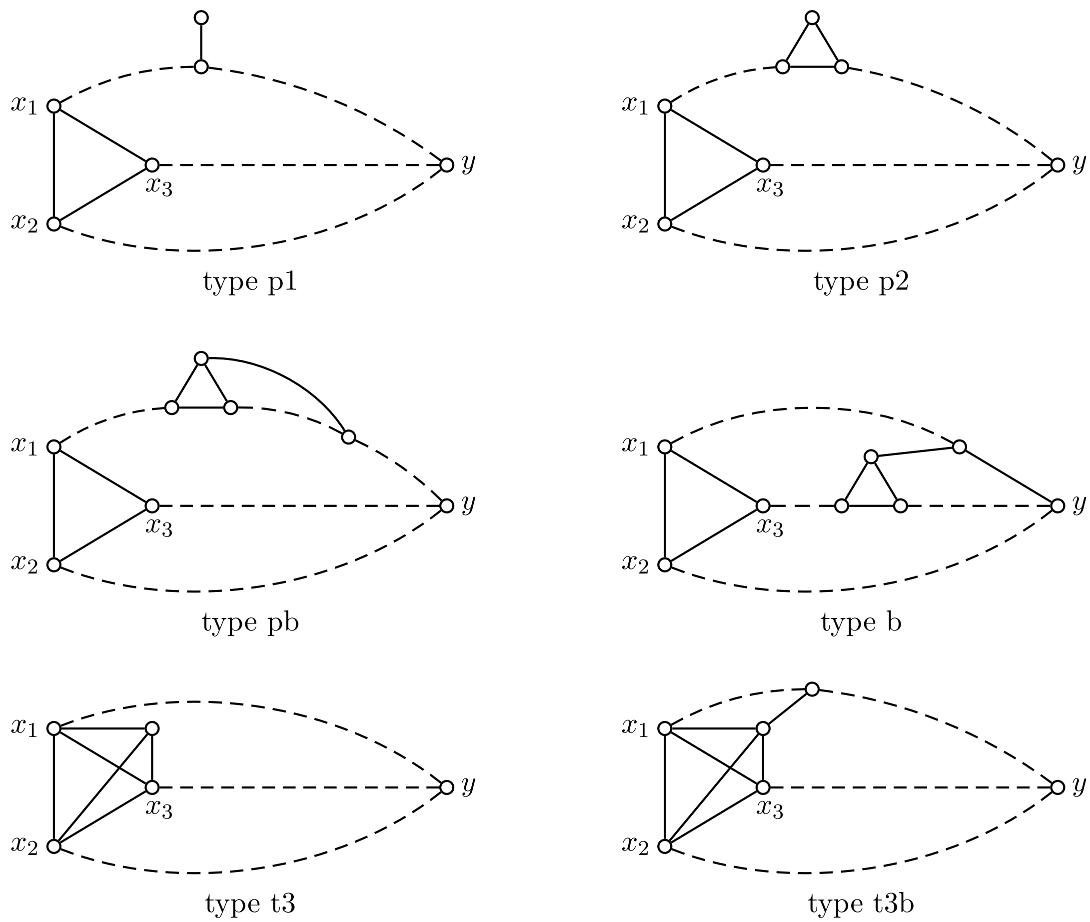


Figure 12: Different types of nodes adjacent to a 3PC($\triangle, \cdot$).

Type t: A node that is of type t3 or t3b w.r.t. Σ.

Definition 4.2 *A* crossing *of* Σ *is a chordless path* $P = p_1, \ldots, p_k$ *in* $G \setminus \Sigma$ *such that either* $k = 1$ *and* p_1 *is of type b w.r.t.* Σ*, or* $k > 1$ *and for some* $i, j \in \{1, 2, 3\}$*,* $i \neq j$*,* $N(p_1) \cap V(\Sigma) \subseteq V(P_{x_i y})$*,* $N(p_k) \cap V(\Sigma) \subseteq V(P_{x_j y})$*,* p_1 *has a neighbor in* $V(P_{x_i y}) \setminus \{y\}$*,* p_k *has a neighbor in* $V(P_{x_j y}) \setminus \{y\}$*, and no node of* $P \setminus \{p_1, p_k\}$ *has a neighbor in* Σ*.*

Definition 4.3 *A crossing* $P = p_1, \ldots, p_k$ *of* Σ *is called a* hat *if* $k > 1$*,* p_1 *and* p_k *are both of type p1 w.r.t.* Σ *adjacent to different nodes of* $\{x_1, x_2, x_3\}$*.*

Definition 4.4 *Let* $P = p_1, \ldots, p_k$ *be a crossing of* Σ *such that one of the following holds:*

(i) $k = 1$ *and* p_1 *is of type b w.r.t.* Σ*, say* p_1 *is adjacent to* y_i *for some* $i \in \{1, 2, 3\}$*, and it has two more neighbors in* $P_{x_j y} \setminus \{y\}$*, for some* $j \in \{1, 2, 3\} \setminus \{i\}$*.*

(ii) $k > 1$*,* p_1 *is of type p1 and* p_k *is of type p2 w.r.t.* Σ*, for some* $i \in \{1, 2, 3\}$*,* p_1 *is adjacent to* y_i*, and for some* $j \in \{1, 2, 3\} \setminus \{i\}$*,* $N(p_k) \cap V(\Sigma) \subseteq V(P_{x_j y}) \setminus \{y\}$*.*

Such a path P *is called a* y_i-crosspath *of* Σ*. We also say that* P *is a* crosspath *from* y_i *to* $P_{x_j y}$*.*

If say $x_3 y$ *is an edge, then* Σ *induces a bug* (H, x)*, where* $x = x_3 = y_3$*. In this case, the* y_3*-crosspath (or* x*-crosspath) of* Σ*, is also called the* center-crosspath *of the bug* (H, x)*.*

Lemma 4.5 *Let* G *be a (diamond, 4-hole)-free odd-signable graph that does not contain a proper wheel. If* P *and* Q *are crossing appendices of a hole* H *of* G*, then their node-attachments are adjacent.*

Proof: Assume not. Then without loss of generality (ii) of Lemma 2.3 holds. Let H'_P be the $u_1 u$-subpath of H that does not contain u_2. Without loss of generality v belongs to H'_P. Note that since G is diamond-free, it is not possible that q_1 is adjacent to both p_1 and u_1. Hence $H'_P \cup P \cup q_1$ induces a proper wheel with center q_1. □

Lemma 4.6 *Let* G *be a (diamond, 4-hole)-free odd-signable graph that does not contain a proper wheel.* $\Sigma = 3\text{PC}(x_1 x_2 x_3, y)$ *of* G *can have a crosspath from at most one of the nodes* y_1, y_2, y_3*.*

Proof: Suppose not and let $P = u_1, \ldots, u_n$ be a y_1-crosspath and $Q = v_1, \ldots, v_m$ a y_2-crosspath. Let u', u'' (resp. v', v'') be adjacent neighbors of u_n (resp. v_m) in Σ. Note that by definition of a crosspath, y does not coincide with any of the nodes u', u'', v', v''.

First suppose that u', u'' belong to $P_{x_2 y}$, and v', v'' belong to $P_{x_1 y}$. Let H be the hole induced by $P_{x_1 y} \cup P_{x_2 y}$. Then P and Q are crossing appendices of H, which contradicts Lemma 4.5.

So without loss of generality we may assume that u', u'' belong to $P_{x_3 y}$.

Suppose that v', v'' also belong to $P_{x_3 y}$. Since there is no diamond, $(P_{x_3 y} \setminus \{x_3, y, y_3\}) \cup P \cup Q \cup \{y_1, y_2\}$ contains a chordless path P' from y_1 to y_2. But then $P' \cup P_{x_1 y} \cup P_{x_2 y}$ induces a $3\text{PC}(y_1, y_2)$.

So v', v'' belong to $P_{x_1 y}$. Let H be the hole induced by $P_{x_1 y} \cup P_{x_2 y}$. Let P' be the chordless path from u_1 to x_3 in $P \cup (P_{x_3 y} \setminus \{y, y_3\})$. Then P' and Q are crossing appendices of H, contradicting Lemma 4.5. □

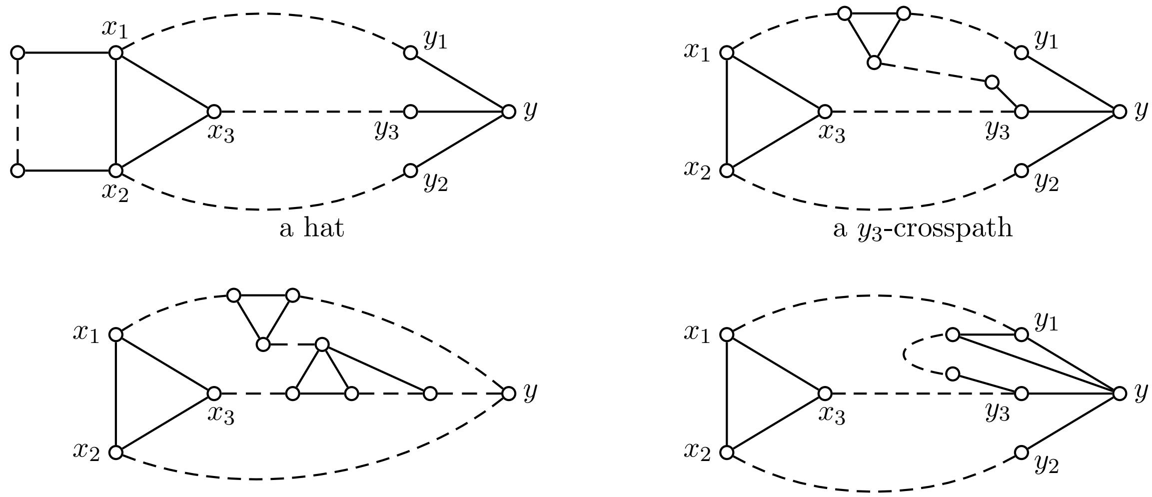


Figure 13: Different crossings of a $3PC(x_1x_2x_3, y)$.

Lemma 4.7 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not contain a proper wheel. If $P = p_1, \ldots, p_k$ is a crossing of a $\Sigma = 3PC(x_1x_2x_3, y)$ of G, then one of the following holds.*

(i) P is a crosspath of Σ.

(ii) P is a hat of Σ.

(iii) One of p_1, p_k is of type pb w.r.t. Σ, and the other is of type p2 w.r.t. Σ.

(iv) One of p_1, p_k is of type p1 w.r.t. Σ, and the other is of type p2 w.r.t. Σ, say p_1 is of type p1 and p_k is of type p2. Furthermore, p_1 is adjacent to a node of $\{y_1, y_2, y_3\}$ and p_k is adjacent to y.

Proof: If $k = 1$ then p_1 is of type b w.r.t. Σ, and hence it is a crosspath. So assume $k > 1$, and without loss of generality $N(p_1) \cap \Sigma \subseteq P_{x_1y}$, $N(p_k) \cap \Sigma \subseteq P_{x_2y}$, p_1 has a neighbor in $P_{x_1y} \setminus y$ and p_k has a neighbor in $P_{x_2y} \setminus y$. Let u_1 (resp. u_2) be the neighbor of p_1 in P_{x_1y} that is closest to x_1 (resp. y). Let v_1 (resp. v_2) be the neighbor of p_k in P_{x_2y} that is closest to x_2 (resp. y). By Lemma 4.1, p_1 and p_k are of type p w.r.t. Σ. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$.

First suppose that p_1 is of type pb. Then (H, p_1) is a bug. Assume (iii) does not hold, *i.e.*, p_k is not of type p2. If p_k is of type p1, then $v_1 \neq y$ and hence $H \cup P$ contains a $3PC(p_1, v_1)$. So p_k is of type pb. If $k > 2$ then $H \cup P$ contains a $3PC(p_1, p_k)$. So $k = 2$. But then (H, p_1) and p_k contradict Lemma 4.1. So by symmetry we may assume that neither p_1 nor p_k is of type pb.

Suppose that p_1 and p_k are both of type p1. Then $u_1, v_1 \neq y$. If $u_1 = x_1$ and $v_1 = x_2$, then P is a hat of Σ. Otherwise $H \cup P$ induces a $3PC(u_1, v_1)$.

Suppose p_1 and p_k are both of type p2. Then $H \cup P$ induces either a $3PC(p_1u_1u_2, p_kv_1v_2)$ or an even wheel with center y.

Therefore we may assume that P is an appendix of H. Without loss of generality u_1 is the node-attachment of P. P and P_{x_3y} are crossing appendices of H, and hence by Lemma 4.5, $u_1 = y_1$. If p_k is not adjacent to y, then P is a y_1-crosspath of Σ. If p_k is adjacent to y, then (iv) holds. $\square$

5 Bugs

For a bug (H, x) we use the following notation in this section. Let x_1, x_2, y be the neighbors of x in H, such that x_1x_2 is an edge. Let H_1 (resp. H_2) be the sector of (H, x) that contains y and x_1 (resp. x_2). Let y_1 (resp. y_2) be the neighbor of y in H_1 (resp. H_2).

Definition 5.1 *An* ear *of a bug (H, x) is a chordless path $P = p_1, \ldots, p_k$ in $G \setminus (V(H) \cup \{x\})$ such that $k > 1$, p_1 is of type p1 w.r.t. (H, x) adjacent to x, p_k is of type p2 w.r.t. (H, x) adjacent to y and a node of $\{y_1, y_2\}$, and no intermediate node of P has a neighbor in (H, x).*

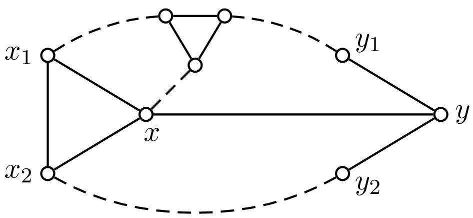


Figure 14: A center-crosspath of a bug (H, x).

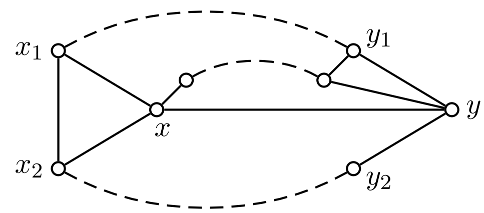


Figure 15: An ear of a bug (H, x).

In this section we decompose bugs with center-crosspaths (see Figure 14), 3PC($\triangle, \cdot$)'s with a hat (see Figure 13) and bugs with ears (see Figure 15). The order in which these decompositions are performed is of the key importance. As a consequence of these decompositions, in a graph that has no clique cutset nor a bisimplicial cutset, the only crossings of a 3PC($\triangle, \cdot$) are the crosspaths. We note that a bug with a center-crosspath is not a nontrivial basic graph, whereas any 3PC($\triangle, \cdot$) with a crosspath, that is not a bug with a center-crosspath, is a nontrivial basic graph.

Theorem 5.2 *Let G be a (diamond, 4-hole)-free odd-signable graph. If G contains a bug with a center-crosspath, or G contains a bug but does not contain a* 3PC($\triangle, \cdot$) *with a crosspath, then G has a bisimplicial cutset.*

Proof: By Theorem 3.2 we may assume that G does not contain a proper wheel.

If G has a bug (H, x) with a center-crosspath P, we choose (H, x) and P so that $|H \cup P|$ is minimized. Otherwise, G does not have a 3PC($\triangle, \cdot$) with a crosspath, and we choose a bug (H, x) so that $|H|$ is minimized.

Suppose (H, x) has a center-crosspath $P = p_1, \ldots, p_k$. Then p_1 is adjacent to x, and let u_1, u_2 be the neighbors of p_k in H. Without loss of generality $u_1, u_2 \in H_2 \setminus y$, and u_1 is the neighbor of p_k in H_2 that is closer to y.

Let X be the node set consisting of x_1, x_2 and all type t nodes w.r.t. (H, x). Let Y be the node set consisting of y and all type p2 nodes w.r.t. (H, x) that are adjacent to x and y. Since there is no diamond, both of the sets X and Y induce a clique. We now show that $S = X \cup Y \cup x$ is a bisimplicial cutset separating H_1 from H_2.

Assume not and let $Q = q_1, \ldots, q_l$ be a direct connection from H_1 to H_2 in $G \setminus S$. By Lemma 4.1 and Lemma 4.6, $l > 1$, and q_1 and q_l are of type p, or of type b adjacent to x. Furthermore, q_1 has a neighbor in $H_1 \setminus \{x_1, y\}$ and q_l has a neighbor in $H_2 \setminus \{x_2, y\}$. Also, the only nodes of H that may have a neighbor in $Q \setminus \{q_1, q_l\}$ are x_1, x_2, y. Since there is no 4-hole nor a diamond, and by definition of S, no node of $Q \setminus \{q_1, q_l\}$ is adjacent to more than one node of $\{x, x_1, x_2, y\}$.

Let H_1' (resp. H_2') be the subpath of H_1 (resp. H_2) whose one endnode is x_1 (resp. x_2), the other endnode is adjacent to q_1 (resp. q_l), and no intermediate node of H_1' (resp. H_2') is adjacent to q_1 (resp. q_l).

Claim 1: *At most one of the sets $\{x_1, x_2\}$ or $\{y\}$ may have a neighbor in $Q \setminus \{q_1, q_l\}$.*

Proof of Claim 1: Assume not. Then there is a subpath Q' of $Q \setminus \{q_1, q_l\}$ such that one endnode of Q' is adjacent to y, the other is adjacent to a node of $\{x_1, x_2\}$, say to x_1, and no intermediate node of Q' has a neighbor in H. Then $H \cup Q'$ induces a 3PC(x_1, y). This completes the proof of Claim 1.

Claim 2: *q_1 cannot be of type pb.*

Proof of Claim 2: Assume q_1 is of type pb, and let H' be the hole of $H \cup q_1$ that contains q_1, x_1, x_2, y. Then (H', x) is a bug.

First assume that P exists. If q_1 is not adjacent to a node of P, then (H', x) and P contradict the minimality of $|H \cup P|$. So q_1 is adjacent to a node of P. Let p_i be the node of P with lowest index adjacent to q_1. Then $H_1 \cup \{x, q_1, p_1, \dots, p_i\}$ contains a 3PC(q_1, x).

Now assume that P does not exist, *i.e.*, G does not contain a 3PC$(\triangle, \cdot)$ with a crosspath. Since $|H'| < |H|$, bug (H', x) contradicts our choice of (H, x). This completes the proof of Claim 2.

Let v_1 (resp. v_2) be the neighbor of q_1 in H_1 that is closest to x_1 (resp. y). By Claim 2, either $v_1 = v_2$ or v_1v_2 is an edge.

By Lemma 2.2 and Lemma 4.6, either y has a neighbor in Q, or a node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$. We now consider the following two cases.

Case 1: No node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$.

Then y has a neighbor in Q. Let q_t be the node of Q with lowest index adjacent to y.

We first show that x cannot have a neighbor in Q. Assume it does. Let H' be the hole induced by $H_1' \cup H_2' \cup Q$. Then (H', x) must be a bug, and hence x has a unique neighbor q_s in Q. Note that q_s is not adjacent to y. If $t < s$, then $(H_2 \setminus x_2) \cup \{x, q_t, \dots, q_l\}$ contains a 3PC(q_s, y). So $t > s$. But then $(H_1 \setminus x_1) \cup \{x, q_1, \dots, q_t\}$ contains a 3PC(q_s, y).

Therefore x does not have a neighbor in Q. In particular, q_1 and q_l are not of type b. By Claim 2, q_1 is of type p1 or p2. We now consider the following two cases.

Case 1.1: Either P does not exist, or it does exist but no node of P is adjacent to or coincident with a node of Q.

If P does exist, let R be the chordless path from q_l to x in $(H_2 \setminus \{x_2, y\}) \cup P \cup \{x, q_l\}$.

First suppose that q_1 is of type p1. Node v_1 is adjacent to y, else $H_1 \cup \{x, q_1, \dots, q_t\}$ induces a 3PC(v_1, y). If P exists, then $H_1 \cup Q \cup R$ induces a proper wheel with center y. So P does not exist, *i.e.*, G has no 3PC$(\triangle, \cdot)$ with a crosspath. $H_1 \cup H_2' \cup Q$ must induce a bug with center y (since it cannot induce a 3PC$(\cdot, \cdot)$ nor a proper wheel). But then x is a crosspath of this bug, a contradiction. Therefore, q_1 must be of type p2.

Suppose that q_1 is adjacent to y. First assume that P exists. Then $H_1 \cup Q \cup R$ must induce a bug with center y, and hence $y_2 \notin R$ and $N(y) \cap Q = \{q_1\}$. In particular, $y_2 \notin H_2'$ (since we are assuming that no node of P is adjacent to or coincident with a node of Q). But then $H_1 \cup H_2' \cup Q \cup x$ induces a 3PC(x_1x_2x, q_1yy_1). So P does not exist. Let H' be the hole induced by $(H_1 \setminus y) \cup H_2' \cup Q$. If (H', y) is a bug, then x is its center-crosspath, a contradiction.

So (H', y) is not a bug, *i.e.*, q_1 is the unique neighbor of y in Q and H' does not contain y_2. But then $H' \cup \{x, y\}$ induces a 3PC(x_1x_2x, y_1q_1y). Therefore, q_1 is not adjacent to y.

Assume P exists. Since $H'_1 \cup Q \cup R \cup y$ cannot induce a 3PC(x, q_t), it must induce a bug, and hence either (i) $y_2 \notin R$ and $N(y) \cap Q = \{q_t, q_{t+1}\}$, or (ii) $y_2 \in R$ and $t = l$. If (i) holds, then $y_2 \notin H'_2$, and hence $H_1 \cup H'_2 \cup Q$ induces a 3PC$(yq_tq_{t+1}, q_1v_1v_2)$. So (ii) holds. So q_l is adjacent to y and y_2. Since there is no 4-hole, q_l is not adjacent to x_2. Since $y_2 \in R$, q_l must be of type p2. But then $H \cup Q$ induces a 3PC$(q_1v_1v_2, q_lyy_2)$.

So P does not exist, *i.e.*, G has no 3PC$(\triangle, \cdot)$ with a crosspath. Let $\Sigma =$ 3PC$(q_1v_1v_2, y)$ induced by $H_1 \cup \{x, q_1, \ldots, q_t\}$. Suppose q_l has a neighbor in $H_2 \setminus \{y, y_2\}$. If $N(y) \cap Q = q_t$, then $q_{t+1}, \ldots, q_l, H'_2$ is a crosspath of Σ, a contradiction. If $N(y) \cap Q = \{q_t, q_{t+1}\}$, then $H_1 \cup H'_2 \cup Q$ induces a 3PC$(q_1v_1v_2, yq_tq_{t+1})$. Let H' be the hole induced by $H'_1 \cup H'_2 \cup Q$. If y has exactly two neighbors in Q, and they are not adjacent, then $H' \cup y$ induces a 3PC$(\cdot, \cdot)$. So (H', y) must be a bug. But then x is a center-crosspath of (H', y), a contradiction. Therefore, q_l does not have a neighbor in $H_2 \setminus \{y, y_2\}$.

Let H' be the hole induced by $H'_1 \cup H'_2 \cup Q$. If (H', y) is a bug, then x is its center-crosspath, a contradiction. So y has exactly two neighbors in H'. These two neighbors are adjacent, else $H' \cup y$ induces a 3PC$(\cdot, \cdot)$. In particular, $t = l$. Hence q_l is of type p2, and so $H \cup Q$ induces a 3PC$(q_1v_1v_2, q_lyy_2)$.

Case 1.2: P does exist, and a node of P is adjacent to or coincident with a node of Q.

Let q_i be the node of Q with lowest index adjacent to a node of P, and let p_j (resp. $p_{j'}$) be the node of P with highest (resp. lowest) index adjacent to q_i. If $i < t$, then by Lemma 2.2, $q_1, \ldots, q_i, p_j, \ldots, p_k$ is a crosspath, contradicting Lemma 4.6. So $i \geq t$.

Suppose $t = 1$. Then q_1 is of type p2. Since $H_1 \cup \{x, y, q_1, \ldots, q_i, p_1, \ldots p_{j'}\}$ cannot induce a proper wheel with center y, q_1 is the unique neighbor of y in $q_1, \ldots, q_i$. But then $H \cup \{q_1, \ldots, q_i, p_j, \ldots, p_k\}$ induces a 3PC$(yy_1q_1, u_1u_2p_k)$. So $t > 1$.

$H'_1 \cup \{x, y, q_1, \ldots, q_i, p_1, \ldots, p_{j'}\}$ must induce a bug with center y, and hence y is not adjacent to H'_1 and $N(y) \cap \{q_1, \ldots, q_i\} = \{q_t, q_{t+1}\}$.

If q_1 is of type p1, then $H_1 \cup \{x, q_1, \ldots, q_t\}$ induces a 3PC(v_1, y). So q_1 is of type p2. If $i < l$ then $(H \setminus y_2) \cup \{q_1, \ldots, q_i, p_j, \ldots, p_k\}$ contains a 3PC$(q_1v_1v_2, yq_tq_{t+1})$. So $i = l$. If q_l has a neighbor in $H_2 \setminus \{y, y_2\}$, then $(H \setminus y_2) \cup Q$ contains a 3PC$(q_1v_1v_2, yq_tq_{t+1})$. So q_l does not have a neighbor in $H_2 \setminus \{y, y_2\}$. Since there is no diamond, $t+1 < l$. Also, if $j' = k$ then p_k is not adjacent to y_2, else there is a diamond. But then $\{x, y, y_2, q_{t+1}, \ldots, q_l, p_1, \ldots p_{j'}\}$ induces a 3PC(y, q_l).

Case 2: A node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$.

By Claim 1, y has no neighbor in $Q \setminus \{q_1, q_l\}$. Let q_i be the node of $Q \setminus q_1$ with lowest index adjacent to a node of $\{x_1, x_2\}$.

Suppose that q_i is adjacent to x_2. If q_1 is of type p1, then $H \cup \{q_1, \ldots, q_i\}$ induces a 3PC$(x_2, \cdot)$. So q_1 is of type p2 or b. But then x and $q_1, \ldots, q_i$ are crossing appendices of H, and hence Lemma 4.5 is contradicted. Therefore, q_i is adjacent to x_1.

Let q_j be the node of Q with highest index adjacent to x_1. Let R be the chordless path from q_l to y in $H_2 \cup q_l$. Let H' be the hole induced by $H_1 \cup R \cup \{q_j, \ldots, q_l\}$. Note that if q_l is adjacent to x, then q_l is of type b, and hence q_l is not adjacent to y. So if x has a neighbor in $\{q_j, \ldots, q_l\}$ then x has three pairwise nonadjacent neighbors in H'. Hence $H' \cup x$ induces either a 3PC(x, y) or a proper wheel. □

Lemma 5.3 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not contain a proper wheel. Assume that G contains a $\Sigma = 3\text{PC}(\triangle, \cdot)$ with a hat, and one of the following holds.*

(i) Σ is not a bug.

(ii) G does not contain a bug with a center-crosspath.

Then G has a clique cutset.

Proof: Assume G contains a $\Sigma = 3\text{PC}(x_1x_2x_3, y)$ with a hat $P = p_1, \ldots, p_k$, and (i) or (ii) holds. Let S be the set comprised of x_1, x_2, x_3 and all type t nodes w.r.t. Σ. Since there is no diamond, S induces a clique. We now show that S is a clique cutset separating P from $\Sigma \setminus S$. Assume not, and let $Q = q_1, \ldots, q_l$ be a direct connection from P to $\Sigma \setminus S$ in $G \setminus S$. We may assume without loss of generality that P and Q are chosen so that $|P \cup Q|$ is minimized. Let p_i (resp. p_j) be the node of P with lowest (resp. highest) index adjacent to q_1.

Without loss of generality p_1 is adjacent to x_1 and p_k to x_2. By Lemma 4.1 and definition of S, x_1, x_2, x_3 are the only nodes of Σ that may have a neighbor in $Q \setminus \{q_1, q_l\}$ and no node of Q is adjacent to more than one node of $\{x_1, x_2, x_3\}$. If x_3 has a neighbor in $Q \setminus q_l$, then either $(P \setminus p_1) \cup Q$ (if q_1 has a neighbor in $P \setminus p_1$) or $(P \setminus p_k) \cup Q$ (otherwise) contains a hat P' of Σ and a path Q' that is a direct connection from P' to $\Sigma \setminus S$ in $G \setminus S$, contradicting the minimality of $|P \cup Q|$. So x_3 has no neighbor in $Q \setminus q_l$, and similarly at most one of x_1, x_2 may have a neighbor in $Q \setminus q_l$. Furthermore, if x_1 (resp. x_2) has a neighbor in $Q \setminus q_l$, then $j = 1$ (resp. $i = k$). By symmetry and Lemma 4.1, it is enough to consider the following cases.

Case 1: q_l is of type p w.r.t. Σ with neighbors in P_{x_3y}.

First suppose that x_1 and x_2 have no neighbors in $Q \setminus q_l$. Without loss of generality x_1y is not an edge. If $i < k$ then $(\Sigma \setminus x_3) \cup Q \cup \{p_1, \ldots, p_i\}$ contains a $3\text{PC}(x_1, y)$. So $i = k$. If x_2y is not an edge, then $P \cup Q \cup P_{x_1y} \cup (P_{x_3y} \setminus x_3) \cup x_2$ contains a $3\text{PC}(x_1, p_k)$. So x_2y is an edge, *i.e.*, Σ is a bug, and hence (ii) must hold. If $N(q_l) \cap \Sigma = y$ then $P \cup Q \cup P_{x_3y} \cup \{x_1, x_2\}$ induces an even wheel with center x_2. So q_l has a neighbor in $P_{x_3y} \setminus y$, and hence $Q' = p_k, Q$ is a crossing of Σ. If Q' is a crosspath of Σ, then it is a center-crosspath of a bug Σ, contradicting (ii). By definition of Q, q_l has a neighbor in $P_{x_3y} \setminus x_3$ and hence Q' cannot be a hat of Σ. So Q' must satisfy (iv) of Lemma 4.7, *i.e.*, q_l is of type p2 w.r.t. Σ and it is adjacent to y. Let H be the hole induced by $P \cup Q \cup (P_{x_3y} \setminus y) \cup x_1$. Then (H, x_2) is a bug and y is its center-crosspath, contradicting (ii).

So without loss of generality x_2 has a neighbor in $Q \setminus q_l$, and hence $i = k$. But then $P \cup Q \cup P_{x_3y} \cup \{x_1, x_2\}$ contains a proper wheel with center x_2.

Case 2: q_l is of type p w.r.t. Σ with neighbors in P_{x_2y}, and it has a neighbor in $P_{x_2y} \setminus y$.

Then x_1y is an edge, else $(\Sigma \setminus x_2) \cup P \cup Q$ contains a $3\text{PC}(x_1, y)$. So Σ is a bug, and hence (ii) holds.

First suppose that $i < k$. Then x_2 has no neighbor in $Q \setminus q_l$. Let $Q' = p_1, \ldots, p_i, Q$. Then Q' is a crossing of Σ. If Q' is a crosspath of Σ, then it is a center-crosspath of bug Σ, contradicting (ii). By definition of Q, Q' cannot be a hat of Σ. So Q' must satisfy (iv) of Lemma 4.7, *i.e.*, q_l is of type p2 w.r.t. Σ and it is adjacent to y. Then $P_{x_3y} \cup Q \cup \{x_2, p_j, \ldots, p_k\}$ induces a hole H', and hence (H', x_1) must be a bug. So $j > 1$ and x_1 has no neighbor in $Q \setminus q_l$. If $i = j$ then $P \cup Q \cup \{x_1, x_2, y\}$ induces a $3\text{PC}(x_1, p_i)$. If p_ip_j is not an edge, then $P \cup Q \cup \{x_1, x_2, y\}$ contains a 3PC(x_1, q_1). So p_ip_j is an edge. But then $p_1, \ldots, p_i$ is a center-crosspath of (H', x_1), contradicting (ii).

Therefore, $i = k$. So x_1 has no neighbor in $Q \setminus q_l$. Let x_2' be the neighbor of x_2 in P_{x_2y}. If x_2 has no neighbor in Q and q_l has a neighbor in $P_{x_2y} \setminus \{x_2, x_2'\}$, then $P \cup Q \cup P_{x_2y} \cup x_1$ contains a 3PC(x_1, p_k). If $N(q_l) \cap P_{x_2y} \subseteq \{x_2, x_2'\}$, then q_l is adjacent to x_2' and hence $P \cup Q \cup P_{x_2y} \cup x_1$ induces a proper wheel with center x_2. Hence q_l has a neighbor in $P_{x_2y} \setminus \{x_2, x_2'\}$. So x_2 does have a neighbor in Q. Let R be the chordless path from q_l to y in $P_{x_2y} \cup q_l$. Then $P \cup Q \cup R \cup x_1$ induces a hole H'. Hence (H', x_2) must be a bug, *i.e.*, $N(x_2) \cap Q = q_1$ and $x_2' \notin R$. But then $Q \cup R \cup P_{x_3y} \cup \{x_1, x_2\}$ induces a bug with center x_1, and P is its center-crosspath, contradicting (ii).

Case 3: q_l is of type b w.r.t. Σ with no neighbor in P_{x_2y}.

If P_{x_1y} or P_{x_3y} is an edge, then Σ induces a bug and q_l is its center-crosspath, a contradiction. So P_{x_1y} and P_{x_3y} are not edges. Hence $(\Sigma \setminus \{x_1, x_3\}) \cup P \cup Q$ contains a 3PC(q_l, y).

Case 4: q_l is of type b w.r.t. Σ with no neighbor in P_{x_3y}.

Without loss of generality $N(q_l) \cap P_{x_1y} = y_1$. If $x_1 = y_1$ then Σ induces a bug with center x_1, and q_l is its center-crosspath, a contradiction. So $x_1 \neq y_1$. If x_1y_1 is not an edge, then $(\Sigma \setminus x_2) \cup P \cup Q$ contains a 3PC(q_l, y). So x_1y_1 is an edge. But then, since there is no 4-hole, q_l is not adjacent to x_2. Therefore, $(\Sigma \setminus x_1) \cup P \cup Q$ contains a 3PC(q_l, y). □

Corollary 5.4 *Let G be a (diamond, 4-hole)-free odd-signable graph. If G contains a* 3PC$(\triangle, \cdot)$ *with a hat, then G has a clique cutset or a bisimplicial cutset.*

Proof: Follows from Theorem 3.2, Theorem 5.2 and Lemma 5.3. □

Lemma 5.5 *Let G be a (diamond, 4-hole)-free odd-signable graph. If G contains a bug with an ear, then G has a clique cutset or a bisimplicial cutset.*

Proof: Assume G has no clique cutset nor a bisimplicial cutset. By Theorem 3.2, Theorem 5.2 and Corollary 5.4, G does not contain a proper wheel, a bug with center-crosspath, nor a 3PC$(\triangle, \cdot)$ with a hat.

Let (H, x) be a bug and $P = p_1, \ldots, p_k$ its ear. Assume (H, x) and P are chosen so that $|H \cup P|$ is minimized. Without loss of generality p_k is adjacent to y_2.

Let X be the set comprised of x_1, x_2 and all type t nodes w.r.t. (H, x). Let Y be the set comprised of y and all type p2 nodes w.r.t. (H, x) adjacent to x and y. Since there is no diamond, both of the sets X and Y induce cliques. We now show that $S = X \cup Y \cup x$ is a cutset separating H_1 from $H_2 \cup P$.

Assume not and let $Q = q_1, \ldots, q_l$ be a direct connection from H_1 to $H_2 \cup P$ in $G \setminus S$. Note that x_1, x_2, x, y are the only nodes of $H \cup P \cup x$ that may have a neighbor in $Q \setminus \{q_1, q_l\}$, and no node of $Q \setminus \{q_1, q_l\}$ is adjacent to more than one node of $\{x_1, x_2, x, y\}$ (by definition of Q and Lemma 4.1).

Let Σ' be the 3PC(yy_2p_k, x) induced by $H_2 \cup P \cup \{x, y\}$. Now, Σ' is a bug with center y.

Claim 1: *q_1 and q_l are not of type b w.r.t. (H, x).*

Proof of Claim 1: Assume q_1 is of type b w.r.t. (H, x). Since q_1 cannot be a center-crosspath of (H, x), q_1 is not adjacent to x.

First suppose that $N(q_1) \cap H_1 = y_1$. If q_1 has a neighbor in $P \setminus p_k$, then $(H_2 \setminus y_2) \cup (P \setminus p_k) \cup \{x, y_1, q_1\}$ contains a 3PC(x, q_1). Node q_1 cannot be adjacent to p_k, else $\{y, y_1, q_1, p_k\}$ induces a 4-hole. So q_1 has no neighbor in P. But then $(H \setminus x_2)) \cup P \cup \{q_1, x\}$ contains an even wheel with center y.

So $N(q_1) \cap H_2 = y_2$. Suppose q_1 has a neighbor in P. Since there is no diamond, q_1 is not adjacent to p_k. But then $(H_1 \setminus y_1) \cup (P \setminus p_k) \cup \{x, q_1, y_2\}$ contains a 3PC(q_1, x). So q_1 has no neighbor in P. Let R be the chordless path in $(H_1 \setminus x_1) \cup q_1$ from q_1 to y_1. Then R is a hat of Σ', a contradiction.

So q_1 cannot be of type b, and by analogous argument neither can q_l. This completes the proof of Claim 1.

Claim 2: *q_1 and q_l are not of type pb w.r.t. (H, x).*

Proof of Claim 2: Suppose q_1 is of type pb w.r.t. (H, x). Note that $N(q_1) \cap H \subseteq H_1$. If q_1 has a neighbor in $P \setminus p_k$, then $H_1 \cup (P \setminus p_k) \cup \{x, q_1\}$ contains a 3PC(x, q_1). Suppose q_1 is adjacent to p_k. Then, since there is no diamond, q_1 is not adjacent to y, and hence $H_1 \cup \{x, q_1, p_k\}$ contains a 3PC(q_1, y). So q_1 has no neighbor in P. Let H' be the hole of $H \cup q_1$ that contains x_1, x_2, y and q_1. Then (H', x) is a bug and P its ear, contradicting the minimality of $|H \cup P|$.

Now suppose that q_l is of type pb. Let H' be the hole of $H \cup q_l$ that contains x_1, x_2, y and q_l. Then (H', x) is a bug. If q_l has a neighbor in P, then by Lemma 4.1 applied to Σ' and q_l, $N(q_l) \cap P = p_k$ and q_l is adjacent to y. But then P is an ear of (H', x), contradicting the minimality of $|H \cup P|$. Hence q_l has no neighbor in P. Suppose q_l is adjacent to y. Then since there is no diamond and q_l is not adjacent to p_k, q_l is not adjacent to y_2. Then q_l is a center-crosspath of bug Σ', a contradiction. So q_l is not adjacent to y. But then P is an ear of (H', x), contradicting the minimality of $|H \cup P|$. This completes the proof of Claim 2.

By Lemma 4.1, Claim 1, Claim 2 and the definition of S, q_1 is of type p1 or p2 w.r.t. (H, x) with neighbors in H_1, and if q_l has a neighbor in H, then q_l is of type p1 or p2 w.r.t. (H, x) with neighbors in H_2.

Claim 3: *At most one of the sets $\{x_1, x_2\}$ and $\{x, y\}$ may have a neighbor in $Q \setminus \{q_1, q_l\}$. Furthermore, at most one of the nodes x_1, x_2 may have a neighbor in $Q \setminus \{q_1, q_l\}$.*

Proof of Claim 3: First suppose that both a node of $\{x_1, x_2\}$ and a node of $\{x, y\}$ have a neighbor in $Q \setminus \{q_1, q_l\}$. Then there is a subpath Q' of $Q \setminus \{q_1, q_l\}$ such that one endnode of Q' is adjacent to a node of $\{x_1, x_2\}$, the other endnode of Q' is adjacent to a node of $\{x, y\}$, and no intermediate node of Q' has a neighbor in $H \cup x$. If x is adjacent to an endnode of Q', then Q' is a hat of (H, x), a contradiction. So y is adjacent to an endnode of Q'. But then $H \cup Q'$ induces a 3PC$(y, \cdot)$.

Now suppose that both x_1 and x_2 have a neighbor in $Q \setminus \{q_1, q_l\}$. Then x and y do not. Hence there is a subpath Q' of $Q \setminus \{q_1, q_l\}$ whose one endnode is adjacent to x_1, the other is adjacent to x_2, and no intermediate node of Q' has a neighbor in $H \cup x$. But then Q' is a hat of (H, x), a contradiction. This completes the proof of Claim 3.

Claim 4: *x has no neighbor in Q.*

Proof of Claim 4: Assume it does. By Claim 3, x_1 and x_2 have no neighbors in $Q \setminus \{q_1, q_l\}$. Let H' be the hole of $(H \setminus y) \cup P \cup Q$ that contains x_1, x_2 and Q. Then (H', x) must be a bug, and hence x has a unique neighbor q_t in Q. Furthermore, $N(q_l) \cap (P \cup H_2) \neq p_1$. Since there is no 4-hole, $N(q_l) \cap (P \cup H_2) \neq \{p_1, x_2\}$, *i.e.*, q_l has a neighbor in $(H_2 \setminus x_2) \cup (P \setminus p_1)$.

Suppose y has a neighbor in $q_1, \ldots, q_t$. Then $(H \setminus \{x_1, x_2\}) \cup (P \setminus p_1) \cup Q \cup x$ contains a 3PC(q_t, y). So y has no neighbor in $q_1, \ldots, q_t$. If q_1 is of type p1, then $H_1 \cup \{x, q_1, \ldots, q_t\}$ induces a 3PC$(x, \cdot)$. So q_1 is of type p2. But then $q_1, \ldots, q_t$ is a center-crosspath of (H, x), a contradiction. This completes the proof of Claim 4.

Claim 5: *x_1 has no neighbor in $Q \setminus q_1$, and x_2 has no neighbor in $Q \setminus q_l$.*

Proof of Claim 5: Suppose x_1 has a neighbor in $Q \setminus q_1$. Let q_i be the node of Q with highest index adjacent to x_1. By Claim 4, x has no neighbor in Q. By Claim 3, y has no neighbor in $Q \setminus \{q_1, q_l\}$. If q_l is adjacent to y or q_l has a neighbor in $P \setminus p_1$, then $H_1 \cup (P \setminus p_1) \cup \{x, q_i, \ldots, q_l\}$ contains a 3PC(x_1, y). So y has no neighbor in $Q \setminus q_1$.

If $N(q_l) \cap (H \cup P) = p_1$, then $q_i, \ldots, q_l, p_1$ is a hat of (H, x), a contradiction. So q_l has a neighbor in H_2. If $N(q_l) \cap H_2 = x_2$, then $q_i, \ldots, q_l$ is a hat of (H, x), a contradiction. So q_l has a neighbor in $H_2 \setminus x_2$. If q_l is of type p1, then $H \cup \{q_i, \ldots, q_l\}$ induces a 3PC$(x_1, \cdot)$. So q_l is of type p2, and hence crossing appendices $q_i, \ldots, q_l$ and x of H contradict Lemma 4.5.

So x_1 has no neighbor in $Q \setminus q_1$. By analogous argument x_2 has no neighbor in $Q \setminus q_l$. This completes the proof of Claim 5.

By Claim 4, x has no neighbor in Q. By Claim 5, x_1 has no neighbor in $Q \setminus q_1$ and x_2 has no neighbor in $Q \setminus q_l$. Suppose $N(q_l) \cap H = x_2$. If y has a neighbor in Q, then $H_2 \cup Q \cup x$ contains a 3PC(x_2, y). So y has no neighbor in Q. If q_1 is of type p1, then $H \cup Q$ induces a 3PC$(x_2, \cdot)$. So q_1 is of type p2. But then crossing appendices Q and x of H contradict Lemma 4.5. So $N(q_l) \cap H \neq x_2$.

Suppose that $N(q_l) \cap P = p_1$. Then $(H_1 \setminus x_1) \cup P \cup Q \cup x$ contains a 3PC(p_1, y). So $N(q_l) \cap P \neq p_1$.

If q_l is adjacent to both p_1 and x_2, then there is a 4-hole. Therefore, q_l has a neighbor in $(H_2 \setminus x_2) \cup (P \setminus p_1)$.

Case 1: y has a neighbor in Q.

Let q_i be the node of Q with highest index adjacent to y.

Suppose q_l does not have a neighbor in H_2. Since x has no neighbor in Q and $q_i, \ldots, q_l$ cannot be a center-crosspath or a hat of bug Σ', either q_l has a unique neighbor in P and that neighbor is in $P \setminus p_k$, or q_l has two nonadjacent neighbors in P. In both cases $P \cup \{x, y, q_i, \ldots, q_l\}$ contains a 3PC$(y, \cdot)$. So q_l has a neighbor in H_2.

Suppose q_l is adjacent to y. First assume that q_l is of type p1 w.r.t. (H, x). Then by Lemma 4.1 applied to Σ' and q_l, node q_l is of type b w.r.t. Σ', and hence it is a center-crosspath of bug Σ', a contradiction. So q_l is of type p2 w.r.t. (H, x).

Since there is no diamond, q_l is adjacent to p_k. If q_l has a neighbor in $P \setminus p_k$, then (H, x) and a subpath of $P \setminus p_k$ contradict the minimality of $|H \cup P|$. So $N(q_l) \cap (H \cup P) = \{y, y_2, p_k\}$. But then $(H \setminus y) \cup P \cup Q \cup x$ contains a 3PC$(p_k y_2 q_l, x x_1 x_2)$. So q_l is not adjacent to y.

Suppose q_l has a neighbor in P. By Lemma 4.1 applied to Σ', q_l is of type b w.r.t. Σ'. But then either $N(q_l) \cap P = p_1$ or $N(q_l) \cap H = x_2$. We have already established that none of these two possibilities can happen. Therefore q_l has no neighbor in P. But then $(H \setminus x_2) \cup P \cup Q \cup x$ contains a proper wheel with center y.

Case 2: y has no neighbor in Q.

Suppose q_l has a neighbor in H. Then Q is an appendix of H, else $H \cup Q$ induces a 3PC$(\cdot, \cdot)$

or a 3PC($\triangle, \triangle$). By Lemma 4.5, Q is in fact a crosspath of (H, x). If Q is a y_1-crosspath of (H, x), then $(H \setminus x_2) \cup P \cup Q \cup x$ contains either a proper wheel with center y (if q_l has no neighbor in $P \setminus p_k$) or a 3PC(y_1, x) (otherwise). So Q is a y_2-crosspath of (H, x). By Lemma 4.1 applied to Σ' and q_l, node q_l has no neighbor in P. But then a subpath of $(H_1 \setminus x_1) \cup Q$ is a hat of Σ', a contradiction.

So q_l has no neighbor in H. Suppose $N(q_l) \cap P = p_k$. Then the chordless path from q_l to y_1 in $(H \setminus x_1) \cup Q$ is a hat of Σ', a contradiction. So q_l has a neighbor in $P \setminus p_k$. If q_1 is of type p1 w.r.t. (H, x), then $H_1 \cup (P \setminus p_k) \cup Q \cup x$ contains a 3PC$(x, \cdot)$. So q_1 is of type p2 w.r.t. (H, x). But then the chordless path from p_1 to q_1 in $(P \setminus p_k) \cup Q$ is a center-crosspath of (H, x), a contradiction. □

Lemma 5.6 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. If P is a crossing of a $\Sigma =$ 3PC$(\triangle, \cdot)$ of G, then P is a crosspath of Σ.*

Proof: Assume G does not contain a clique cutset nor a bisimplicial cutset. By Theorem 3.2 G does not contain a proper wheel. Let $P = p_1, \ldots, p_k$ be a crossing of a $\Sigma =$ 3PC$(x_1x_2x_3, y)$ of G. Suppose that P is not a crosspath of Σ. Then $k > 1$ and (ii), (iii) or (iv) of Lemma 4.7 holds. P cannot be a hat of Σ by Lemma 5.4.

Suppose that (iii) of Lemma 4.7 holds. Without loss of generality p_1 is of type pb w.r.t. Σ, with neighbors in P_{x_1y} and p_k is of type p2 w.r.t. Σ, with neighbors in P_{x_2y}. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then (H, p_1) is a bug, and $p_2, \ldots, p_k$ is either a center-crosspath or an ear of (H, p_1), contradicting Theorem 5.2 or Lemma 5.5.

Suppose that (iv) of Lemma 4.7 holds. Without loss of generality p_1 is of type p1 w.r.t. Σ, adjacent to y_1, and p_k is of type p2 w.r.t. Σ, adjacent to y and y_2. If $y_1 = x_1$ then Σ is a bug and P its ear, contradicting Lemma 5.5. So $y_1 \neq x_1$. Let H' be the hole induced by $((P_{x_1y} \cup P_{x_2y}) \setminus y) \cup P$. Then (H', y) is a bug and P_{x_3y} is its center-crosspath, contradicting Theorem 5.2. □

6 Attachments to a 3PC($\triangle, \cdot$)

We now examine how certain types of nodes adjacent to a $\Sigma =$ 3PC$(\triangle, \cdot)$ attach to Σ in graphs that have no clique cutsets nor bisimplicial cutsets.

In this section we consider a (diamond, 4-hole)-free odd-signable graph G. For a 3PC$(x_1x_2x_3, y)$ of G we use the notation $P_{x_1y}, P_{x_2y}, P_{x_3y}, y_1, y_2, y_3$ as defined in Section 4.

Definition 6.1 *Let u be a type t3 node w.r.t. a $\Sigma =$ 3PC$(x_1x_2x_3, y)$ of G. Let X be the set comprised of x_1, x_2, x_3 and all type t nodes w.r.t. Σ. Note that since G is diamond-free, set X induces a clique. Suppose that $X \setminus \{u\}$ is not a clique cutset of G, and let $P = p_1, \ldots, p_k$ be a direct connection from u to Σ in $G \setminus (X \setminus \{u\})$. Path P is called an* attachment *of u to Σ. If such a path exists, we say that u is* attached *to Σ.*

Lemma 6.2 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let u be a type t3 node w.r.t. a $\Sigma =$ 3PC$(x_1x_2x_3, y)$. Then u is attached to Σ. Let $P = p_1, \ldots, p_k$ be an attachment of u to Σ. Then no node of Σ has a neighbor in $P \setminus p_k$ and p_k is of type p1 w.r.t. Σ.*

Proof: By Theorem 3.2, G does not contain a proper wheel. Since G has no clique cutset, there exists a direct connection $P = p_1, \ldots, p_k$ from u to Σ in $G \setminus (X \setminus u)$, *i.e.*, u is attached. By definition of P and Lemma 4.1, no node of P has more than one neighbor in $\{x_1, x_2, x_3\}$. The only nodes of Σ that may have a neighbor in $P \setminus p_k$ are x_1, x_2, x_3. If at least two nodes of $\{x_1, x_2, x_3\}$ have a neighbor in $P \setminus p_k$, then a subpath of $P \setminus p_k$ is a hat of Σ, contradicting Lemma 5.6. So without loss of generality x_2 and x_3 do not have neighbors in $P \setminus p_k$. If x_1 has a neighbor in $P \setminus p_k$, let p_i be such a neighbor with highest index. By Lemma 4.1 and definition of P, p_k is of type p or b w.r.t. Σ.

Case 1: p_k is of type b w.r.t. Σ.

Let $l \in \{1, 2, 3\}$ such that $N(p_k) \cap P_{x_l y} = y_l$. If $x_l = y_l$ then Σ is a bug and p_k is its center-crosspath, contradicting Theorem 5.2. So $x_l \neq y_l$.

Let H be the hole of Σ such that (H, p_k) is a bug. By Theorem 5.2, path $u, p_1, \ldots, p_{k-1}$ cannot be a center-crosspath of (H, p_k), so H must contain $P_{x_1 y}$ and x_1 must have a neighbor in $P \setminus p_k$. In particular $k > 1$. If $i < k - 1$ then $p_i, \ldots, p_{k-1}$ and (H, p_k) contradict Lemma 5.6. So $i = k - 1$, *i.e.*, x_1 is adjacent to p_{k-1}. Since $x_l \neq y_l$, Lemma 4.1 applied to (H, p_k) and p_{k-1} is contradicted.

Case 2: p_k is of type pb w.r.t. Σ.

If the neighbors of p_k in Σ are contained in $P_{x_2 y} \cup P_{x_3 y}$, then let H be the hole induced by $P_{x_2 y} \cup P_{x_3 y}$. Otherwise, let H be the hole induced by $P_{x_1 y} \cup P_{x_2 y}$. Note that (H, p_k) is a bug. By Theorem 5.2, path $u, p_1, \ldots, p_{k-1}$ cannot be a center-crosspath of (H, p_k) and by Lemma 5.5 it cannot be an ear of (H, p_k), so H must contain $P_{x_1 y}$ (*i.e.*, p_k has neighbors in $P_{x_1 y}$) and x_1 must have a neighbor in $P \setminus p_k$. In particular $k > 1$. If x_1 is adjacent to p_k, then $H \cup P \cup u$ contains a proper wheel with center x_1. So x_1 is not adjacent to p_k. If $i = k - 1$ then p_i contradicts Lemma 4.1. So $i < k - 1$. But then $p_i, \ldots, p_{k-1}$ and (H, p_k) contradict Lemma 5.6.

Case 3: p_k is of type p2 w.r.t. Σ.

The neighbors of p_k in Σ must be contained in $P_{x_1 y}$, else $P_{x_2 y} \cup P_{x_3 y} \cup P \cup u$ induces a 3PC($\triangle, \triangle$) or an even wheel. Node p_k cannot be adjacent to x_1, since otherwise $P_{x_1 y} \cup P_{x_2 y} \cup P \cup u$ induces a proper wheel with center x_1. If x_1 does not have a neighbor in $P \setminus p_k$, then $P_{x_1 y} \cup P_{x_2 y} \cup P \cup u$ induces a 3PC($\triangle, \triangle$). So x_1 has a neighbor in $P \setminus p_k$. Since G does not contain a proper wheel , $P_{x_1 y} \cup P_{x_2 y} \cup P \cup u$ induces a bug (with center x_1) together with a center-crosspath, contradicting Theorem 5.2.

Case 4: p_k is of type p1 w.r.t. Σ.

Suppose x_1 has a neighbor in $P \setminus p_k$. Let a be the neighbor of p_k in Σ. If $a \notin P_{x_1 y}$ then $p_i, \ldots, p_k$ contradicts Lemma 5.6. So $a \in P_{x_1 y}$. If $a x_1$ is not an edge, then $P_{x_1 y} \cup P_{x_2 y} \cup \{p_i, \ldots, p_k\}$ induces a 3PC(x_1, a). So $a x_1$ is an edge. But then $P_{x_1 y} \cup P_{x_2 y} \cup P \cup u$ induces a proper wheel with center x_1. Therefore x_1 does not have a neighbor in $P \setminus p_k$, which proves the lemma. □

Lemma 6.3 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let u be a type t3 node w.r.t. a $\Sigma = $ 3PC$(x_1 x_2 x_3, y)$. Then all attachments of u to Σ end in the same path of Σ.*

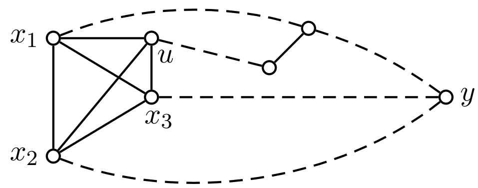


Figure 16: An attachment of a type t3 node u w.r.t. a $3\text{PC}(x_1x_2x_3, y)$.

Proof: Let $P = p_1, \dots, p_k$ and $Q = q_1, \dots, q_l$ be two attachments of u to Σ. By Lemma 6.2, p_k and q_l are both of type p1 w.r.t. Σ, and no node of Σ has a neighbor in $(P \setminus p_k) \cup (Q \setminus q_l)$. Let p (resp. q) be the neighbor of p_k (resp. q_l) in Σ. Suppose that $p \in P_{x_1y} \setminus y$ and $q \in P_{x_2y} \setminus y$. Note that by definition of attachment, $p \neq x_1$ and $q \neq x_2$. If a node of P is adjacent to or coincident with a node of Q, then there is a chordless path in $P \cup Q$ from p_k to q_l, that contradicts Lemma 5.6. So no node of P is adjacent to or coincident with a node of Q. But then $(\Sigma \setminus \{x_1, x_2\}) \cup P \cup Q \cup u$ induces a $3\text{PC}(u, y)$. $\square$

Definition 6.4 *Let u be a type t3b node w.r.t. a $\Sigma = 3\text{PC}(x_1x_2x_3, y)$, and suppose that u has a neighbor in $P_{x_iy} \setminus x_i$. Let Σ' be the $3\text{PC}(\triangle, \cdot)$ contained in $(\Sigma \setminus \{x_i\}) \cup \{u\}$. We say that Σ' is obtained by* substituting u into Σ.

Definition 6.5 *Let u be a type t3 node w.r.t. a $\Sigma = 3\text{PC}(x_1x_2x_3, y)$, and let $P = p_1, \dots, p_k$ be an attachment of u to Σ. By Lemma 6.2, p_k is of type p1 w.r.t. Σ. If the neighbor of p_k in Σ is in path P_{x_iy}, then we say that attachment P* ends in P_{x_iy}. *Suppose that P ends in P_{x_iy}. Let Σ' be the $3\text{PC}(\triangle, \cdot)$ contained in $(\Sigma \setminus \{x_i\}) \cup P \cup \{u\}$. We say that Σ' is obtained by* substituting u and its attachment P into Σ.

Lemma 6.6 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let u be a type t node w.r.t. a $\Sigma = 3\text{PC}(x_1x_2x_3, y)$. If u is of type t3b, then assume that u is not adjacent to y, and let Σ' be the $3\text{PC}(\triangle, \cdot)$ obtained by substituting u into Σ. If u is of type t3, then let $P = p_1, \dots, p_k$ be an attachment of u to Σ such that p_k is not adjacent to y, and let Σ' be the $3\text{PC}(\triangle, \cdot)$ obtained by substituting u and P into Σ. Then Q is a crosspath of Σ if and only if Q is a crosspath of Σ'.*

Proof: By Theorem 3.2, G does not contain a proper wheel. Without loss of generality assume that if u is of type t3b (resp. t3) w.r.t. Σ then u (resp. p_k) has a neighbor $p \in P_{x_1y} \setminus \{x_1, y\}$. Let $Q = q_1, \dots, q_l$ be a crosspath of Σ. Note that if Q is a y_t-crosspath, then by Theorem 5.2, $x_t \neq y_t$. We now show that Q is a crosspath of Σ'.

Case 1: u is of type t3b w.r.t. Σ.

First suppose that Q is a y_2-crosspath of Σ that ends in P_{x_3y}. If u does not have a neighbor in Q, then clearly Q is a crosspath of Σ'. So assume that u does have a neighbor in Q, and let q_i be such a neighbor with lowest index. Then $(P_{x_1y} \setminus x_1) \cup P_{x_2y} \cup \{u, q_1, \dots, q_i\}$ contains a $3\text{PC}(u, y_2)$.

Next suppose that Q is a y_1-crosspath that ends in P_{x_3y}. If u does not have a neighbor in Q, then clearly Q is a crosspath of Σ'. So assume that u does have a neighbor in Q, and let q_i be such a neighbor with highest index. Then $P_{x_2y} \cup P_{x_3y} \cup \{u, q_i, \dots, q_l\}$ contains a $3\text{PC}(\triangle, \triangle)$ or an even wheel with center x_3.

Finally, by symmetry, we may assume that Q is a y_2-crosspath that ends in P_{x_1y}. If u has a neighbor in $Q \setminus q_l$, then $(\Sigma \setminus \{x_1, x_2\}) \cup (Q \setminus q_l) \cup u$ contains a 3PC(u, y). So u does not have a neighbor in $Q \setminus q_l$. Suppose that u is adjacent to q_l. Let H be the hole induced by $P_{x_1y} \cup P_{x_3y}$. Then (H, u) is a bug, and by Lemma 4.1, q_l is of type b w.r.t. (H, u), contradicting Theorem 5.2. So u does not have a neighbor in Q. If the neighbors of q_l in P_{x_1y} are contained in the py-subpath of P_{x_1y}, then clearly Q is a crosspath of Σ'. So assume that the neighbors of q_l in P_{x_1y} are contained in the x_1p-subpath of P_{x_1y}, call it P'. Then $(P' \setminus x_1) \cup Q$ contains a path R from q_1 to p that contradicts Lemma 5.6 applied to Σ'.

Case 2: u is of type t3 w.r.t. Σ.

$P = p_1, \ldots, p_k$ is an attachment of u to Σ such that p_k is not adjacent to y. By Lemma 6.2, p_k is of type p1 w.r.t. Σ, and no node of Σ has a neighbor in $P \setminus p_k$. Then $\Sigma' = $ 3PC(ux_2x_3, y). Let r_1 and r_2 be the adjacent neighbors of q_l in Σ. Suppose that u has a neighbor in $Q \setminus q_l$, and let q_i be such a neighbor with highest index. Then $q_i, \ldots, q_l$ is an attachment of u to Σ that contradicts Lemma 6.2. So u does not have a neighbor in $Q \setminus q_l$. Now suppose that u is adjacent to q_l. If Q is a y_1-crosspath of Σ, then $P_{x_2y} \cup P_{x_3y} \cup Q \cup u$ induces a 3PC$(\triangle, \triangle)$ or an even wheel. Analogous contradiction is obtained if Q is a y_2-crosspath or a y_3-crosspath of Σ. So u does not have a neighbor in Q.

First suppose that Q is a y_1-crosspath of Σ. If Q is not a y_1-crosspath of Σ', then some node of Q is adjacent to or coincident with a node of P. Let q_i be the node of Q with highest index adjacent to a node of $P \cup u$. If $i < l$ then path $q_i, \ldots, q_l$ contradicts Lemma 5.6 applied to Σ'. So $i = l$, and hence q_l and Σ' contradict Lemma 4.1.

Now assume without loss of generality that Q is a y_2-crosspath of Σ. Suppose that a node of P is adjacent to or coincident with a node of Q. Let q_i be the node of Q with lowest index adjacent to a node of P, and let p_j be the node of P with highest index adjacent to q_i. If $i \neq l$ then path $q_1, \ldots, q_i, p_j, \ldots, p_k$ contradicts Lemma 5.6 applied to Σ. So $i = l$. But then, by Lemma 4.1 applied to Σ' and q_l, r_1 and r_2 are contained in P_{x_1y}. Hence by Lemma 5.6 and Lemma 4.1, Q is a y_2-crosspath of Σ'. So we may assume that no node of P is adjacent to or coincident with a node of Q. If q_l has a neighbor in Σ', then by Lemma 5.6 and Lemma 4.1, Q is a y_2-crosspath of Σ'. Otherwise, $r_1, r_2 \in \Sigma \setminus \Sigma'$. But then Q together with an appropriate subpath of $P_{x_1y} \setminus x_1$ contradicts Lemma 5.6 applied to Σ'.

Therefore, if Q is a crosspath of Σ, then it is a crosspath of Σ'. The converse holds by symmetry, since x_1 is either of type t3b w.r.t. Σ' or of type t3 w.r.t. Σ' attached to Σ' by path $\Sigma \setminus \Sigma'$. $\square$

Lemma 6.7 *Let G be a (diamond, 4-hole)-free odd-signable graph. If G contains a bug (H, x) with a type p1 or p2 node that is adjacent to x, then G has a clique cutset or a bisimplicial cutset.*

Proof: Let (H, x) be a bug. Let x_1, x_2, y be the neighbors of x in H such that x_1x_2 is an edge. Let H_1 (resp. H_2) be the sector of (H, x) with endnodes x_1 (resp. x_2) and y.

Let U be the set of type p1 and p2 nodes w.r.t. (H, x) that are adjacent to x, and assume that $U \neq \varnothing$. Assume G does not have a clique cutset nor a bisimplicial cutset. Since $\{x, y\}$ cannot be a clique cutset separating U from H, there exists a path $P = p_1, \ldots, p_k$ in $G \setminus \{x, y\}$ such that $p_1 \in U$, p_k has a neighbor in $H \setminus \{x, y\}$, and no node of $P \setminus \{p_1, p_k\}$ has a neighbor in $(H \cup x) \setminus y$. We may assume without loss of generality that bug (H, x) and path P are chosen so that $|P|$ is minimized. By Theorem 3.2, G does not contain a proper wheel. So by Lemma 4.1 we need to consider the following cases.

Case 1: p_k is of type b w.r.t. (H, x).

By Theorem 5.2, p_k cannot be adjacent to x. Without loss of generality p_k is adjacent to y_1 (and hence has two neighbors in H_2). Let H' be the hole induced by $P \cup (H_1 \setminus y) \cup x$. Since $H' \cup y$ cannot induce a 3PC(x, y_1), y must have a neighbor in P. So (H', y) is a wheel, and hence it is a bug. In particular, $N(y) \cap P = p_1$. But then $H_2 \cup P$ induces a 3PC$(xyp_1, \triangle)$.

Case 2: p_k is of type p w.r.t. (H, x).

By definition of P, p_k is not adjacent to x. So without loss of generality the neighbors of p_k in $H \cup x$ are contained in H_1. Note that p_k has a neighbor in $H_1 \setminus y$. If y has no neighbor in $P \setminus \{p_1, p_k\}$, then by Lemma 5.6, P is a center-crosspath of (H, x), contradicting Theorem 5.2. So y has a neighbor in $P \setminus \{p_1, p_k\}$. Let H' be the hole contained in $(H_1 \setminus y) \cup P \cup x$ that contains x_1. Since $H' \cup y$ cannot induce a 3PC$(\cdot, \cdot)$, (H', y) is a wheel, and hence it is a bug. But then H_2 is either a center-crosspath of (H', y) (contradicting Theorem 5.2) or an ear of (H', y) (contradicting Lemma 5.5).

Case 3: p_k is of type t3b w.r.t. (H, x).

Then without loss of generality p_k has a neighbor in $H_1 \setminus x_1$. Let H' be the hole contained in $H \cup p_k$ that contains p_k and H_2. Then (H', x) is a bug. By Lemma 4.1, p_1 is of type p1 or p2 w.r.t. (H', x) adjacent to x. In particular, $k > 2$. So (H', x) and $P \setminus p_k$ contradict our choice of (H, x) and P.

Case 4: p_k is of type t3 w.r.t. (H, x).

By Lemma 6.2, there exists an attachment $Q = q_1, \ldots, q_l$ of p_k to (H, x), q_l is of type p1 w.r.t. (H, x) and no node of $H \cup x$ has a neighbor in $Q \setminus q_l$. Without loss of generality the neighbor of q_l in $H \cup x$ is contained in H_1. Let H' be the hole contained in $(H \setminus x_1) \cup Q \cup p_k$ that contains p_k and H_2. Then (H', x) is a bug. Note that $p_1 \notin Q$ (by definition of attachment and Lemma 6.2). By Lemma 4.1 and Theorem 5.2, p_1 is of type p1 or p2 w.r.t. (H', x). Let p_i be the node of P with lowest index adjacent to a node of Q (note that such a node exists since p_k is adjacent to q_1). If $i = k$ then $p_1, \ldots, p_{k-1}$ is a hat of (H', x), contradicting Corollary 5.4. So $i < k$ and hence (H', x) and $p_1, \ldots, p_i$ contradict our choice of (H, x) and P. □

Definition 6.8 *Let u be a type p2 node w.r.t. a 3PC$(x_1x_2x_3, y)$ of G, that is adjacent to y and y_1. Assume that $x_1 \neq y_1$. Let $S = (N[y_1] \cap N[y]) \setminus \{u\}$. Note that since G is diamond-free, S induces a clique. Suppose that S is not a clique cutset of G, and let $P = p_1, \ldots, p_k$ be a direct connection from u to Σ in $G \setminus S$. Such a path P is called an* attachment *of u to Σ. If such a path exists, we say that u is* attached *to Σ.*

Lemma 6.9 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let u be a type p2 node w.r.t. a $\Sigma =$ 3PC$(x_1x_2x_3, y)$ of G that is adjacent to y and y_1, and assume that $x_1 \neq y_1$. Then u is attached to Σ. Let $P = p_1, \ldots, p_k$ be an attachment of u to Σ. Then no node of Σ has a neighbor in $P \setminus p_k$ and p_k is of type p1 w.r.t. Σ, with a neighbor in $P_{x_1y} \setminus \{y, y_1\}$.*

Proof: By Theorem 3.2, G does not contain a proper wheel. Since G has no clique cutset, there exists a direct connection P from u to Σ in $G \setminus S$, where $S = (N[y_1] \cap N[y]) \setminus u$. So u

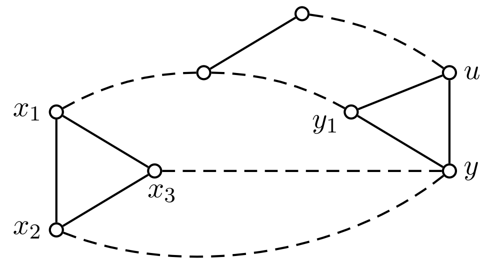


Figure 17: An attachment of a type p2 node u w.r.t. a 3PC$(x_1x_2x_3, y)$, when u is adjacent to y and y_1, and $x_1 \neq y_1$.

is attached to Σ. By definition of P, the only nodes of Σ that may have a neighbor in $P \setminus p_k$ are y and y_1, no node of P has more than one neighbor in $\{y, y_1\}$, and p_k has a neighbor in $\Sigma \setminus \{y, y_1\}$. If y or y_1 has a neighbor in $P \setminus p_k$, then let p_i be such a neighbor with highest index. By Lemma 4.1, we now consider the following cases.

Case 1: p_k is of type pb or b w.r.t. Σ.

First suppose that $N(p_k) \cap \Sigma \subseteq P_{x_1y} \cup P_{x_2y}$. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then (H, p_k) is a bug. Suppose that $k = 1$. Then $N(u) \cap (H \cup p_k) = \{p_k, y, y_1\}$. By Lemma 4.1 applied to (H, p_k) and u, node u must be of type b w.r.t. (H, p_k). But then u is a center-crosspath of (H, p_k), contradicting Theorem 5.2. So $k > 1$. Since p_{k-1} is adjacent to p_k and $N(p_{k-1}) \cap (H \cup p_k) \subseteq \{p_k, y, y_1\}$, by Lemma 4.1, node p_{k-1} is of type p1, p2 or b w.r.t. (H, p_k). If p_{k-1} is of type b w.r.t. (H, p_k), then Theorem 5.2 is contradicted. If p_{k-1} is of type p1 or p2 w.r.t. (H, p_k), then Lemma 6.7 is contradicted.

Now without loss of generality we may assume that p_k is of type b w.r.t. Σ and $N(p_k) \cap \Sigma \subseteq P_{x_2y} \cup P_{x_3y}$. Let H be the hole induced by $P_{x_2y} \cup P_{x_3y}$. Then (H, p_k) is a bug. If $k = 1$ then u and (H, p_k) contradict Lemma 4.1. So $k > 1$. By Lemma 4.1, p_{k-1} is of type p1 w.r.t. (H, p_k). But then (H, p_k) and p_{k-1} contradict Lemma 6.7

Case 2: p_k is of type p1 or p2 w.r.t. Σ with neighbors in P_{x_2y} or P_{x_3y}.

Without loss of generality $N(p_k) \cap \Sigma \subseteq P_{x_2y}$. If y and y_1 do not have neighbors in $P \setminus p_k$, then path P, u contradicts Lemma 5.6. So y or y_1 has a neighbor in $P \setminus p_k$.

Suppose that p_i is adjacent to y. Let p be the neighbor of p_k in P_{x_2y} that is closest to x_2. Let H be the hole contained in $((P_{x_1y} \cup P_{x_2y}) \setminus y) \cup P \cup u$ that contains the x_2p-subpath of P_{x_2y}, x_1y_1-subpath of P_{x_1y} and $p_1, \dots, p_i$. Note that y has at least two nonadjacent neighbors in H, p_i and y_1. Since $H \cup y$ cannot induce a 3PC(y_1, p_i), (H, y) must be a wheel, and hence it is a bug. In particular py is not an edge. Node y_2 is adjacent to y, and hence by Lemma 4.1, it is either of type p1 or b w.r.t. (H, y). If y_2 is of type p1 w.r.t. (H, y), then Lemma 6.7 is contradicted. If y_2 is of type b w.r.t. (H, y), then Theorem 5.2 is contradicted.

So p_i must be adjacent to y_1. Then since G is diamond-free, $i > 1$. By Lemma 5.6, path $p_i, \dots, p_k$ is a y_1-crosspath of Σ. Let Σ' be the 3PC$(\triangle, y_1)$ induced by $P_{x_1y} \cup P_{x_2y} \cup \{p_i, \dots, p_k\}$. Let $u = p_0$, and let p_j be the node of $p_0, p_1, \dots, p_{i-1}$ with highest index adjacent to y. Let H be the hole contained in $(P_{x_2y} \setminus x_2) \cup \{p_j, \dots, p_k\}$. Since $H \cup y_1$ cannot induce a 3PC(p_i, y), (H, y_1) is a wheel, and hence it must be a bug. Let y_1' be the neighbor of y_1 in $P_{x_1y} \setminus y$. Then y_1' is of type p1 w.r.t. (H, y_1), contradicting Lemma 6.7.

Case 3: p_k is of type p1 or p2 w.r.t. Σ with neighbors in P_{x_1y}.

Then p_k is not adjacent to y. First suppose that y_1 and y do not have neighbors in $P \setminus p_k$.

If p_k is of type p2, then $P_{x_1y} \cup P_{x_2y} \cup P \cup u$ induces a 3PC$(uy_1y, \triangle)$ or an even wheel with center y_1. So p_k is of type p1, and the lemma holds. So we may assume that y_1 or y has a neighbor in $P \setminus p_k$. Let p be the neighbor of p_k in P_{x_1y} that is closest to x_1.

Suppose that p_i is adjacent to y_1. Let $u = p_0$, and let p_j be the node of $p_0, \dots, p_{i-1}$ with highest index adjacent to y. Let H be the hole induced by x_1p-subpath of P_{x_1y}, P_{x_2y} and $p_j, \dots, p_k$. Note that y_1 is adjacent to two nonadjacent nodes of H, p_i and y. Since $H \cup y_1$ cannot induce a 3PC(p_i, y), (H, y_1) is a wheel, and hence it is a bug. In particular, py_1 is not an edge. Let y_1' be the neighbor of y_1 in $P_{x_1y} \setminus y$. By Lemma 4.1, y_1' is of type p1 or b w.r.t. (H, y_1), contradicting Lemma 6.7 or Theorem 5.2. So we may assume that p_i is adjacent to y. If p_k is of type p1 w.r.t. Σ, then $P_{x_1y} \cup P_{x_2y} \cup \{p_i, \dots, p_k\}$ induces a 3PC(p, y). So p_k is of type p2 w.r.t. Σ. Let p' be the neighbor of p_k in P_{x_1y} distinct from p, and let P' be the $p'y_1$-subpath of P_{x_1y}. Let H be the hole contained in $P \cup P' \cup u$ that contains P' and $p_i, \dots, p_k$. Node y has two nonadjacent neighbors in H, p_i and y_1. Since $H \cup y$ cannot induce a 3PC(p_i, y_1), (H, y) is a wheel, and hence it is a bug. But then y_2 is of type p1 w.r.t. (H, y), adjacent to y, contradicting Lemma 6.7.

Case 4: p_k is of type t3 w.r.t. Σ.

If y and y_1 do not have neighbors in $P \setminus p_k$, then $P_{x_1y} \cup P_{x_2y} \cup P \cup u$ induces a 3PC$(y_1yu, p_kx_1x_2)$. So y or y_1 has a neighbor in $P \setminus p_k$.

First suppose that p_i is adjacent to y. Let H be the hole contained in $(P_{x_1y} \setminus y) \cup P \cup u$ that contains $P_{x_1y} \setminus y$ and $p_i, \dots, p_k$. As before, (H, y) is a bug. Note that either $y_2 \neq x_2$ or $y_3 \neq x_3$. Without loss of generality $y_2 \neq x_2$. But then y_2 is of type p1 w.r.t. (H, y) adjacent to y, contradicting Lemma 6.7.

Hence p_i must be adjacent to y_1. Let H be the hole contained in $P_{x_2y} \cup P \cup u$ that contains P_{x_2y} and $p_i, \dots, p_k$. As before, (H, y_1) is a bug. Let y_1' be the neighbor of y_1 in $P_{x_1y} \setminus y$. By Lemma 4.1, y_1' is of type p1 or b w.r.t. (H, y_1), contradicting Lemma 6.7 or Theorem 5.2.

Case 5: p_k is of type t3b w.r.t. Σ.

Let H be a hole of Σ such that (H, p_k) is a bug. If $k = 1$, let $z = u$, and otherwise let $z = p_{k-1}$. By Lemma 4.1, z is of type p1, p2 or b w.r.t. (H, p_k), adjacent to p_k, contradicting Lemma 6.7 or Theorem 5.2. □

7 Blocking sequences for 2-joins

In this section we consider an induced subgraph H of G that contains a 2-join $H_1|H_2$. We say that a 2-join $H_1|H_2$ *extends* to G if there exists a 2-join $H_1'|H_2'$ of G, with $H_1 \subseteq H_1'$ and $H_2 \subseteq H_2'$. We characterize the situation in which the 2-join of H does not extend to a 2-join of G.

Definition 7.1 *A* blocking sequence *for a 2-join $H_1|H_2$ of an induced subgraph H of G is a sequence of distinct nodes $x_1, \dots, x_n$ in $G \setminus H$ with the following properties:*

(1) (i) $H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$,

(ii) $H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, and

(iii) if $n > 1$ then, for $i = 1, \dots, n-1$, $H_1 \cup x_i|H_2 \cup x_{i+1}$ is not a 2-join of $H \cup \{x_i, x_{i+1}\}$.

(2) $x_1, \dots, x_n$ is minimal w.r.t. property (1), in the sense that no sequence $x_{j_1}, \dots, x_{j_k}$ with $\{x_{j_1}, \dots, x_{j_k}\} \subset \{x_1, \dots, x_n\}$, satisfies (1).

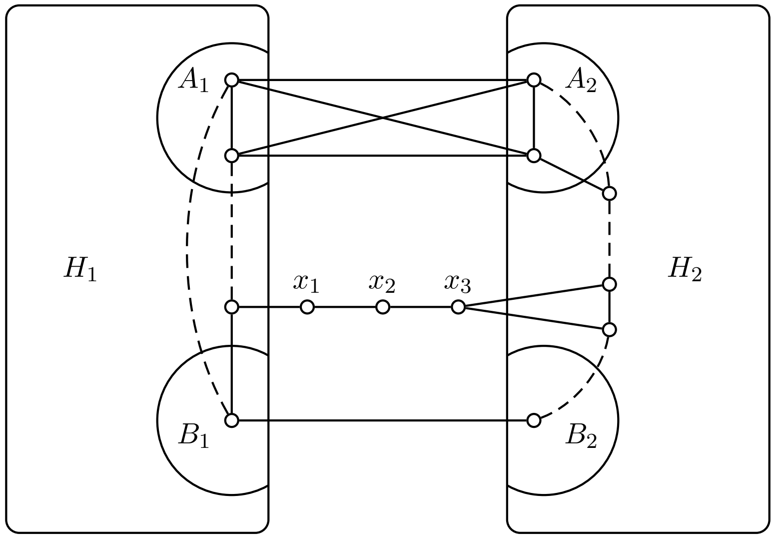


Figure 18: A blocking sequence x_1, x_2, x_3 for the 2-join $H_1|H_2$.

Blocking sequences for 2-joins were introduced in [7], where the following results are obtained.

Let H be an induced subgraph of G with 2-join $H_1|H_2$ and special sets (A_1, A_2, B_1, B_2). In the following results we let $S = x_1, \dots, x_n$ be a blocking sequence for the 2-join $H_1|H_2$ of an induced subgraph H of G.

Remark 7.2 ([7]) *$H_1|H_2 \cup u$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_1 = \varnothing, A_1$ or B_1. Similarly, $H_1 \cup u|H_2$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_2 = \varnothing, A_2$ or B_2.*

Lemma 7.3 ([7]) *If $n > 1$ then, for every node x_j, $j \in \{1, \dots, n-1\}$, $N(x_j) \cap H_2 = \varnothing, A_2$ or B_2, and for every node x_j, $j \in \{2, \dots, n\}$, $N(x_j) \cap H_1 = \varnothing, A_1$ or B_1.*

Theorem 7.4 ([7]) *Let H be an induced subgraph of a graph G that contains a 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G if and only if there exists no blocking sequence for $H_1|H_2$ in G.*

Lemma 7.5 ([7]) *If x_j is the node of lowest index adjacent to a node of H_2, then $x_1, \dots, x_j$ is a chordless path. Similarly, if x_j is the node of highest index adjacent to a node of H_1, then $x_j, \dots, x_n$ is a chordless path.*

Theorem 7.6 ([7]) *Let G be a graph and H an induced subgraph of G with a 2-join $H_1|H_2$ and special sets (A_1, A_2, B_1, B_2). Let H' be an induced subgraph of G with 2-join $H_1'|H_2$ and special sets (A_1', A_2, B_1', B_2) such that $A_1' \cap A_1 \neq \varnothing$ and $B_1' \cap B_1 \neq \varnothing$. If S is a blocking sequence for $H_1|H_2$ and $H_1' \cap S \neq \varnothing$, then a proper subset of S is a blocking sequence for $H_1'|H_2$.*

8 Decomposable 3PC($\triangle, \cdot$)

In this section we decompose certain 3PC($\triangle, \cdot$)'s (called the decomposable 3PC($\triangle, \cdot$)'s). This will allow s to prove Lemma 1.4.

Definition 8.1 *Let $\Sigma = 3\text{PC}(x_1x_2x_3, y)$, with the neighbors of y on paths P_{x_1y}, P_{x_2y} and P_{x_3y} being nodes y_1, y_2 and y_3 respectively. Σ is a* decomposable $3\text{PC}(\triangle, \cdot)$ *if the following hold.*

(1) $x_3 \neq y_3$.

(2) If G contains a $3\text{PC}(\triangle, \cdot)$ with a crosspath, then Σ has a y_1-crosspath and all crosspaths of Σ are from y_1 to P_{x_2y}.

(3) One of the following holds:

(i) There exists a node u of type t3 w.r.t. Σ such that every attachment of u to Σ ends in P_{x_3y}.

(ii) There exists a node u of type t3b w.r.t. Σ that has a neighbor in $P_{x_3y} \setminus \{x_3, y\}$.

(iii) There exists a node u of type p w.r.t. Σ that has a neighbor in $P_{x_3y} \setminus \{y\}$.

$\Sigma \cup u$ is called an extension *of the decomposable $3\text{PC}(x_1x_2x_3, y)$.*

Lemma 8.2 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let $\Sigma = 3\text{PC}(x_1x_2x_3, y)$ and let u be a type t3b node w.r.t. Σ that is adjacent to y. A node $v \in G \setminus (\Sigma \cup u)$ is adjacent to u if and only if v is of type t w.r.t. Σ.*

Proof: Suppose that v is adjacent to u. If v does not have a neighbor in $\Sigma \setminus y$, then bug induced by $P_{x_1y} \cup P_{x_2y} \cup u$ and v contradict Lemma 6.7. So v does have a neighbor in $\Sigma \setminus y$. Without loss of generality v has a neighbor in $P_{x_3y} \setminus y$. Let Σ' be the $3\text{PC}(ux_2x_3, y)$ obtained by substituting u into Σ. Suppose that v is not of type t w.r.t. Σ. So by Lemma 4.1, v is of type p or b w.r.t. Σ. By Lemma 4.1, v is of type b w.r.t. Σ'. But then v is a center-crosspath of bug Σ', contradicting Theorem 5.2.

Now suppose that v is of type t w.r.t. Σ. If v is not adjacent to u, then $\{u, v, x_1, x_2\}$ induces a diamond. □

Lemma 8.3 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let Σ be a $3\text{PC}(x_1x_2x_3, y)$ of G such that if G has a $3\text{PC}(\triangle, \cdot)$ with a crosspath, then Σ has a y_1-crosspath and all crosspaths of Σ are from y_1 to P_{x_2y}. Then there cannot exist a path $P = p_1, \ldots, p_k$ in $G \setminus \Sigma$ such that p_1 is of type p w.r.t. Σ with a neighbor in $(P_{x_1y} \cup P_{x_2y}) \setminus y$, p_k is of type p w.r.t. Σ with a neighbor in $P_{x_3y} \setminus y$, and no node of $P \setminus \{p_1, p_k\}$ has a neighbor in $\Sigma \setminus y$.*

Proof: Assume such a path P exists. Let $j \in \{1, 2\}$ such that $N(p_1) \cap \Sigma \subseteq P_{x_jy}$. By Theorem 3.2, G does not contain a proper wheel. If no node of $P \setminus \{p_1, p_k\}$ is adjacent to y, then by Lemma 5.6, P is a crosspath, contradicting the assumption that all crosspaths of Σ are from y_1 to P_{x_2y}. So a node of $P \setminus \{p_1, p_k\}$, say p_i, is adjacent to y. Let H be the hole of $(\Sigma \setminus y) \cup P$ that contains $P \cup \{x_j, x_3\}$. Suppose that y has at least three neighbors in H. Then since (H, y) cannot be a proper wheel, it must be a bug. Let $j' \in \{1, 2\} \setminus j$. Then (H, y) and $y_{j'}$ contradict either Theorem 5.2 or Lemma 6.7. So y has at most two neighbors in H. Suppose y has two neighbors in H. Since $H \cup y$ cannot induce a $3\text{PC}(\cdot, \cdot)$, these two neighbors are adjacent. In particular, H does not contain y_j nor y_3. But then $H \cup P_{x_{j'}y}$ induces a $3\text{PC}(x_1x_2x_3, \triangle)$.

Therefore y has exactly one neighbor in H. In particular, p_k has a neighbor in $P_{x_3y} \setminus \{y, y_3\}$ and p_k is not adjacent to y. Let H' be the hole induced by $P_{x_{j'}y} \cup P_{x_3y}$. If p_k is of type p1 w.r.t. Σ, then $H' \cup \{p_1, \dots, p_k\}$ induces a $3PC(y, \cdot)$. Suppose p_k is of type pb w.r.t. Σ. Then (H', p_k) is a bug. By Lemma 4.1, p_{k-1} is of type p1 w.r.t. (H', p_k), contradicting Lemma 6.7. Therefore p_k is of type p2 w.r.t. Σ. By symmetry, p_1 is also of type p2 w.r.t. Σ and it is not adjacent to y.

Let Σ' be the $3PC(x_1x_2x_3, y)$ contained in $(\Sigma \setminus y_3) \cup \{p_i, \dots, p_k\}$. Then $p_1, \dots, p_{i-1}$ is a p_i-crosspath of Σ'. So Σ has a y_1-crosspath $Q = q_1, \dots, q_l$. If no node of Q is adjacent to or coincident with a node of P, then Q is a y_1-crosspath of Σ', contradicting Lemma 4.6. So a node of Q is adjacent to or coincident with a node of P. Note that p_k and q_1 are distinct nodes. Let P' be a chordless path from q_1 to p_k in $P \cup Q$. If P' does not contain p_i nor q_l, then P' is a y_1-crosspath of Σ that ends in P_{x_3y}, contradicting our assumption. So P' contains p_i or q_l.

Suppose P' does not contain q_l. Then it contains p_i. If P' does not contain p_1, then path $P' \setminus \{p_i, \dots p_k\}$ and Σ' contradict either Lemma 4.1 (if this path consist of a single node) or Lemma 5.6 (otherwise). So P' contains p_1, *i.e.*, p_1 has a neighbor in Q, and it does not belong to Q (since if p_1 belongs to Q, then $p_1 = q_l$, and this cannot be since we are assuming that P' does not contain q_l). Let H' be the hole contained in $((P_{x_1y} \cup P_{x_2y}) \setminus y) \cup Q$ that contains $Q \cup \{x_1, x_2\}$. Then (H', p_1) is a bug. By Lemma 4.1, p_2 is of type p1, p2 or b w.r.t. (H', p_1). Since p_2 is adjacent to p_1, bug (H', p_1) and p_2 contradict either Theorem 5.2 (if p_2 is of type b w.r.t. (H', p_1)) or Lemma 6.7 (otherwise).

So P' contains q_l. Then no node of $Q \setminus q_l$ has a neighbor in P, and q_l does have a neighbor in P. Let $p_{l'}$ be the neighbor of q_l in P with highest index. If $l' > i$ then $q_l, p_{l'}, \dots, p_k$ contradicts Lemma 5.6 applied to Σ. Suppose that $l' = i$. Let H' be the hole induced by $\Sigma' \setminus (P_{x_1y} \setminus y)$. Then (H', q_l) is a bug. If $l > 1$ then (H', q_l) and q_{l-1} contradict Lemma 4.1 or Lemma 6.7. So $l = 1$. But then (H', q_l) and y_1 contradict Lemma 4.1. Therefore $l' < i$. If $l' = 1$ then p_1 and q_l are distinct nodes. If $j = 1$ then $P_{x_1y} \cup P_{x_2y} \cup \{p_1, q_l\}$ induces a $3PC(\triangle, \triangle)$. So $j = 2$. Recall that p_1, p_k and q_l cannot be adjacent to y. Hence $P_{x_1y} \cup (P_{x_3y} \setminus y_3) \cup P \cup Q$ contains a $3PC(y_1, p_i)$. So $l' > 1$. But then $q_l, p_{l'}, \dots, p_{i-1}$ is a p_i-crosspath of Σ' and Q is a y_1-crosspath of Σ', contradicting Lemma 4.6. $\square$

Lemma 8.4 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let Σ be a $3PC(x_1x_2x_3, y)$ of G such that if G has a $3PC(\triangle, \cdot)$ with a crosspath, then Σ has a y_1-crosspath and all crosspaths of Σ are from y_1 to P_{x_2y}. Then there cannot exist a path $P = p_1, \dots, p_k$ in $G \setminus \Sigma$ such that p_1 is of type p w.r.t. Σ with a neighbor in $(P_{x_1y} \cup P_{x_2y}) \setminus y$, p_k is of type t3 w.r.t. Σ such that all attachments of p_k to Σ end in P_{x_3y}, and no node of $P \setminus \{p_1, p_k\}$ has a neighbor in $\Sigma \setminus y$.*

Proof: Assume such a path P exists. By Theorem 3.2 G does not contain a proper wheel. Let $j \in \{1, 2\}$ such that $N(p_1) \cap \Sigma \subseteq P_{x_jy}$. First suppose that y does not have a neighbor in $P \setminus \{p_1, p_k\}$. Since all attachments of p_k to Σ end in P_{x_3y}, path $p_1, \dots, p_{k-1}$ cannot be an attachment of p_k to Σ. In particular, $N(p_1) \cap \Sigma = x_j$. By Lemma 6.2, p_k is attached to Σ with attachment $Q = q_1, \dots, q_l$ such that q_l is of type p1 w.r.t. Σ and no node of $Q \setminus q_l$ has a neighbor in Σ. Let Σ' be the $3PC(x_1x_2p_k, y)$ obtained by substituting p_k and Q into Σ. If no node of $P \setminus p_k$ is adjacent to or coincident with a node of Q, then path $p_1, \dots, p_{k-1}$ and Σ' contradict either Lemma 4.1 or Lemma 5.6. So a node of $P \setminus p_k$ is adjacent to or coincident with a node of Q. Then $(P \setminus p_k) \cup Q$ contains a chordless path P' from p_1 to q_l. Since p_1 and

q_l are both of type p1 w.r.t. Σ, P' contradicts Lemma 5.6 applied to Σ. Therefore, y must have a neighbor in $P \setminus \{p_1, p_k\}$.

Let p_i be the neighbor of y in $P \setminus \{p_1, p_k\}$ with highest index. Let H be the hole contained in $P \cup (P_{x_j y} \setminus y)$ that contains $P \cup x_j$. Suppose that y has at least three neighbors in H. Then since (H, y) cannot be a proper wheel, it must be a bug. Let $j' \in \{1, 2\} \setminus j$. Then (H, y) and $y_{j'}$ contradict either Theorem 5.2 or Lemma 6.7. So y has at most two neighbors in H. Suppose y has two neighbors in H. Since $H \cup y$ cannot induce a 3PC$(\cdot, \cdot)$, these two neighbors are adjacent. In particular, H does not contain y_j. But then $H \cup P_{x_{j'} y}$ induces a 3PC$(x_1 x_2 p_k, \triangle)$. Therefore, y has exactly one neighbor in H (namely p_i).

Let Σ' be the 3PC$(x_1 x_2 p_k, y)$ induced by $P_{x_1 y} \cup P_{x_2 y} \cup \{p_i, \dots, p_k\}$. Then $p_{i-1}, \dots, p_1$ is a p_i-crosspath of Σ'. So Σ has a y_1-crosspath $Q = q_1, \dots, q_l$. Note that by Theorem 5.2, $y_1 \neq x_1$. If no node of Q is adjacent to or coincident with a node of P, then Q is a y_1-crosspath of Σ', contradicting Lemma 4.6. So a node of Q is adjacent to or coincident with a node of P. Let P' be a chordless path from q_1 to p_k in $P \cup Q$. If P' does not contain p_i nor q_l, then $P' \setminus p_k$ is an attachment of p_k to Σ that ends in $P_{x_1 y} \setminus y$, contradicting our assumption. So P' contains p_i or q_l.

Suppose P' does not contain q_l. Then it contains p_i. If P' does not contain p_1, then path $P' \setminus \{p_i, \dots, p_k\}$ and Σ' contradict Lemma 4.1 (if this path consists of a single node) or Lemma 5.6 (otherwise). So P' contains p_1, *i.e.*, p_1 has a neighbor in Q (and it is not contained in Q). Let H' be the hole contained in $((P_{x_1 y} \cup P_{x_2 y}) \setminus y) \cup Q$ that contains $Q \cup \{x_1, x_2\}$. Then (H', p_1) is a bug. By Lemma 4.1, p_2 is of type p1, p2 or b w.r.t. (H', p_1). Since p_2 is adjacent to p_1, (H', p_1) and p_2 contradict either Theorem 5.2 (if p_2 is of type b w.r.t. (H', p_1)) or Lemma 6.7 (otherwise).

Therefore P' contains q_l. Then no node of $Q \setminus q_l$ has a neighbor in P and q_l does have a neighbor in P. Let $p_{l'}$ be the neighbor of q_l in P with highest index. If $l' > i$ then $p_{k-1}, \dots, p_{l'}, q_l$ is an attachment of p_k to Σ that contradicts Lemma 6.2. Suppose that $l' = i$. Let H' be the hole induced by $\Sigma' \setminus (P_{x_1 y} \setminus y)$. Then (H', q_l) is a bug. If $l > 1$ then (H', q_l) and q_{l-1} contradict Lemma 4.1 or Lemma 6.7. So $l = 1$, and hence (H', q_l) and y_1 contradict Lemma 4.1. Therefore $l' < i$. But then Q is a y_1-crosspath of Σ'. Since Σ' has a p_i-crosspath, Lemma 4.6 is contradicted. $\square$

Theorem 8.5 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. If G contains a decomposable* 3PC$(\triangle, \cdot)$*, then G has a 2-join.*

Proof: Let $\Sigma =$ 3PC$(x_1 x_2 x_3, y)$ be a decomposable 3PC$(\triangle, \cdot)$, and $\Sigma \cup u$ its extension. Let Y be the set of all type t3b nodes w.r.t. Σ that are adjacent to y. Let $H_1 = P_{x_1 y} \cup P_{x_2 y} \cup Y$, $H_2 = P_{x_3 y_3} \cup u$ and $H = H_1 \cup H_2$. Let $A_1 = \{x_1, x_2\} \cup Y$ and $B_1 = \{y\}$. If u is of type t w.r.t. Σ, then let $A_2 = \{x_3, u\}$ and $B_2 = \{y_3\}$. If u is of type p w.r.t. Σ, then let $A_2 = \{x_3\}$ and let B_2 contain y_3 and possibly u (if u is of type p2 or p3 adjacent to y). By Lemma 8.2, $H_1 | H_2$ is a 2-join of H with special sets (A_1, A_2, B_1, B_2). We now show that 2-join $H_1 | H_2$ of H extends to a 2-join of G (which proves the theorem). Assume it does not. By Theorem 7.4, there exists a blocking sequence $S = p_1, \dots, p_n$. Without loss of generality we assume that H and S are chosen so that the size of S is minimized. By Definition 7.1 and Remark 7.2, a node of S is adjacent to a node of H_2. Let p_j be the node of S with lowest index that is adjacent to a node of H_2. By Lemma 7.5, $p_1, \dots, p_j$ is a chordless path.

Claim 1: *Node p_j cannot be of type t3b w.r.t. Σ.*

Proof of Claim 1: Assume it is. Since p_j is a node of $G \setminus H$, p_j is not adjacent to y. Suppose that p_j has a neighbor in $P_{x_3y} \setminus \{x_3, y\}$. Then $\Sigma \cup p_j$ is an extension of a decomposable 3PC$(\triangle, \cdot)$. Let $H' = \Sigma \cup Y \cup p_j$ and $H_2' = H' \setminus H_1$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A_1' = A_1, A_2' = \{x_3, p_j\}, B_1' = B_1, B_2' = \{y_3\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S.

Therefore, without loss of generality p_j has a neighbor in $P_{x_1y} \setminus \{x_1, y\}$. Let Σ' be the 3PC$(p_jx_2x_3, y)$ obtained by substituting p_j into Σ. If u is of type t3 (resp. t3b) w.r.t. Σ, then by Lemma 4.1, u is of the same type w.r.t. Σ'. Suppose that u is of type p w.r.t. Σ and that it is not of the same type w.r.t. Σ'. Then u is adjacent to p_j. Since $p_j \notin Y$, p_j is not adjacent to y, and hence u and Σ' contradict Lemma 4.1. So u is of the same type w.r.t. Σ' as it is w.r.t. Σ.

Suppose that u is of type t3 w.r.t Σ (and Σ'). Since every attachment of u to Σ ends in P_{x_3y}, it follows that every attachment of u to Σ' ends in P_{x_3y}.

By Lemma 6.6, any crosspath w.r.t. Σ' is also a crosspath w.r.t. Σ. So Σ' is decomposable with extension $\Sigma' \cup u$. Let Y' be the set of all nodes of type t3b w.r.t. Σ' that are adjacent to y. Note that by Lemma 4.1, $Y = Y'$. Let $H' = \Sigma' \cup Y \cup u$ and $H_1' = H' \setminus H_2$. H' has a 2-join $H_1'|H_2$ with special sets $A_1' = \{p_j, x_2\} \cup Y, A_2' = A_2, B_1' = B_1, B_2' = B_2$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1'|H_2$ of H', contradicting our choice of H and S. This completes the proof of Claim 1.

Claim 2: *Node p_j cannot be of type t3 w.r.t. Σ.*

Proof of Claim 2: Assume it is. By Lemma 6.2, p_j is attached to Σ and every attachment of p_j to Σ ends in a type p1 node w.r.t. Σ.

Suppose that p_j has an attachment $Q = q_1, \ldots, q_m$ to Σ such that q_m is of type p1 w.r.t Σ adjacent to a node of $H_1 \setminus y$. Without loss of generality q_m is adjacent to a node of $P_{x_1y} \setminus y$. Note that by Lemma 6.2, no node of Σ has a neighbor in $Q \setminus q_m$. Let Σ' be the 3PC$(p_jx_2x_3, y)$ obtained by substituting p_j and Q into Σ.

We now show that u is of the same type w.r.t. Σ' as it is w.r.t. Σ. If u is of type t3b w.r.t. Σ, then by Lemma 4.1, u is of type t3b w.r.t. Σ'. Suppose that u is of type p w.r.t. Σ. Suppose that u has a neighbor in p_j, Q. By Lemma 4.1, u is of type b w.r.t. Σ'. But then $N(u) \cap P_{x_3y} = y_3$, and u has a neighbor in Q, and hence the chordless path from u to q_m in $Q \cup u$ contradicts Lemma 5.6 applied to Σ. So u cannot be adjacent to a node of p_j, Q, and hence u is of type p w.r.t. Σ'. Finally suppose that u is of type t3 w.r.t. Σ. Then by Lemma 4.1, u is of type t w.r.t. Σ'. Suppose that u is of type t3b w.r.t. Σ'. Then u has a neighbor q_i in Q, and hence $q_i, \ldots, q_m$ is an attachment of u to Σ that does not end in P_{x_3y}, a contradiction. Therefore, u is of the same type w.r.t. Σ' as it is w.r.t. Σ.

Suppose that u is of type t3 w.r.t Σ (and Σ'), and that it has an attachment that ends in a type p1 node w.r.t. Σ' adjacent to a node of Q. Then clearly u has an attachment to Σ that ends in $P_{x_1y} \setminus y$, contradicting our assumption. Therefore, every attachment of u to Σ' ends in P_{x_3y}.

By Lemma 6.6, any crosspath w.r.t. Σ' is also a crosspath w.r.t. Σ. So Σ' is decomposable with extension $\Sigma' \cup u$. Let Y' be the set of all nodes of type t3b w.r.t. Σ' that are adjacent to y. Note that by Lemma 4.1, $Y = Y'$. Let $H' = \Sigma' \cup Y \cup u$ and $H_1' = H' \setminus H_2$. H' has a 2-join $H_1'|H_2$ with special sets $A_1' = \{p_j, x_2\} \cup Y, A_2' = A_2, B_1' = B_1, B_2' = B_2$. By Theorem

7.6, a proper subset of S is a blocking sequence for the 2-join $H_1'|H_2$ of H', contradicting our choice of H and S.

Therefore, p_j cannot have an attachment to Σ that ends in a type p1 node w.r.t. Σ adjacent to a node of $H_1 \setminus y$. So every attachment of p_j to Σ ends in a type p1 node w.r.t. Σ adjacent to a node of P_{x_3y}. But then $\Sigma \cup p_j$ is an extension of a decomposable 3PC($\triangle, \cdot$). Let $H' = \Sigma \cup Y \cup p_j$ and $H_2' = H' \setminus H_1$. H' has a 2-join $H_1|H_2'$ with special sets $A_1' = A_1, A_2' = \{x_3, p_j\}, B_1' = B_1, B_2' = \{y_3\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S. This completes the proof of Claim 2.

Claim 3: *Node p_j cannot be of type b w.r.t. Σ.*

Proof of Claim 3: Assume it is. Since Σ is decomposable, $N(p_j) \cap P_{x_1y} = y_1$ and p_j has two adjacent neighbors in $P_{x_2y} \setminus y$. Let H^* be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then (H^*, p_j) is a bug. Since p_j has a neighbor in H_2, it must be adjacent to u. If u is of type t w.r.t. Σ, then u is of type b w.r.t. (H^*, p_j). Since u is adjacent to p_j, it is a center-crosspath of (H^*, p_j), contradicting Theorem 5.2. So u is of type p w.r.t. Σ with a neighbor in $P_{x_3y} \setminus y$. If u is adjacent to y, then Lemma 4.1 applied to (H^*, p_j) is contradicted. So u is not adjacent to y, and hence u is of type p1 w.r.t. (H^*, p_j) adjacent to p_j, contradicting Lemma 6.7. This completes the proof of Claim 3.

Claim 4: *Node p_j does not have a neighbor in $\Sigma \setminus y$, it is adjacent to u and u is of type t3 or p w.r.t. Σ.*

Proof of Claim 4: First suppose that p_j does not have a neighbor in $\Sigma \setminus y$. Since p_j has a neighbor in H_2, it must be adjacent to u. Suppose that u is of type t3b w.r.t. Σ. Let H^* be the hole induced by $P_{x_2y} \cup P_{x_3y}$. Then (H^*, u) is a bug. By Lemma 4.1, p_j is of type p1 w.r.t. (H^*, u) adjacent to u, contradicting Lemma 6.7.

Now suppose that p_j has a neighbor in $\Sigma \setminus y$. By Lemma 4.1 and Claims 1, 2 and 3, p_j is of type p w.r.t. Σ. Suppose that the neighbors of p_j in Σ are contained in P_{x_1y}. Since p_j has a neighbor in H_2, it must be adjacent to u. If u is of type p w.r.t. Σ, then by Lemma 5.6, u, p_j must be a crosspath of Σ, contradicting the assumption that Σ is decomposable. If u is of type t3 w.r.t. Σ, then since G is diamond-free, p_j is not adjacent to x_1, and hence p_j is an attachment of u that has a neighbor in $P_{x_1y} \setminus y$, contradicting the assumption that all attachments of u to Σ end in P_{x_3y}. So u is of type t3b w.r.t. Σ. Let H^* be the hole induced by $P_{x_2y} \cup P_{x_3y}$. Then (H^*, u) is a bug. By Lemma 4.1, p_j is of type p1 w.r.t. (H^*, u), contradicting Lemma 6.7. Therefore, the neighbors of p_j in Σ cannot be contained in P_{x_1y}, and by symmetry they cannot be contained in P_{x_2y}. So p_j is of type p w.r.t. Σ and it has a neighbor in $P_{x_3y} \setminus y$.

So $\Sigma \cup p_j$ is an extension of a decomposable 3PC($\triangle, \cdot$). Let $H' = \Sigma \cup Y \cup p_j$ and $H_2' = H' \setminus H_1$. H' has a 2-join $H_1|H_2'$ with special sets $A_1' = A_1, A_2' = \{x_3\}, B_1' = B_1, B_2'$ consists of y_3 and possibly p_j (if p_j is adjacent to y). By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S. This completes the proof of Claim 4.

Claim 5: *Node p_1 is of type p w.r.t. Σ with a neighbor in $(P_{x_1y} \cup P_{x_2y}) \setminus y$.*

Proof of Claim 5: By Claims 1 and 2, p_1 cannot be of type t w.r.t. Σ. So by Lemma 8.2, p_1 is not adjacent to a node of Y. Since $H_1|H_2 \cup p_1$ is not a 2-join of $H \cup p_1$, p_1 must have a neighbor in H_1. Since p_1 is not adjacent to any node of Y and by Remark 7.2, p_1 must have a neighbor in $(P_{x_1y} \cup P_{x_2y}) \setminus y$.

Suppose that p_1 is of type b w.r.t. Σ. Since Σ is decomposable, $N(p_1) \cap P_{x_1y} = y_1$, and p_1 has two adjacent neighbors in $P_{x_2y} \setminus y$. By Claim 3, $j > 1$. Let H^* be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then (H^*, p_1) is a bug. Since $p_1, \ldots, p_j$ is a chordless path and $j > 1$, p_2 is adjacent to p_1. By Lemma 4.1, Lemma 6.7 and Theorem 5.2 applied to (H^*, p_1) and p_2, node p_2 must be of type t w.r.t. (H^*, p_1). So p_2 has two adjacent neighbors in $P_{x_2y} \setminus y$, and hence by Claim 4, $j > 2$. But then p_2 contradicts Lemma 7.3 (since $N(p_2) \cap H_1 \neq \varnothing, A_1$ or B_1). Therefore p_1 cannot be of type b w.r.t. Σ, and hence by Lemma 4.1, p_1 is of type p w.r.t. Σ with a neighbor in $(P_{x_1y} \cup P_{x_2y}) \setminus y$. This completes the proof of Claim 5.

Claim 6: *$j > 1$ and nodes of $p_2, \ldots, p_{j-1}$ are either not adjacent to any node of H or are of type p1 w.r.t. Σ adjacent to y.*

Proof of Claim 6: By Claims 4 and 5, $j > 1$. Let $i \in \{2, \ldots, j-1\}$. By definition of p_j, $N(p_i) \cap H_2 = \varnothing$. The result now follows from Lemma 4.1 and Lemma 7.3. This completes the proof of Claim 6.

By Claims 4, 5 and 6, path $p_1, \ldots, p_j, u$ contradicts Lemma 8.3 or Lemma 8.4. □

Proof of Lemma 1.4: Assume G contains a $\Sigma = 3PC(x_1x_2x_3, y)$, but does not contain a $3PC(\triangle, \cdot)$ with a crosspath. Assume also that G does not have a clique cutset, a bisimplicial cutset nor a 2-join. By Theorem 3.2 and Theorem 5.2 G does not contain a wheel. In particular, Σ is long. Assume $G \neq \Sigma$. So $G \setminus \Sigma$ contains a node that has a neighbor in Σ. By Lemma 4.1 any such node is of type p1, p2 or t3 w.r.t. Σ.

First suppose that there exists $u \in G \setminus \Sigma$ that is either of type p1 w.r.t. Σ that is not adjacent to y, or of type p2 w.r.t. Σ. Then Σ is decomposable with extension $\Sigma \cup u$, contradicting Theorem 8.5. Therefore, nodes of $G \setminus \Sigma$ that have a neighbor in Σ are either of type p1 w.r.t. Σ adjacent to y, or of type t3 w.r.t. Σ.

Next suppose that there exists $u \in G \setminus \Sigma$ that is of type t3 w.r.t. Σ. By Lemma 6.2, every attachment of u to Σ ends in a type p1 node w.r.t. Σ. Since all type p1 nodes w.r.t. Σ are adjacent to y, every attachment of u to Σ ends in P_{x_3y}, and hence Σ is decomposable with extension $\Sigma \cup u$, contradicting Theorem 8.5.

Therefore, nodes of $G \setminus \Sigma$ that have a neighbor in Σ are all of type p1 w.r.t. Σ adjacent to y. Let u be any such node. Then u and $\Sigma \setminus y$ are contained in different connected components of $G \setminus y$, *i.e.*, $\{y\}$ is a clique cutset of G, contradicting our assumption. □

9 Connected triangles

In this section we decompose certain connected triangles.

Definition 9.1 *A* connected triangles $T(a_1a_2c, b_1b_2d, x, y)$ *consists of a* $3PC(a_1a_2c, x)$, *with node $y \in P_{a_2x}$ adjacent to node x, together with a y-crosspath P with endnode b_2 adjacent to $b_1, d \in P_{cx}$, where d lies on the cb_1-subpath of P_{cx}. Note that $c = d$ is allowed in this* *definition. All other nodes must be distinct. When $c = d$, we say that the connected triangles are* degenerate*.*

So if $T(a_1a_2c, b_1b_2d, x, y)$ is a connected triangles, then the graph obtained from T by removing the edge xy is a 3PC(a_1a_2c, b_1b_2d) or a 4-wheel with center $c = d$. $T(a_1a_2c, b_1b_2d, x, y)$ is a connected triangles if and only if $T(b_1b_2d, a_1a_2c, x, y)$ is a connected triangles. For $\{z, z'\} = \{x, y\}$ and triangle $\triangle_T = a_1a_2c$ or b_1b_2d, $T(a_1a_2c, b_1b_2d, x, y)$ contains a 3PC$(\triangle_T, z)$ with a z'-crosspath.

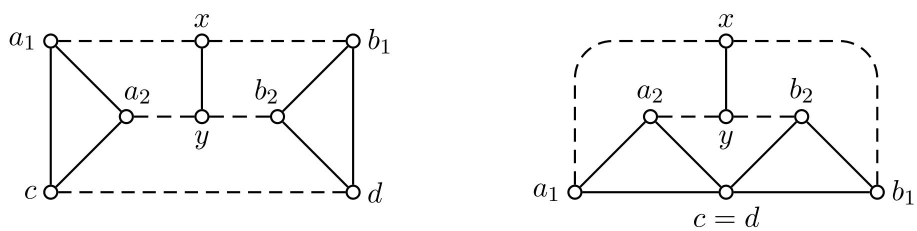


Figure 19: A connected triangles $T(a_1a_2c, b_1b_2d, x, y)$.

Definition 9.2 *Let $T(a_1a_2c, b_1b_2d, x, y)$ be connected triangles. Note that T is a nontrivial basic graph with special nodes x and y. Let P_{cd} be the cd-path of T that does not contain any node of $\{a_1, a_2, b_1, b_2, x, y\}$. Similarly define $P_{a_1x}, P_{a_2y}, P_{b_1x}, P_{b_2y}$. The path P_{cd} is the* internal segment *of T and paths $P_{a_1x}, P_{a_2y}, P_{b_1x}, P_{b_2y}$ are the* leaf segments *of T.*

Lemma 9.3 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let $T(\triangle, \triangle, x, y)$ be a connected triangles of G. If a node $u \in G \setminus T$ has a neighbor in T, then one of the following holds.*

(i) For some segment P of T, $\varnothing \neq N(u) \cap T \subseteq P$, and u is of type p w.r.t. some 3PC$(\triangle, \cdot)$ *contained in T.*

(ii) For some big clique K of T, $N(u) \cap T = K$.

(iii) For some big clique K of T and for some segment P of T that contains a node of K, $K \subseteq N(u) \cap T \subseteq K \cup P$, and $|N(u) \cap (T \setminus K)| = 1$.

(iv) $N(u) \cap T = \{x, y\}$.

(v) For some $z \in \{x, y\}$ and for some segment P of T that does not contain z, $N(u) \cap T = \{z, u_1, u_2\}$, where u_1u_2 is an edge of $P \setminus \{x, y\}$.

Proof: By Theorem 3.2, G does not contain a proper wheel. Let $T = T(a_1a_2c, b_1b_2d, x, y)$. Let Σ_x be the 3PC(a_1a_2c, x) contained in T and Σ'_x the 3PC(b_1b_2d, x) contained in T. We may assume without loss of generality that u has a neighbor in Σ_x. Then by Lemma 4.1, u is of type p, b, or t w.r.t. Σ_x.

Suppose that u is of type t3b w.r.t. Σ_x. We first show that u cannot have a neighbor in $P_{b_1x} \setminus x$. Assume it does. Then by Lemma 4.1, u is of type pb w.r.t. Σ'_x, and hence u does not have a neighbor in P_{b_2y}. But then $(T \setminus (P_{a_1x} \setminus a_1)) \cup u$ contains either a 3PC(ua_2c, b_1b_2d) or an even wheel with center $c = d$. Therefore, u does not have a neighbor in $P_{b_1x} \setminus x$. If u has a neighbor in $P_{a_1x} \setminus a_1$ or $P_{cd} \setminus c$, then by Lemma 4.1, u is of type pb w.r.t. Σ'_x, and hence u does not have a neighbor in $P_{b_2y} \setminus y$, *i.e.*, u satisfies (iii). So assume u has a neighbor in $P_{a_2y} \setminus a_2$. Then by Lemma 4.1 applied to u and Σ'_x, u cannot have a neighbor in $P_{b_2y} \setminus y$, and hence (iii) holds. So by symmetry we may now assume that u is not of type t3b w.r.t. neither Σ_x nor Σ'_x.

Next suppose that u is of type t3 w.r.t. Σ_x. By Lemma 4.1 applied to u and Σ'_x, u cannot have a neighbor in $P_{b_2y} \setminus y$. Hence u satisfies (ii). So by symmetry we may now assume that u is not of type t3 w.r.t. neither Σ_x nor Σ'_x.

Suppose that u is of type b w.r.t. Σ_x. Let u_1 and u_2 be the two adjacent neighbors of u in Σ_x and let u' be the third neighbor of u in Σ_x. Since by our assumption u cannot be of type t3b w.r.t. Σ'_x, u_1 and u_2 are contained in a segment of T. First suppose that $u' = y$. So u must be of type b w.r.t. Σ'_x. In particular, u does not have a neighbor in $P_{b_2y} \setminus y$, *i.e.,* (v) holds. Next suppose that $u' \in P_{a_1x} \setminus x$. If u has a neighbor in $P_{b_2y} \setminus y$, then u must be of type b w.r.t. Σ'_x. But then $u_1, u_2 \in P_{a_2y}$, and hence $(T \setminus (P_{a_1x} \cup P_{b_1x})) \cup u$ induces an even wheel with center u. So u does not have a neighbor in $P_{b_2y} \setminus y$, *i.e.,* $N(u) \cap T = \{u', u_1, u_2\}$. If u_1, u_2 are contained in P_{a_2y} or P_{cd}, then $(T \setminus P_{b_1x}) \cup u$ contains a 3PC(a_1a_2c, uu_1u_2) (if u is not adjacent to a_2 nor c) or an even wheel with center a_2 (if u is adjacent to a_2) or an even wheel with center c (if u is adjacent to c). So u_1, u_2 are contained in P_{b_1x}. But then $(T \setminus x) \cup u$ contains a 3PC(a_1a_2c, b_1b_2d) or an even wheel with center $c = d$. Finally suppose that $u' \in P_{b_1x} \setminus x$. If u has a neighbor in $P_{b_2y} \setminus y$, then u must be of type b w.r.t. Σ'_x. But then $u_1, u_2 \in P_{a_2y}$, and hence $(T \setminus (P_{a_1x} \cup P_{b_1x})) \cup u$ induces an even wheel with center u. So u does not have a neighbor in $P_{b_2y} \setminus y$, *i.e.,* $N(u) \cap T = \{u', u_1, u_2\}$. If $u_1, u_2 \in P_{a_2y}$, then $(T \setminus P_{a_1x}) \cup u$ contains a 3PC(uu_1u_2, b_1b_2d). So $u_1, u_2 \in P_{a_1x}$. But then $(T \setminus x) \cup u$ contains a 3PC(a_1a_2c, b_1b_2d) (if $c \neq d$) or an even wheel with center c (if $c = d$). So by symmetry we may now assume that u is not of type b w.r.t. neither Σ_x nor Σ'_x.

Next suppose that u is adjacent to both x and y. Assume (iv) does not hold. Then u has a neighbor $u' \in T \setminus \{x, y\}$. We may assume without loss of generality that $u' \in \Sigma_x \setminus \{x, y\}$. Then u must be of type pb w.r.t. Σ_x, *i.e.,* $u' \in P_{a_2y}$ and u has no neighbor in $(P_{a_1x} \cup P_{b_1x} \cup P_{cd}) \setminus x$. If u has a neighbor in $P_{b_2y} \setminus y$, then u is of type pb w.r.t. Σ'_x, and hence $P_{cd} \cup P_{a_2y} \cup P_{b_2y} \cup u$ induces a proper wheel with center u, a contradiction. So u does not have a neighbor in $P_{b_2y} \setminus y$. But then $(T \setminus P_{cd}) \cup u$ induces a bug with center u and a hat, contradicting Corollary 5.4. So we may assume that u is not adjacent to both x and y.

By our assumptions u is of type p w.r.t. Σ_x and Σ'_x. Since u is not adjacent to both x and y, $N(u) \cap \Sigma_x \subseteq P$, where P is a segment of T. Similarly $N(u) \cap \Sigma'_x \subseteq Q$, where Q is a segment of T. Suppose that (i) does not hold. Then $P = P_{a_2y}$, $Q = P_{b_2y}$, node u has a neighbor in $P_{b_2y} \setminus y$ and u has a neighbor in $P_{a_2y} \setminus y$. But then $(T \setminus y) \cup u$ contains a 3PC(a_1a_2c, b_1b_2d) (if $c \neq d$) or an even wheel with center c (if $c = d$). □

Theorem 9.4 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. Let $T(a_1a_2c, b_1b_2c, x, y)$ be a degenerate connected triangles. Then there exists no node $u \notin T$ such that c is the unique neighbor of u in T.*

Proof: Assume $T(a_1a_2c, b_1b_2c, x, y)$ is a degenerate connected triangles. Let Σ_x (resp. Σ'_x) be the 3PC(a_1a_2c, x) (resp. 3PC(b_1b_2c, x)) contained in T. Let U be the set of nodes $u \in G \setminus T$ such that $N(u) \cap T = c$. Assume $U \neq \varnothing$. Let S be the set of nodes comprised of a_1, a_2, c and all type (ii) and (iii) nodes w.r.t. T that are adjacent to a_1, a_2 and c. Note that since G is diamond-free, S induces a clique. Since S cannot be a clique cutset, there exists a direct connection $p_1, \dots, p_k$ from U to $T \setminus S$ in $G \setminus S$. Let $p_0 \in U$ be such that p_0p_1 is an edge, and let $P = p_0, \dots, p_k$. Note that by Lemma 9.3, if $u \in G \setminus (T \cup S)$ is adjacent to c and

$N(u) \cap T \subseteq \{a_1, a_2, c\}$, then $u \in U$. So by Lemma 9.3 and the definition of P, the following hold: a_1 and a_2 are the only nodes of T that may have a neighbor in $P \setminus \{p_0, p_k\}$, and a node of $P \setminus \{p_0, p_k\}$ may be adjacent to at most one node of $\{a_1, a_2\}$. Suppose that a node of $P \setminus \{p_0, p_k\}$ is adjacent to a node of $\{a_1, a_2\}$. Let p_i be such a node with lowest index. Then $p_0, \ldots, p_i$ is a hat of Σ_x, contradicting Lemma 5.6. Therefore, no node of $P \setminus \{p_0, p_k\}$ has a neighbor in T. By Lemma 9.3, we now consider the following cases.

Case 1: p_k is of type (i) w.r.t. T.

Then without loss of generality $N(p_k) \cap T \subseteq P_{b_1x}$. If $N(p_k) \cap T = b_1$, then P is a hat of Σ'_x, contradicting Lemma 5.6. So p_k has a neighbor in $P_{b_1x} \setminus b_1$. But then $(\Sigma'_x \setminus b_1) \cup P$ contains a 3PC(c, x).

Case 2: p_k is of type (iv) w.r.t. T.

Then $P_{a_1x} \cup P_{a_2y} \cup P \cup c$ induces a 3PC(a_1a_2c, xyp_k).

Case 3: p_k is of type (v) w.r.t. T.

Without loss of generality p_k is adjacent to y and has two adjacent neighbors in P_{a_1x}. Let H be the hole induced by $P_{a_1x} \cup P_{a_2y}$. Then (H, p_k) is a bug, and p_{k-1} is of type p1 w.r.t. (H, p_k), contradicting Lemma 6.7.

Case 4: p_k is of type (iii) w.r.t. T.

Then by definition of P, p_k is adjacent to b_1, b_2, c and without loss of generality it has a neighbor in $P_{b_1x} \setminus b_1$. Let Σ be the 3PC(cb_2p_k, x) contained in $T \cup p_k$. Then by Lemma 4.1 applied to Σ, $k > 1$, and hence $p_0, \ldots, p_{k-1}$ is a hat of Σ, contradicting Lemma 5.6.

Case 5: p_k is of type (ii) w.r.t. T.

Then by definition of P, $N(p_k) \cap T = \{b_1, b_2, c\}$. Let S' be the set of nodes comprised of b_1, b_2, c and all type (ii) and (iii) nodes w.r.t. T that are adjacent to b_1, b_2 and c. Note that since G is diamond-free, S' induces a clique. Since S' cannot be a clique cutset, there exists a direct connection $Q = q_1, \ldots, q_l$ from $P \setminus p_k$ to $T \setminus S'$ in $G \setminus S'$. So q_1 has a neighbor in $P \setminus p_k$ and q_l has a neighbor in $T \setminus S'$. By Lemma 9.3 and the definition of Q, the following hold: b_1, b_2 and c are the only nodes of T that may have a neighbor in $Q \setminus q_l$, and a node of $Q \setminus q_l$ may be adjacent to at most one node of $\{b_1, b_2, c\}$. Suppose that b_1 or b_2 has a neighbor in $Q \setminus q_l$. Then $(Q \setminus q_l) \cup (P \setminus p_k)$ contains a path P' whose one endnode is adjacent to c and no other node of T, whose other endnode is adjacent to exactly one node of $\{b_1, b_2\}$ and no other node of T, and whose intermediate nodes have no neighbors in T. But then P' is a hat of Σ'_x, contradicting Lemma 5.6. So b_1 and b_2 have no neighbors in $Q \setminus q_l$. But then $(P \setminus p_k) \cup Q$ contains a path whose one endnode is adjacent to c and no other node of T, whose other endnode is q_l (and hence is adjacent to a node of $T \setminus \{b_1, b_2, c\}$), and whose intermediate nodes have no neighbors in T. By symmetry and Cases 1, 2, 3 and 4, $N(q_l) \cap T = \{a_1, a_2, c\}$. Let R be the chordless path from p_k to q_l in $P \cup Q$. Then q_l is of type t3 w.r.t. Σ_x and $R \setminus q_l$ is an attachment of q_l to Σ_x that contradicts Lemma 6.2. $\square$

Definition 9.5 *Let $T(a_1a_2c, b_1b_2d, x, y)$ be connected triangles. A path $P = p_1, \ldots, p_k$ in $G \setminus T$ is an x-*crosspath *of T if one of the following holds:*

(i) $k = 1$ and p_1 is of type (v) w.r.t. T, it is adjacent to x and has two adjacent neighbors in P_{cd}.

(ii) $k > 1$, $N(p_1) \cap T = x$, $N(p_k) \cap T$ *consists of two adjacent nodes of* P_{cd}*, and no node of* $P \setminus \{p_1, p_k\}$ *has a neighbor in* T.

A y-crosspath *of* T *is defined analogously. A* crosspath *of* T *is either an* x*-crosspath or a* y*-crosspath of* T*. Note that if* P *is an* x*-crosspath (resp.* y*-crosspath) of* T*, then* P *is an* x*-crosspath (resp.* y*-crosspath) of any* 3PC$(\triangle, y)$ *(resp.* 3PC$(\triangle, x)$*) contained in* T.

Definition 9.6 *Connected triangles* $T(a_1a_2c, b_1b_2d, x, y)$ *are* decomposable *if they are non-degenerate and there is no crosspath w.r.t.* T*. Furthermore, there exists* $u \notin T$ *that satisfies one of the following:*

(i) $\varnothing \neq N(u) \cap V(T) \subseteq V(P_{cd})$.

(ii) $N(u) \cap V(T) = \{a_1, a_2, c, v\}$ *where* v *is a node of* $P_{cd} \setminus c$*, or* $N(u) \cap V(T) = \{b_1, b_2, d, v\}$ *where* v *is a node of* $P_{cd} \setminus d$.

The graph $H = T \cup u$ *is an* extension *of decomposable connected triangles* T.

Theorem 9.7 *Let* G *be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset nor a bisimplicial cutset. If* G *contains a decomposable connected triangles, then* G *has a 2-join.*

Proof: Let $T(a_1a_2c, b_1b_2d, x, y)$ be decomposable connected triangles of G, and let $H = T \cup u$ be its extension. Let $H_2 = P_{cd} \cup u$ and $H_1 = H \setminus H_2$. Let $A_1 = \{a_1, a_2\}, B_1 = \{b_1, b_2\}$, A_2 contains c and possibly u (if u is adjacent to a_1, a_2, c), and B_2 contains d and possibly u (if u is adjacent to b_1, b_2, d). Then $H_1|H_2$ is a 2-join of H with special sets (A_1, A_2, B_1, B_2). We now show that 2-join $H_1|H_2$ of H extends to a 2-join of G (which proves the theorem). Assume it does not. By Theorem 7.4, there exists a blocking sequence $S = p_1, \ldots, p_n$. Without loss of generality we assume that H and S are chosen so that the size of S is minimized. Let p_j be the node of S with lowest index that is adjacent to a node of H_2. Let Σ_x be the 3PC(a_1a_2c, x) contained in T and let Σ_y be the 3PC(a_1a_2c, y) contained in T.

Claim 1: *No node of* S *is of type (iii) w.r.t.* T.

Proof of Claim 1: Assume p_i is a vertex of type (iii) w.r.t. T.

First suppose that $N(p_i) \cap T = \{a_1, a_2, c, v\}$, where v is a node of $P_{cd} \setminus c$. Then $H' = T \cup p_i$ is an extension of a decomposable connected triangles. Let $H_2' = P_{cd} \cup p_i$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A_1' = A_1, A_2' = \{c, p_i\}, B_1' = B_1, B_2' = \{d\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S.

We may now assume without loss of generality that $N(p_i) \cap T = \{a_1, a_2, c, v\}$, where v is a node of $P_{a_1x} \setminus a_1$. Let $T'(p_ia_2c, b_1b_2d, x, y)$ be the connected triangles contained in $(T \setminus a_1) \cup p_i$. Suppose that T' has a crosspath $Q = q_1, \ldots, q_l$. If no node of $P_{a_1x} \setminus x$ is adjacent to or coincident with a node of Q, then Q is a crosspath of T, contradicting the assumption that T is decomposable. So a node of $P_{a_1x} \setminus x$ is adjacent to or coincident with a node of Q. Let q_t be the node of Q with highest index that has a neighbor in $P_{a_1x} \setminus x$. If $t > 1$ then $q_t, \ldots, q_l$ and Σ_y contradict Lemma 5.6. So $t = 1$. Since q_t is adjacent to a node of $P_{a_1x} \setminus x$, by Lemma 9.3, $l > 1$. But then, since q_t has a neighbor in $P_{a_1x} \setminus x$ and q_l has two adjacent neighbors in P_{cd}, Q and Σ_y contradict Lemma 5.6. Therefore, T' has no crosspath.

By Lemma 9.3, u is of the same type w.r.t. T' as it is w.r.t. T. So T' is a decomposable connected triangles with extension $H' = T' \cup u$. Let $H_1' = H' \setminus H_2$. Then $H_1'|H_2$ is a 2-join of H' with special sets $A_1' = \{p_1, a_2\}, A_2' = A_2, B_1' = B_1, B_2' = B_2$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1'|H_2$ of H', contradicting our choice of H and S. This completes the proof of Claim 1.

Claim 2: *Node p_j is either of type (ii) w.r.t. T, or it does not have a neighbor in T, it is adjacent to u and u is of type (i) w.r.t. T.*

Proof of Claim 2: First suppose that p_j is of type (i) w.r.t. T. If $N(p_j) \cap T \subseteq P_{cd}$, then $H' = T \cup p_j$ is an extension of a decomposable connected triangles. Let $H_2' = P_{cd} \cup p_j$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A_1' = A_1, A_2' = \{c\}, B_1' = B_1, B_2' = \{d\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S. So without loss of generality we may assume that $N(p_j) \cap T \subseteq P_{a_1x}$. Since p_j has a neighbor in H_2, it must be adjacent to u. If u is of type (i) w.r.t. T, then by Lemma 5.6, p_j, u must be an x-crosspath w.r.t. Σ_y. But then p_j, u is an x-crosspath w.r.t. T, a contradiction. So u is of type (iii) w.r.t. T. Let H^* be the hole induced by $P_{cd} \cup P_{a_1x} \cup P_{b_1x}$. Then (H^*, u) is a bug. By Lemma 4.1, p_j is of type b w.r.t. (H^*, u), *i.e.*, it is a center-crosspath of (H^*, u), contradicting Theorem 5.2. Therefore, p_j cannot be of type (i) w.r.t. T.

Next suppose that p_j is of type (iv) w.r.t. T. Since p_j has a neighbor in H_2, it must be adjacent to u. But then u is an attachment of p_j to Σ_x that contradicts Lemma 6.9.

Now suppose that p_j is of type (v) w.r.t. T. Since T is decomposable, without loss of generality p_j is adjacent to y and has two adjacent neighbors in P_{a_1x}. Since p_j has a neighbor in H_2, it must be adjacent to u. Let H^* be the hole induced by $P_{a_1x} \cup P_{a_2y}$. Then (H^*, p_j) is a bug. By Lemma 6.7, u cannot be of type p1 w.r.t. (H^*, p_j), and hence u has a neighbor in H^*. But then u is adjacent to a_1, a_2, c, and hence u is a center-crosspath of (H^*, p_j), contradicting Theorem 5.2.

Therefore, by Lemma 9.3, if p_j has a neighbor in T, then it is of type (ii) w.r.t. T. Now assume that p_j has no neighbor in T. Then p_j is adjacent to u. Suppose u is of type (iii) w.r.t. T. Let H^* be the hole induced by $P_{cd} \cup P_{a_1x} \cup P_{b_1x}$. Then (H^*, u) is a bug, and hence p_j and (H^*, u) contradict Lemma 6.7. This completes the proof of Claim 2.

By Lemma 7.5, $p_1, \ldots, p_j$ is a chordless path. By Lemma 7.3, Lemma 9.3, Claim 1 and definition of p_j, for $1 < i < j$, $N(p_i) \cap T = \varnothing$.

Claim 3: *Node p_1 is of type (i) or (iv) w.r.t. T and $N(p_1) \cap T \subseteq H_1$.*

Proof of Claim 3: By definition of a blocking sequence, $H_1|H_2 \cup p_1$ is not a 2-join of $H \cup p_1$. So by Remark 7.2, p_1 has a neighbor in H_1 and p_1 is not of type (ii) w.r.t. T. Suppose that p_1 is of type (v) w.r.t. T. Since T is decomposable, $N(p_1) \cap T \subseteq H_1$. Without loss of generality $N(p_1) \cap T = \{y, r, s\}$, where r and s are two adjacent nodes of P_{a_1x}. Let H^* be the hole induced by $P_{a_1x} \cup P_{a_2y}$. Then (H^*, p_1) is a bug. By Lemma 6.7, p_2 cannot be of type p1 w.r.t. (H^*, p_1). So p_2 has a neighbor in H^*, and hence $j = 2$. By Claim 2, p_2 is of type (ii) w.r.t. T adjacent to a_1, a_2, c (since p_2 has a neighbor in H^*). But then p_2 is a center-crosspath of (H^*, p_1), contradicting Theorem 5.2. Therefore, p_1 cannot be of type (v) w.r.t. T. So by Claim 1 and Lemma 9.3, p_1 is of type (i) or (iv) w.r.t. T and $N(p_1) \cap T \subseteq H_1$. This completes the proof of Claim 3.

Claim 4: *If $N(p_j) \cap T = \{a_1, a_2, c\}$, then the following hold:*

(i) There exists a chordless path $Q = q_1, \dots, q_l$ in $G \setminus T$ such that q_1 is adjacent to p_j, $N(q_l) \cap T = r$, where r is a node of $P_{cd} \setminus c$, and no node of $Q \setminus q_l$ has a neighbor in T.

(ii) There does not exists a chordless path $Q = q_1, \dots, q_l$ in $G \setminus T$ such that q_1 is adjacent to p_j, $N(q_l) \cap T = r$, where r is a node of $H_1 \setminus \{a_1, a_2\}$, and no node of $Q \setminus q_l$ has a neighbor in T.

Proof of Claim 4: Suppose that $N(p_j) \cap T = \{a_1, a_2, c\}$. Let K be the set of nodes that consists of a_1, a_2, c and all type (ii) and (iii) nodes w.r.t. T that are adjacent to a_1, a_2 and c. Since G is diamond-free, K induces a clique. Since $K \setminus p_j$ cannot be a clique cutset separating p_j from T, $G \setminus (K \setminus p_j)$ contains a direct connection $Q = q_1, \dots, q_l$ from p_j to T. So q_1 is adjacent to p_j, q_l has a neighbor in $T \setminus \{a_1, a_2, c\}$, and no node of $Q \setminus q_l$ has a neighbor in $T \setminus \{a_1, a_2, c\}$. Without loss of generality q_l has a neighbor in $\Sigma_x \setminus \{a_1, a_2, c\}$. Then p_j is of type t3 w.r.t. Σ_x and Q is an attachment of p_j to Σ_x. By Lemma 6.2, no node of $Q \setminus q_l$ has a neighbor in T and q_l is of type p1 w.r.t. Σ_x. By symmetry, if q_l has a neighbor in $\Sigma_y \setminus \{a_1, a_2, c\}$, then q_l is of type p1 w.r.t. Σ_y. Therefore by Lemma 9.3, $N(q_l) \cap T = r$, where r is a node of $T \setminus \{a_1, a_2, c\}$. If $r \in P_{cd} \setminus c$ then (i) holds. We now show that r cannot be contained in $H_1 \setminus \{a_1, a_2\}$, proving (i) and (ii). Suppose $r \in H_1 \setminus \{a_1, a_2\}$. If $r \in P_{b_2y} \setminus y$ then $(T \setminus y) \cup Q \cup p_j$ contains a 3PC$(a_1 p_j c, b_1 b_2 d)$. So $r \notin P_{b_2y} \setminus y$, and by symmetry $r \notin P_{b_1x} \setminus x$. So without loss of generality $r \in P_{a_1x} \setminus a_1$. Let $T'(p_j a_2 c, b_1 b_2 d, x, y)$ be the connected triangles contained in $(T \setminus a_1) \cup Q \cup p_j$. By Lemma 9.3, u is of the same type w.r.t. T' as it is w.r.t. T. We now show that T' cannot have a crosspath.

Suppose $R = r_1, \dots, r_t$ is a y-crosspath of T'. Since T is decomposable, R cannot be a crosspath of T, and hence a node of P_{a_1x} is adjacent to or coincident with a node of R. Let r_i be the node of R with highest index adjacent to a node of P_{a_1x}. Note that x does not have a neighbor in R, so r_i has a neighbor in $P_{a_1x} \setminus x$. By Lemma 9.3, $i < t$. If $i > 1$ then $r_i, \dots, r_t$ and Σ_y contradict Lemma 5.6. So $i = 1$. By Lemma 9.3, r_1 is of type (v) w.r.t. T. Let H^* be the hole induced by $P_{a_1x} \cup P_{a_2y}$. Then (H^*, r_1) is a bug, and r_2 is of type p1 w.r.t. (H^*, r_1), contradicting Lemma 6.7.

Now suppose that R is an x-crosspath of T'. Then R is an x-crosspath w.r.t. $\Sigma'_y =$ 3PC$(p_j a_2 c, y)$ contained in T'. By Lemma 6.6, R is a crosspath of Σ_y, and hence it is a crosspath of T, contradicting the assumption that T is decomposable.

Therefore, T' cannot have a crosspath. Hence T' is a decomposable connected triangles with extension $H' = T' \cup u$. Let $H'_1 = H' \setminus H_2$. Then $H'_1 | H_2$ is a 2-join of H' with special sets $A'_1 = \{p_j, a_2\}, A'_2 = A_2, B'_1 = B_1, B'_2 = B_2$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H'_1 | H_2$ of H', contradicting our choice of H and S. This completes the proof of Claim 4.

By Claim 2 we now consider the following two cases.

Case 1: $N(p_j) \cap T = \varnothing$, p_j is adjacent to u and u is of type (i) w.r.t. T.

Note that $p_1, \dots, p_j, u$ is a chordless path whose intermediate nodes have no neighbors in T. By Lemma 5.6 applied to $p_1, \dots, p_j, u$ and Σ_x, p_1 cannot be of type (iv) w.r.t. T. So by Claim 3, p_1 is of type (i) w.r.t. T, and without loss of generality $N(p_1) \cap T \subseteq P_{a_2y}$. By Lemma 5.6, $p_1, \dots, p_j, u$ is a y-crosspath w.r.t. Σ_x, and hence it is a y-crosspath of T, contradicting the assumption that T is decomposable.

Case 2: Node p_j is of type (ii) w.r.t. T.

Without loss of generality $N(p_j) \cap T = \{a_1, a_2, c\}$. By Claim 3 and Lemma 6.2 applied to Σ_x or Σ_y, p_j and $p_{j-1}, \dots, p_1$, $N(p_1) \cap T = r$, where r is a node of H_1. By Claim 4 (ii), $r \in \{a_1, a_2\}$. Without loss of generality $r = a_1$. By Claim 4 (i), there exists a chordless path $Q = q_1, \dots, q_l$ in $G \setminus T$ such that q_1 is adjacent to p_j, $N(q_l) \cap T = r'$, where r' is a node of $P_{cd} \setminus c$, and no node of $Q \setminus q_l$ has a neighbor in T. Let Σ'_y be the 3PC$(a_1 a_2 p_j, y)$ contained in $(T \setminus c) \cup Q \cup p_j$. Let p_i be the node of $p_1, \dots, p_{j-1}$ with highest index that has a neighbor in $Q \cup p_j$. By Lemma 4.1, $i > 1$. Then $p_1, \dots, p_i$ and Σ'_y contradict Lemma 5.6. □

10 Basic graphs

In this section we analyze properties of nontrivial basic graphs, and prove Lemma 1.5 (thus completing the proof of Theorem 1.2).

Lemma 10.1 ([7]) *Let K be a big clique of a nontrivial basic graph R with special nodes x and y, and let u, v be two distinct nodes of K. Then R contains a hole H, that contains nodes u, v, x, y and no other node of K.*

Lemma 10.2 ([7]) *Every leaf (resp. internal) segment of a nontrivial basic graph R with special nodes x and y is the leaf (resp. internal) segment of a connected triangles $T(\triangle, \triangle, x, y)$ contained in R.*

Lemma 10.3 ([7]) *For any pair of segments P and Q of a nontrivial basic graph R with special nodes x and y, R contains a $\Sigma =$ 3PC$(\triangle, z)$, where $z \in \{x, y\}$, that contains $P \cup Q \cup \{x, y\}$ such that P and Q belong to distinct paths of Σ. Furthermore, R contains a z'-crosspath w.r.t. Σ, where $z' \in \{x, y\} \setminus \{z\}$.*

In particular, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T.

Definition 10.4 *A graph R contained in G is a* maximum nontrivial basic graph of G, *if it is nontrivial basic and out of all nontrivial basic graphs in G, R has the largest number of segments, and out of all nontrivial basic graphs of G that have the same number of segments as R, R has the largest number of nodes.*

Lemma 10.5 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset, a bisimplicial cutset, nor a 2-join. Let R be a maximum nontrivial basic graph of G, with special nodes x and y.*

(1) If P is a leaf segment of R containing x, then R contains a $\Sigma =$ 3PC$(\triangle, x)$ in which P is one of the paths and y is contained in one of the other two paths. Furthermore, R contains a y-crosspath w.r.t. Σ and all crosspaths of Σ in G are y-crosspaths that do not end in P.

(2) If P is an internal segment of R, then R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P is the internal segment of T and in G there is no crosspath w.r.t. T.

Proof: Let P be a leaf segment of R containing x. By Lemma 10.2, R contains a connected triangles $T(\triangle, \triangle, x, y)$ with P being a leaf segment of T. So T contains a $\Sigma = 3\text{PC}(\triangle, x)$ in which P is one of the paths and y is contained in one of the other two paths. Also T contains a y-crosspath w.r.t. Σ. By Lemma 4.6, all crosspaths of Σ are y-crosspaths. Suppose there exists a y-crosspath $Y = y_1, \dots, y_m$ such that y_m has neighbors r and s in P. Note that since Y is a crosspath of Σ, $r, s \in P \setminus x$. In fact, no node of Y is adjacent to x and no node of $Y \setminus y_m$ has a neighbor in P. Since P is a segment of R, $y_m \notin R$. If no node of Y is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$, then $R' = R \cup Y$ is a nontrivial basic graph. (Note that in this case, $R' \setminus \{x, y\}$ is a line graph of a tree in which Y is a leaf segment and it is easy to check that all conditions for R' to be nontrivial basic are satisfied). Since this would contradict the maximality of R, we may assume that some node of Y is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$. Let y_j be a node of Y with highest index that is adjacent to a node, say u, of $R \setminus \{r, s, y\}$. Node u belongs to some segment Q ($\neq P$) of R. By Lemma 10.3, R contains a connected triangles $T'(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T'. Since P is a leaf segment of R that contains x, T' contains a $\Sigma' = 3\text{PC}(\triangle, x)$ that contains y, and is such that P and Q belong to different paths of Σ'. Furthermore, R contains a y-crosspath w.r.t. Σ'. Let P' (resp. Q') be the path of Σ' that contains P (resp. Q).

Suppose that $j = m$. Then by Lemma 4.1, y_m is of type b w.r.t. Σ', and hence y_m is a u-crosspath of Σ'. Then by Lemma 4.6 and since Σ' has a y-crosspath, $u = y$, contradicting our choice of u. So $j < m$, *i.e.*, y_m is of type p2 w.r.t. Σ'. Note also that y_j cannot have a neighbor in P.

Suppose y_j is of type b w.r.t. Σ'. If y_j has a neighbor in P', then P' together with one other path of Σ' induces a bug (with center y_j) and $y_j, \dots, y_m$ is its center-crosspath, contradicting Theorem 5.2. So y_j does not have a neighbor in P'. But then $(\Sigma' \setminus P') \cup \{x, y_j\}$ induces a bug Σ'', with center y_j. Recall that y_m is not adjacent to x, and hence $P' \cup \{y_{j+1}, \dots, y_m\}$ contains a center-crosspath of this bug, contradicting Theorem 5.2. So y_j cannot be of type b w.r.t. Σ'.

Suppose y_j is of type t3b w.r.t. Σ'. Then $\Sigma' \cup \{y_j, \dots, y_m\}$ contains a bug (with center y_j) and a path that either contradicts Lemma 5.6 or is a center-crosspath, contradicting Theorem 5.2.

Suppose that y_j is of type t3 w.r.t. Σ'. Then $y_{j+1}, \dots, y_m$ is an attachment of y_j to Σ' that contradicts Lemma 6.2.

Therefore, by Lemma 4.1, y_j is of type p w.r.t. Σ'. Recall that y_j is not adjacent to x. Hence $y_j, \dots, y_m$ is a crossing of Σ'. By Lemma 5.6, $y_j, \dots, y_m$ is a u-crosspath of Σ'. Hence by Lemma 4.6 and since Σ' has a y-crosspath, $u = y$, contradicting our choice of u. Therefore (1) holds.

Now let P be an internal segment of R. By Lemma 10.2, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P is the internal segment of T. Suppose without loss of generality that there is a y-crosspath $Y = y_1, \dots, y_m$ w.r.t. T. Let r and s be the neighbors of y_m in P. Since P is a segment of R, $y_m \notin R$. If no node of Y is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$, then (as before) $R' = R \cup Y$ is a nontrivial basic graph, contradicting the maximality of R. So a node of Y is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$. Let y_j be the node of Y with highest index that has a neighbor, say u, in $R \setminus \{r, s, y\}$. Node u belongs to some segment Q ($\neq P$) of R. Note that no node of Y is adjacent to x, and no node of $Y \setminus y_m$ has a neighbor in P. In particular, $u \notin \{x, y\}$. By Lemma 10.3, R contains a connected triangles $T'(\triangle, \triangle, x, y)$, such that P and Q belong to different segments of T'. So

T' contains a $\Sigma' = 3\text{PC}(\triangle, z)$, where $z \in \{x, y\}$, that contains both x and y, and such that P and Q belong to different paths of Σ'. Furthermore, T' contains a z'-crosspath of Σ', where $z' \in \{x, y\} \setminus z$. If $z = x$ then a contradiction is obtained in exactly the same way as in the proof of (1). So we may assume that P and Q both belong to segments of T' that have y as an endnode. Then $z = y$.

If y_j is of type b w.r.t. Σ', then y_j is a crosspath of Σ'. Since Σ' has an x-crosspath, by Lemma 4.6, y_j must be an x-crosspath of Σ', but this contradicts the fact that y_j cannot be adjacent to x. So y_j cannot be of type b w.r.t. Σ'.

If $j = m$ then by Lemma 4.1, y_j is of type b w.r.t. Σ', a contradiction. So $j < m$, and hence by Lemma 4.1, y_m is of type p2 w.r.t. Σ'.

If y_j is of type p w.r.t. Σ', then since y_j is not adjacent to y and by Lemma 5.6, $y_j, \ldots, y_m$ is a u-crosspath of Σ'. Since Σ' has an x-crosspath and $u \neq x$, Lemma 4.6 is contradicted.

If y_j is of type t3 w.r.t. Σ', then $y_{j+1}, \ldots, y_m$ is an attachment of y_j to Σ' that contradicts Lemma 6.2. So by Lemma 4.1, y_j is of type t3b w.r.t. Σ'. But then $\Sigma' \cup \{y_j, \ldots, y_m\}$ contains a bug with center y_j and a path that either contradicts Lemma 5.6 or is a center-crosspath of this bug, contradicting Theorem 5.2. Therefore (2) holds. □

Lemma 10.6 *Let G be a (diamond, 4-hole)-free odd-signable graph that does not have a clique cutset, a bisimplicial cutset, nor a 2-join. Let R be a maximum nontrivial basic graph of G, with special nodes x and y. If u is a node of $G \setminus R$ that has a neighbor in R, then one of the following holds.*

(i) *For some segment P of R, $\varnothing \neq N(u) \cap R \subseteq P$.*

(ii) *For some big clique K of R, $N(u) \cap R = K$.*

(iii) *For some big clique K of R and for some segment P of R that contains a node of K, $K \subseteq N(u) \cap R \subseteq K \cup P$, $|N(u) \cap (R \setminus K)| = 1$ and $N(u) \cap \{x, y\} = \varnothing$.*

(iv) $N(u) \cap R = \{x, y\}$.

(v) *For some big clique K of R and for some $z \in \{x, y\}$, $N(u) \cap R = K \cup \{z\}$.*

Proof: Let u be a node of $G \setminus R$ that has a neighbor in R.

Claim 1: *If for some big clique K of R, $|N(u) \cap K| \geq 2$, then $N(u) \cap K = K$.*

Proof of Claim 1: Follows from the fact that G is diamond-free. This completes the proof of Claim 1.

Claim 2: *Let K_1 and K_2 be two distinct big cliques of R. If $|N(u) \cap K_1| \geq 2$, then $|N(u) \cap K_2| \leq 1$.*

Proof of Claim 2: Assume $|N(u) \cap K_1| \geq 2$ and $|N(u) \cap K_2| \geq 2$. Then by Claim 1, $N(u) \cap (K_1 \cup K_2) = K_1 \cup K_2$. Note that $K_1 \cap K_2 = \varnothing$, else there is a diamond in $K_1 \cup K_2 \cup u$. Let P be a segment of R that contains a node $u_1 \in K_1$. Let Q be a segment of R, distinct from P, that contains a node $u_2 \in K_2$. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to distinct segments of T. Then (by definition of a nontrivial basic graph) T contains at least two nodes of K_1, say u_1 and v_1, and at least two nodes of K_2, say u_2 and v_2. But then u is adjacent to all endnodes of two distinct edges of T that do not have a common endnode, contradicting Lemma 9.3. This completes the proof of Claim 2.

Claim 3: *If $N(u) \cap \{x, y\} = \{x, y\}$, then (iv) holds.*

Proof of Claim 3: Assume not. Then for some $v \in R \setminus \{x, y\}$, u is adjacent to x, y and v. Let P be a segment of R that contains v. By Lemma 10.2, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P is one of the segments of T. But then T and u contradict Lemma 9.3. This completes the proof of Claim 3.

Claim 4: *R cannot contain two distinct edges u_1v_1 and u_2v_2, that do not both belong to the same big clique of R, such that u is adjacent to all of $\{u_1, u_2, v_1, v_2\}$.*

Proof of Claim 4: Assume not. Since u_1v_1 and u_2v_2 are distinct edges and they do not belong to the same big clique of R, either $\{u_1, v_1, u_2, v_2\}$ induces a chordless path of length 2 or 3, or no node of $\{u_1, v_1\}$ is adjacent to a node of $\{u_2, v_2\}$. Since G is diamond-free, no node of $\{u_1, v_1\}$ is adjacent to a node of $\{u_2, v_2\}$, in particular all nodes u_1, v_1, u_2, v_2 are distinct. By Claim 3, u_1v_1 (resp. u_2v_2) belongs to either a segment of R or a big clique of R. By Claim 2, it is not possible that both u_1v_1 and u_2v_2 belong to big cliques of R. So without loss of generality u_1v_1 belongs to a segment P of R.

Suppose that u_2v_2 also belongs to P. Then by Lemma 10.2, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P is one of the segments of T. But then T and u contradict Lemma 9.3. So it is not possible that both u_2 and v_2 belong to P. Without loss of generality u_2 belongs to a segment Q of R that is distinct from P. Also without loss of generality $u_2 \notin \{x, y\}$ (since u_2v_2 belongs to either a big clique of R or a segment of R). By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. But then u is adjacent to an edge of some segment of T and has a neighbor $u_2 \notin \{x, y\}$ in another segment of T, contradicting Lemma 9.3. This completes the proof of Claim 4.

Claim 5: *If for some big clique K of R, $|N(u) \cap K| \geq 2$, then (ii), (iii) or (v) holds.*

Proof of Claim 5: Assume that K is a big clique of R such that $|N(u) \cap K| \geq 2$. By Claim 1, $N(u) \cap K = K$. If u does not have a neighbor in $R \setminus (K \cup \{x, y\})$, then (ii) or (v) holds by Claim 3. So we may assume that u has a neighbor $v \in R \setminus (K \cup \{x, y\})$. Let P be the segment of R that contains v.

Suppose that u has a neighbor in $\{x, y\}$. Then by Claim 3, without loss of generality $N(u) \cap \{x, y\} = x$. Let Q be a segment of R, distinct from P, that contains a node of K. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. Since u is adjacent to x and has neighbors in two distinct segments of $T \setminus \{x, y\}$, by Lemma 9.3, u has exactly four neighbors in T: x and the three nodes of a big clique K_1 of T. So $K_1 \subseteq K$ and $v \in K$, contradicting our assumption. Therefore, u is not adjacent to a node of $\{x, y\}$.

Assume that $|N(u) \cap (R \setminus K)| > 1$. Then u has a neighbor $w \in R \setminus (K \cup \{x, y, v\})$. First suppose that $w \in P$. Let Q be a segment of R, distinct from P, that contains a node of K. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. If a big clique of T is contained in K, then u has at least five neighbors in T, contradicting Lemma 9.3. So a big clique of T is not contained in K, and hence a segment of T contains an edge of K. Since u is adjacent to an edge of one segment of T and has at least two more neighbors in another segment of T, Lemma 9.3 is contradicted. Hence $w \notin P$.

So w belongs to a segment Q of R that is distinct from P. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. Since u is not adjacent to x nor y, by Lemma 9.3, u has exactly four neighbors in T and it is adjacent to all three nodes of a big clique K_1 of T. By Claim 2, $K_1 \subseteq K$, contradicting the assumption that $v, w \in R \setminus K$. Therefore $|N(u) \cap (R \setminus K)| = 1$.

Suppose that (iii) does not hold. Then P does not contain a node of K. Let Q be any segment of R that contains a node of K. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. By Lemma 9.3 and since u is not adjacent to a node of $\{x, y\}$, for some big clique K_1 of T, $K_1 \subseteq K$. Let K_2 be the other big clique of T. Let P' be the segment of T that contains P. By Lemma 9.3, P' contains a node w of K_1. Since P does not contain a node of K, the vw-subpath of P' contains an edge of a big clique K_3 of R. Assume K_3 is chosen so that the subpath of P' from v to a node of K_3 is shortest possible. Let p be a node of K_3 that does not belong to P'.

First suppose that P' is the internal segment of T. Let P^* be a path from p to z, where $z \in \{x, y\}$, in $R \setminus (T \setminus \{x, y\})$ that does not contain a node of $\{x, y\} \setminus z$ (note that such a path exists by the definition of a nontrivial basic graph). Without loss of generality $z = y$. Let T' be the connected triangles $T'(\triangle, \triangle, x, y)$ contained in $T \cup P^*$ that contains K_1, P' and P^*. Then T' and u contradict Lemma 9.3.

Now assume without loss of generality that P' is a leaf segment of T that contains x. First suppose that p does not belong to a leaf segment of R that contains x. We now show that $R \setminus (T \setminus \{x, y\})$ contains a path P^* from p to y such that P^* does not contain x. If p belongs to a leaf segment of R, then such a path P^* clearly exists. So assume that p belongs to an internal segment S of R. Let K_4 be the big clique of R, distinct from K_3, that contains an endnode of S. Let s_1 be the node of K_4 that belongs to S, and let s_2 and s_3 be two nodes of $K_4 \setminus s_1$. By Lemma 10.1, R contains a hole H that contains s_2, s_3, x, y and no other node of K_4. So H is contained in $R \setminus (T \setminus \{x, y\})$, and hence the desired path P^* exists (it consists of S and the appropriate subpath of H). Let T' be the connected triangles $T'(\triangle, \triangle, x, y)$ contained in $T \cup P^*$ that contains K_2, P' and P^*. Then T' and u contradict Lemma 9.3.

Hence p belongs to a leaf segment P^* of R that contains x. Let p_x (resp. p_w) be the neighbor of p in P' that is closest to x (resp. w). Let $\Sigma' = 3\text{PC}(pp_xp_w, x)$ induced by P', P^* and the leaf segment of T that contains y and a node of K_1. By Lemma 4.1, u is of type b w.r.t. Σ', and hence vx is an edge. So by the choice of K_3, the p_xx-subpath $\bar{P}$ of P' is a leaf segment of R. But then K_3 contains two distinct nodes that belong to leaf segments of R that both contain x, contradicting the definition of a nontrivial basic graph. Therefore P must contain a node of K_1, *i.e.*, (iii) holds. This completes the proof of Claim 5.

By Claim 5, we may assume that for every big clique K of R, $|N(u) \cap K| \leq 1$. By Claim 3, we may assume without loss of generality that u is not adjacent to x. Assume that (i) does not hold. Then u has neighbors v and w in distinct segments of R, say P and Q. By Lemma 10.3, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P and Q belong to different segments of T. By Lemma 9.3, u has exactly three neighbors in T: y and endnodes of an edge of without loss of generality P. Suppose that u has a neighbor in $R \setminus (P \cup y)$. Then by the same argument, for some sector Q' of R, distinct from P, u is adjacent to endnodes of an edge of Q', contradicting Claim 4. Therefore $|N(u) \cap R| = 3$. But then $R \cup u$ is a nontrivial basic graph, contradicting the maximality of R. □

Proof of Lemma 1.5: Assume G does not have a clique cutset, a bisimplicial cutset nor a 2-join. Assume G contains a $\Sigma = 3\text{PC}(\triangle, \cdot)$ with a crosspath P. By Theorem 5.2, it is not possible that Σ is a bug and P its center-crosspath. Hence $\Sigma \cup P$ induces a nontrivial basic graph. Let R be a maximum nontrivial basic graph of G, and let R^* be its extension. Let x, y be the special nodes of R. Assume that $G \neq R^*$. Then there exists a node $u \in G \setminus R^*$ that has a neighbor in R^*.

Claim 1: *u has a neighbor in R.*

Proof of Claim 1: Assume it does not. Then u is adjacent to a node $v \in R^* \setminus R$. Without loss of generality $N(v) \cap R = K \cup x$, where K is a big clique of R. Let v_1 and v_2 be two distinct nodes of K. By Lemma 10.1, R contains a hole H that contains nodes v_1, v_2, x, y. Since G is diamond-free, (H, v) is a bug. But then (H, v) and u contradict Lemma 6.7. This completes the proof of Claim 1.

By Claim 1, u must satisfy one of (i)-(iv) of Lemma 10.6 (note that nodes that satisfy (v) of Lemma 10.6 are in $R^* \setminus R$), and hence we consider the following cases.

Case 1: There exists $u \in G \setminus R^*$ such that $\varnothing \neq N(u) \cap R \subseteq P$, where P is an internal segment of R.

By Lemma 10.5, R contains a connected triangles $T(\triangle, \triangle, x, y)$ such that P is the internal segment of T, and in G there is no crosspath w.r.t. T. By Theorem 9.4, T is nondegenerate. Hence T is decomposable with extension $T \cup u$, contradicting Theorem 9.7.

Case 2: There exists $u \in G \setminus R^*$ such that $\varnothing \neq N(u) \cap R \subseteq P$, where P is a leaf segment of R, and $N(u) \cap R \not\subseteq \{x, y\}$.

Without loss of generality P contains x. By Lemma 10.5, R contains a $\Sigma = 3\text{PC}(\triangle, x)$ in which P is one of the paths and y is contained in one of the other two paths. Also Σ has a y-crosspath and all crosspaths of Σ are y-crosspaths that do not end in P. Suppose that P is of length 1. Then Σ is a bug, u is adjacent to the center of this bug, and hence Σ and u contradict Lemma 6.7. So P is of length greater than 1. But then Σ is decomposable with extension $\Sigma \cup u$, contradicting Theorem 8.5.

Case 3: There exists $u \in G \setminus R^*$ such that for some big clique K of R and for some segment P of R that contains a node of K, $K \subseteq N(u) \cap R \subseteq K \cup P$, $|N(u) \cap (R \setminus K)| = 1$ and $N(u) \cap \{x, y\} = \varnothing$.

First suppose that P is an internal segment of R. Then by Lemma 10.5, R contains a connected triangles $T(a_1a_2c, b_1b_2d, x, y)$ such that P is the internal segment of T, and in G there is no crosspath w.r.t. T. Without loss of generality $\{a_1, a_2, c\} \subseteq K$. Since u has a neighbor in $P \setminus c$, T is nondegenerate. Hence T is decomposable with extension $T \cup u$, contradicting Theorem 9.7.

Now suppose that P is a leaf segment of R. Without loss of generality P contains x. Note that u is not adjacent to x. In particular, since G is diamond-free, P is of length greater than 1. By Lemma 10.5, R contains a $\Sigma = 3\text{PC}(\triangle, x)$ in which P is one of the paths and

y is contained in one of the other two paths of Σ. Furthermore, R contains a y-crosspath w.r.t. Σ, and all crosspath of Σ in G are y-crosspaths that do not end in P. Therefore, Σ is decomposable with extension $\Sigma \cup u$, contradicting Theorem 8.5.

Case 4: There exists $u \in G \setminus R^*$ such that $N(u) \cap R = \{x, y\}$.

By Lemma 10.5, R contains a $\Sigma = 3\text{PC}(a_1a_2a_3, x)$ such that y is contained in P_{a_3x} path of Σ. Note that by definition of nontrivial basic graph, $y \neq a_3$. By Lemma 6.9, u is attached to Σ. Let $P = p_1, \ldots, p_k$ be an attachment of u to Σ. Then by Lemma 6.9, p_k is of type p1 w.r.t. Σ, adjacent to a node of $P_{a_3x} \setminus \{x, y\}$. Note that $p_k \notin R^*$. Suppose that p_k satisfies (ii) or (iii) of Lemma 10.6. Then p_k is adjacent to all nodes of some big clique K of R, and Σ must contain at least one edge of K. But then p_k has at least two neighbors in Σ, contradicting the assumption that p_k is of type p1 w.r.t. Σ. So p_k cannot satisfy (ii) nor (iii) of Lemma 10.6. Hence p_k satisfies (i) of Lemma 10.6. But then Case 1 or 2 holds, and we are done.

Case 5: There exists $u \in G \setminus R^*$ such that $N(u) \cap R = x$ or $N(u) \cap R = y$.

Let U be the set of nodes $u \in G \setminus R^*$ such that $N(u) \cap R = x$ or $N(u) \cap R = y$. So $U \neq \varnothing$. Since $\{x, y\}$ cannot be a clique cutset separating U from R^*, there exists a chordless path $P = u_1, \ldots, u_n$ in $G \setminus \{x, y\}$ such that $u_1 \in U$ and u_n has a neighbor in $R^* \setminus \{x, y\}$. Assume P is shortest such path. We may assume that Cases 1, 2, 3 and 4 do not hold. Hence by Lemma 10.6 and Claim 1, for every $u \in G \setminus R^*$ if u has a neighbor in R^*, then either $u \in U$ or $N(u) \cap R = K$ for some big clique K of R. So no node of $P \setminus u_n$ has a neighbor in R^*, and $N(u_n) \cap R = K$ for some big clique K of R. Without loss of generality u_1 is adjacent to x.

If K does not contain an endnode of a leaf segment whose other endnode is x, then $R \cup P$ is a nontrivial basic graph, contradicting the maximality of R. So there exists a leaf segment Q of R with endnodes x and $r \in K$. If xr is an edge, then $(R \setminus r) \cup P$ is a nontrivial basic graph that contradicts the maximality of R (since $n > 1$). So xr is not an edge, *i.e.*, Q is of length greater than 1. By Lemma 10.5, R contains a $\Sigma = 3\text{PC}(\triangle, x)$ in which Q is one of the paths and y is contained in one of the other two paths of Σ. Furthermore, R contains a y-crosspath w.r.t. Σ, and all crosspath of Σ in G are y-crosspaths that do not end in Q. Note that u_n is of type t3 w.r.t. Σ. If all attachments of u_n to Σ end in Q, then Σ is decomposable with extension $\Sigma \cup u_n$, contradicting Theorem 8.5.

So we may assume that there is an attachment $P' = x_1, \ldots, x_k$ of u_n to Σ such that x_k has a neighbor in $\Sigma \setminus Q$. By Lemma 6.2, x_k is of type p1 w.r.t. Σ. Suppose that x_k is not adjacent to y. Then $x_k \notin R^*$. By our assumption that cases 1, 2, 3 and 4 do not hold, and since x_k is not adjacent to x nor y, x_k satisfies (ii) of Lemma 10.6. So for some big clique K of R, $N(x_k) \cap R = K$. Since x_k is adjacent to a node of Σ, Σ contains at least one node of K, and hence (by definition of nontrivial basic graph) it must contain at least one edge of K. But then x_k would have to have more than one neighbor in Σ, a contradiction. So $N(x_k) \cap \Sigma = y$.

Next we show that no node of P' is adjacent to or coincident with a node of $R \setminus y$. Suppose not and let x_i be the node of P' with lowest index that is adjacent to or coincident with a node of $R \setminus y$, say u.

Suppose that $x_i \in R^*$. If $i < k$ then x_i has no neighbor in $\{x, y\}$ and hence $x_i \in R \setminus y$. By the choice of x_i, $i = 1$, but this contradicts the assumption that $N(u_n) \cap R = K$ (since u_n is adjacent to x_1). So $i = k$. If $x_k \in R$, then by the choice of x_i, $k = 1$, and again the assumption that $N(u_n) \cap R = K$ is contradicted. So $x_k \in R^* \setminus R$. By the choice of x_i and Claim 1, $k = 1$. Let K' be the big clique of R such that $N(x_k) \cap R = K' \cup y$. Note that

$K \neq K'$. Let k_1 and k_2 be two distinct nodes of K'. By Lemma 10.1, let H be the hole of R that contains k_1, k_2, x and y. Then (H, x_k) is a bug. By Lemma 4.1 and since $K \neq K'$, u_n is of type p1 or b w.r.t. (H, x_k), contradicting Lemma 6.7 or Theorem 5.2. Therefore, $x_i \in G \setminus R^*$.

So x_i is adjacent to u. Note that since x_i is not adjacent to x, $u \in R \setminus \{x, y\}$. By Lemma 10.6 and since Cases 1, 2, 3 and 4 do not hold, $N(x_i) \cap R = K'$ for some big clique K' of R. Note that $K \neq K'$ and $i < k$. Node u is contained in some segment Q' ($\neq Q$) of R. By Lemma 10.3, R contains a $\Sigma'' = 3\text{PC}(a_1a_2a_3, x)$ that contains y such that Q and Q' belong to different paths of Σ''. Note that by the choice of x_i, no node of $x_1, \ldots, x_{i-1}$ is adjacent to or coincident with a node of R. If a_1, a_2, a_3 are not contained in K (resp. K'), then the path of Σ'' that contains Q (resp. Q') contains an edge of K (resp. K') and hence u_n (resp. x_i) is of type p2 w.r.t. Σ''. So u_n and x_i are of type p2 or t3 w.r.t. Σ''. If u_n and x_i are both of type p2 w.r.t. Σ'', then path $u_n, x_1, \ldots, x_i$ contradicts Lemma 5.6 applied to Σ''. So without loss of generality u_n is of type t3 w.r.t. Σ''. Then since $K \neq K'$, x_i is of type p2 w.r.t. Σ'', and hence $x_1, \ldots, x_i$ is an attachment of u_n to Σ'' that contradicts Lemma 6.2.

Therefore no node of P' is adjacent to or coincident with a node of $R \setminus y$.

Let Σ' be the $3\text{PC}(\triangle, x)$ obtained from Σ by substituting P, x for Q. Let x_j be the node of P' with highest index that is adjacent to a node of P. By Lemma 5.6, $x_j, \ldots, x_k$ is a y-crosspath w.r.t. Σ', and hence x_j is adjacent to two adjacent nodes u_t, u_{t+1} of P. Let R' be the graph obtained from R by replacing Q with paths P, x and $x_j, \ldots, x_k$. Clearly R' is a nontrivial basic graph that contradicts the maximality of R.

Case 6: There exists $u \in G \setminus R^*$ such that $N(u) \cap R = K$, for some big clique K of R.

We may assume that Cases 1, 2, 3, 4 and 5 do not hold. Hence by Lemma 10.6 and Claim 1, for every $u \in G \setminus R^*$, if u has a neighbor in R^*, then $N(u) \cap R$ is a big clique of R. Let K be a big clique of R and $u \in G \setminus R^*$ such that $N(u) \cap R = K$. Let v_1 and v_2 be two nodes of K. Let $K' = ((N(v_1) \cap N(v_2)) \cup \{v_1, v_2\}) \setminus u$. Since G is diamond-free, K' is a clique. Since K' is not a clique cutset that separates u from R^*, in $G \setminus K'$ there exists a direct connection $P = u_1, \ldots, u_n$ from u to R^*. So $P \subseteq G \setminus R^*$, no node of $P \setminus u_n$ has a neighbor in $R^* \setminus K'$ and u_n is adjacent to a node of R^*. Suppose that a node u_i in $P \setminus u_n$ has a neighbor v in R. Since u_i does not have a neighbor in $R^* \setminus K'$, $v \in K' \cap R$, and hence $v \in K$. Since $u_i \in G \setminus R^*$, $N(u_i) \cap R$ is a big clique of R. So since u_i is adjacent to $v \in K$, $N(u_i) \cap R = K$. But then $u_i \in K'$, a contradiction. Therefore, no node of $P \setminus u_n$ has a neighbor in R.

Since $u_n \in G \setminus R^*$, $N(u_n) \cap R = K''$, where K'' is a big clique of R. So since $u_n \notin K'$, $K \neq K''$ and hence u_n is adjacent to a node $v \in R \setminus K$. Let Q be the segment of R that contains v. Without loss of generality $v_1 \notin Q$. Let Q' be the segment of R that contains v_1. By Lemma 10.3, R contains a $\Sigma = 3\text{PC}(\triangle, \cdot)$ such that Q and Q' belong to different paths of Σ. Then u and u_n are of type p2 or t3 w.r.t. Σ.

If u and u_n are both of type p2 w.r.t. Σ, then path $u, u_1, \ldots, u_n$ contradicts Lemma 5.6. So without loss of generality u is of type t3 w.r.t. Σ. Since $K \neq K''$, u_n is of type p2 w.r.t. Σ. But then $u_1, \ldots, u_n$ is an attachment of u to Σ that contradicts Lemma 6.2. $\square$

11 Proof of Theorem 1.11

Recall that a vertex u of a graph G is a *simplicial extreme* of G if either u is of degree 2 or $N(u)$ induces a clique. We say that a graph G satisfies *property* $*$ if the following holds:

(i) G is a clique, or

(ii) G contains two nonadjacent simplicial extremes.

Let $\mathcal{C}$ be the class of graphs that are (even-hole, diamond)-free. We want to show that for every $G \in \mathcal{C}$, G satisfies property $*$. Assume that this statement does not hold, and let G^* be a minimum counterexample, *i.e.*, $G^* \in \mathcal{C}$, G^* does not satisfy property $*$, and for every $G \in \mathcal{C}$ such that $|V(G)| < |V(G^*)|$, property $*$ holds for G.

Since $G^* \in \mathcal{C}$, by Theorem 1.2, it must be either basic or it has a clique cutset, a bisimplicial cutset or a 2-join. In the following lemmas (Lemmas 11.1, 11.2 11.4 and 11.7) we show that none of these options can actually happen, which proves Theorem 1.11.

Lemma 11.1 *G^* cannot be a basic graph.*

Proof: Suppose G^* is basic. Then clearly G^* cannot be a clique, a hole nor a long 3PC($\triangle, \cdot$), and hence G^* is an extended nontrivial basic graph. So G^* consists of a nontrivial basic graph R with special nodes x and y, such that for all $u \in G^* \setminus R$, for some big clique K of R and for some $z \in \{x, y\}$, $N(u) \cap R = K \cup z$.

Claim 1: *R contains at least two big cliques K_1 and K_2 such that, for $i = 1, 2$, K_i contains two distinct nodes that both belong to leaf segments of R.*

Proof of Claim 1: Let P be a chordless path in $L = R \setminus \{x, y\}$ that contains the largest number of nodes that belong to big cliques of L. Let u and v be the endnodes of P. By the choice of P, u and v belong to leaf segments of L, say P_u and P_v. Let u' (resp. v') be the node of P_u (resp. P_v) that belongs to a big clique of L. Let K_1 (resp. K_2) be the big clique of L that u' (resp. v') belongs to. By the choice of P, since L contains at least two big cliques, $K_1 \neq K_2$. Let u'' (resp. v'') be a node of K_1 (resp. K_2) that does not belong to P. By the choice of P, u'' and v'' must both belong to leaf segments of L. Hence K_1 and K_2 are the desired two big cliques. This completes the proof of Claim 1.

By Claim 1, let K_1 and K_2 be two distinct big cliques of R such that, for $i = 1, 2$, K_i contains nodes u_i and v_i that both belong to leaf segments of R, say P_{u_i} and P_{v_i}. Since R is a nontrivial basic graph, for $i = 1, 2$, without loss of generality $x \in P_{u_i}$ and $y \in P_{v_i}$. Since $P_{u_i} \cup P_{v_i}$ cannot induce a 4-hole, at least one of P_{u_i} or P_{v_i} is of length greater than 1. Hence a node $w_1 \in (P_{u_1} \cup P_{v_1}) \setminus \{x, y, u_1, v_1\}$ is of degree 2 in R. Similarly a node $w_2 \in (P_{u_2} \cup P_{v_2}) \setminus \{x, y, u_2, v_2\}$ is of degree 2 in R. Therefore R contains two nonadjacent nodes of degree 2, and hence so does G^*, contradicting the assumption that G^* is a minimum counterexample to property $*$. □

Lemma 11.2 *Let G be an (even-hole, diamond)-free graph such that for every (even-hole, diamond)-free graph G' such that $|V(G')| < |V(G)|$, property $*$ holds for G'. If S is a clique cutset of G and $C_1, \ldots, C_k$ are the connected components of $G \setminus S$, then for every $i = 1, \ldots, k$, C_i contains a simplicial extreme of G.*
In particular, G^* does not have a clique cutset.

Proof: Assume S is a clique cutset of G, and let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$. Since $k \geq 2$, $G_i = G[C_i \cup S]$ has fewer nodes than G, and hence G_i satisfies property $*$. Hence C_i contains a simplicial extreme of G_i, say x_i. But then x_i is also a simplicial extreme of G. Since x_1 and x_2 are two nonadjacent simplicial extremes of G, it follows that G^* cannot have a clique cutset. □

Lemma 11.3 *Let G be an (even-hole, diamond)-free graph such that for every (even-hole, diamond)-free graph G' such that $|V(G')| < |V(G)|$, property $*$ holds for G'. If $S = \{x, y\}$ is a 2-node cutset of G, then either G has a clique cutset or the following hold.*

- *(i) xy is not an edge, $G \setminus S$ has exactly two connected components C_1 and C_2, and for $i = 1, 2$ every node of S has a neighbor in C_i.*
- *(ii) Either both C_1 and C_2 contain a simplicial extreme of G, or for some $i \in \{1, 2\}$, $G[V(C_i) \cup S]$ induces a path of length 2 or 3.*

Proof: Let $S = \{x, y\}$ be any 2-node cutset of G, and let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$. Assume G does not have a clique cutset. Then xy is not an edge, and for every $i = 1, \ldots, k$, every node of S has a neighbor in C_i. If $k \geq 3$, then there is a 3PC(x, y). So $k = 2$, and hence (i) holds.

We now define blocks G_1 and G_2 of decomposition of G by S. For $i = 1, 2$ let Q_i be any chordless path from x to y in $G[C_i \cup S]$ (note that such a path exists by (i)). Block G_1 consists of $G[C_1 \cup S]$ together with the marker path P_2 from x to y such that no node of $P_2 \setminus \{x, y\}$ has a neighbor in C_1. If Q_2 is of even length, then P_2 is of length 2, and otherwise P_2 is of length 3. Block G_2 is defined analogously.

Claim 1: *G_1 and G_2 are both (even-hole, diamond)-free graphs.*

Proof of Claim 1: By definition of G_1, since G is diamond-free, so is G_1. Suppose G_1 contains an even hole H. H must contain P_2, else it is an even hole of G. But then $(H \setminus P_2) \cup Q_2$ is an even hole of G, a contradiction. So G_1 is (even-hole, diamond)-free, and by symmetry so is G_2. This completes the proof of Claim 1.

Assume $G[C_2 \cup S]$ is not a chordless path of length 2 or 3. We now show that C_1 contains a simplicial extreme of G. Let $S' = \{x', y'\}$ be a 2-node cutset of G such that if C'_1 and C'_2 are the two connected components of $G \setminus S'$, then $C'_1 \subseteq C_1$ and $C_2 \subseteq C'_2$. Out of all such 2-node cutsets assume S' is chosen so that $|C'_1|$ is minimized. Since $G[C_2 \cup S]$ is not a chordless path of length 2 or 3, and $C_2 \subseteq C'_2$, it follows that $G[C'_2 \cup S']$ is not a chordless path of length 2 or 3. Let G'_1 and G'_2 be the blocks of decomposition of G by S'. Then $|V(G'_1)| < |V(G)|$ and by Claim 1, G'_1 is (even-hole, diamond)-free. Hence G'_1 satisfies property $*$. In particular G'_1 contains two nonadjacent simplicial extremes. Suppose that a node c_1 of C'_1 is a simplicial extreme of G'_1. Then c_1 is also a simplicial extreme of G, and since $C'_1 \subseteq C_1$, it follows that C_1 contains a simplicial extreme of G.

So we may assume that no node of C'_1 is a simplicial extreme of G'_1. Then without loss of generality x' is a simplicial extreme of G'_1. In particular, x' has the unique neighbor x'' in C'_1. If $|C'_1| = 1$ then x'' is a simplicial extreme of G'_1, a contradiction. So $|C'_1| \geq 2$. But then $S'' = \{x'', y'\}$ is a 2-node cutset of G, with connected components of $G \setminus S''$ being $C''_1 = C'_1 \setminus x''$ and $C''_2 = C'_2 \cup x'$. This contradicts our choice of S'.

Therefore, either $G[C_2 \cup S]$ is a chordless path of length 2 or 3, or C_1 contains a simplicial extreme of G. By symmetry it follows that either $G[C_1 \cup S]$ is a chordless path of length 2 or 3, or C_2 contains a simplicial extreme of G. So (ii) holds. □

Lemma 11.4 *Let G be an (even-hole, diamond)-free graph such that for every (even-hole, diamond)-free graph G' such that $|V(G')| < |V(G)|$, property $*$ holds for G'. Assume G does not have a clique cutset. If $V_1|V_2$ is a 2-join of G, then for some $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is not an edge, and v_1 and v_2 are both simplicial extremes of G.*
In particular, G^* does not have a 2-join.

Proof: Assume G has a 2-join $V_1|V_2$ with special sets (A_1, A_2, B_1, B_2). Assume there are no nodes $v_1 \in V_1$, $v_2 \in V_2$ such that v_1v_2 is not an edge and v_1 and v_2 are both simplicial extremes of G. For $i = 1, 2$, let Q_i be a chordless path of $G[V_i]$ with one endnode in A_i, the other in B_i, and no intermediate node in $A_i \cup B_i$. Blocks of decomposition by this 2-join, G_1 and G_2, are defined as follows. Block G_1 consists of $G[V_1]$ together with a chordless path $P_2 = a_2, \ldots, b_2$ such that a_2 is adjacent to every node of A_1, b_2 is adjacent to every node of B_1, and these are the only adjacencies between $G[V_1]$ and P_2. If Q_2 is of odd length, then P_2 is an edge, and otherwise P_2 is of length 2. Block G_2 is defined analogously.

Claim 1: *Both blocks G_1 and G_2 are (even-hole, diamond)-free, and satisfy property $*$.*

Proof of Claim 1: By definition of 2-join, since G does not contain a diamond, neither do G_1 and G_2. Suppose G_1 contains an even hole H. Since H cannot be contained in G, it must contain path P_2. But then $(H \setminus P_2) \cup Q_2$ induces an even hole of G, a contradiction. So G_1 is even-hole-free, and by symmetry so is G_2. By definition of 2-join, for $i = 1, 2$, $G[V_i]$ does not induce a chordless path and hence $|V(G_i)| < |V(G)|$, *i.e.*, G_i satisfies property $*$. This completes the proof of Claim 1.

Claim 2: *The following hold:*

(i) A_1 is either a clique or $|A_2| = 1$.

(ii) Every node of A_1 has a neighbor in $V_1 \setminus A_1$.

Analogous statements hold for other special sets.

Proof of Claim 2: Suppose A_1 is not a clique. Let x_1 and x_2 be two nonadjacent nodes of A_1. If $|A_2| > 1$ then $A_2 \cup \{x_1, x_2\}$ contains a diamond or a 4-hole. So $|A_2| = 1$, *i.e.*, (i) holds.

Let $x_1 \in A_1$ and suppose that x_1 does not have a neighbor in $V_1 \setminus A_1$. By definition of 2-join, some node of A_1 must have a neighbor in $V_1 \setminus A_1$, and hence $|A_1| \geq 2$. By (i) and symmetry, A_2 induces a clique. If $N(x_1) \cap A_1$ also induces a clique, then $(N(x_1) \cap A_1) \cup A_2$ is a clique cutset of G, contradicting our assumption. So $N(x_1) \cap A_1$ contains two nonadjacent nodes, x_1' and x_1''. But then $A_2 \cup \{x_1, x_1', x_1''\}$ contains a diamond. Therefore (ii) holds. This completes the proof of Claim 2.

Claim 3: *If $|A_1| = 1$ then $|B_1| > 1$ and $|B_2| > 1$.*

Proof of Claim 3: Assume $|A_1| = 1$. Suppose $|B_1| = 1$. Then by definition of 2-join, $A_1 \cup B_1$ is a 2-node cutset. By Lemma 11.3 and our assumption, either $G[V_1]$ or $G[V_2 \cup A_1 \cup B_1]$ must be a path of length 2 or 3, but this contradicts the definition of a 2-join. So $|B_1| > 1$. By analogous argument $|B_2| > 1$. This completes the proof of Claim 3.

Claim 4: *For $i = 1, 2$, V_i contains a simplicial extreme of G.*

Proof of Claim 4: We first show that V_1 contains a simplicial extreme of G_1. Assume not. By Claim 1, G_1 satisfies property $*$, and hence it contains two nonadjacent simplicial extremes. So P_2 must be of length 2, and a_2 and b_2 are both simplicial extremes of G_1. But then $|A_1| = |B_1| = 1$, contradicting Claim 3. Therefore V_1 contains a simplicial extreme of G_1, say x_1.

We now show that V_1 contains a simplicial extreme of G. If $x_1 \in V_1 \setminus (A_1 \cup B_1)$ then x_1 is a simplicial extreme of G and we are done. So assume without loss of generality that $x_1 \in A_1$ and that x_1 is not a simplicial extreme of G. Then $|A_2| \geq 2$. By Claim 2 (i), A_1 is a clique. By Claim 2 (ii), x_1 has a neighbor x_1' in $V_1 \setminus A_1$. So x_1 is not a simplicial vertex of G_1, *i.e.*, it is of degree 2 in G_1. In particular $|A_1| = 1$. By Claim 3, $|B_1| > 1$ and $|B_2| > 1$. By Claim 2 (i) and symmetry, $B_1 \cup B_2$ induces a clique. If $x_1' \in B_1$ then $B_2 \cup x_1'$ is a clique cutset of G separating x_1 from a node of $V_1 \setminus x_1$ (since x_1 is of degree 2 in G_1, *i.e.*, x_1' is the only neighbor of x_1 in V_1), contradicting our assumption that G has no clique cutset. So $x_1' \notin B_1$. Let $A_1' = \{x_1'\}$ and $A_2' = \{x_1\}$. Then $V_1 \setminus x_1 | V_2 \cup x_1$ is a 2-join of G with special sets (A_1', A_2', B_1, B_2). By the first paragraph $V_1 \setminus x_1$ contains a simplicial extreme y_1 of block G_1'. Clearly y_1 is also a simplicial extreme of G_1. If $y_1 \notin B_1$ then $y_1 \in V_1 \setminus (A_1 \cup B_1)$ and hence it is a simplicial extreme of G. So $y_1 \in B_1$. Since $|B_1| > 1$ and B_1 induces a clique, y_1 is a simplicial vertex of G_1, but then y_1 cannot have a neighbor in $V_1 \setminus B_1$, contradicting Claim 2. Therefore V_1 contains a simplicial extreme of G, and by symmetry so does V_2. This completes the proof of Claim 4.

By Claim 4, there exist nodes $v_1 \in V_1$ and $v_2 \in V_2$ that are both simplicial extremes of G. By our assumption v_1v_2 must be an edge, and hence without loss of generality $v_1 \in A_1$ and $v_2 \in A_2$. By Claim 2 (ii), $|A_1| = |A_2| = 1$. By Claim 3, $|B_1| > 1$ and $|B_2| > 1$. By Claim 2 (i), $B_1 \cup B_2$ induces a clique. Let v_1' be the neighbor of v_1 in V_1. If $v_1' \in B_1$ then $B_2 \cup v_1'$ is a clique cutset of G, a contradiction. So $v_1' \notin B_1$. Let $A_1' = \{v_1'\}$ and $A_2' = \{v_1\}$. Then $V_1 \setminus v_1 | V_2 \cup v_1$ is a 2-join of G with special sets (A_1', A_2', B_1, B_2). By Claim 4, $V_1 \setminus v_1$ contains a simplicial extreme y_1 of G. But then v_2 and y_1 are two nonadjacent simplicial extremes of G with $y_1 \in V_1$ and $v_2 \in V_2$, a contradiction.

Therefore for some $v_1 \in V_1$, $v_2 \in V_2$, v_1v_2 is not an edge and v_1 and v_2 are both simplicial extremes of G. Since G^* is a minimum counterexample to property $*$, it follows that G^* cannot have a 2-join. $\square$

Lemma 11.5 *Suppose S is a bisimplicial cutset of G^* with center x such that for a wheel (H, x) of G^* and a long sector S_1 of (H, x), S separates S_1 from $H \setminus S_1$. Then the following hold.*

(i) If (H, x) is a proper wheel, then S_1 is of length 3 and all intermediate nodes of S_1 are of degree 2 in G^*.

(ii) If (H, x) is a bug, then one of the two long sectors of (H, x) is of length 3 and all its intermediate nodes are of degree 2 in G^*.

Proof: Let x_1 and x_2 be the endnodes of sector S_1 of (H, x). Then $S = X_1 \cup X_2 \cup x$, where X_1 consists of x_1 and all nodes adjacent to both x and x_1, and X_2 consists of x_2 and all nodes adjacent to both x and x_2. Let C_1 be the connected component of $G \setminus S$ that contains $S_1 \setminus S$, and C_2 the connected component of $G \setminus S$ that contains $H \setminus (S_1 \cup S)$. Note that since $S_1 \cup \{x, x_1, x_2\}$ cannot induce an even hole, S_1 is of odd length greater than 1. For $i = 1, 2$

let X_1^i (resp. X_2^i) be the nodes of X_1 (resp. X_2) that have a neighbor in C_i. For $i = 1, 2$ let block $G_i = G^*[C_i \cup X_1^i \cup X_2^i \cup x]$.

Claim 1: *For $i = 1, 2$, either C_i contains a simplicial extreme of G^*, or $|X_1^i| = |X_2^i| = 1$ and the two nodes of $X_1^i \cup X_2^i$ are both of degree 2 in G_i.*

Proof of Claim 1: G_i satisfies property $*$, so G_i contains two nonadjacent simplicial extremes. If a node of C_i is a simplicial extreme of G_i, then it is also a simplicial extreme of G^*. So assume that no node of C_i is a simplicial extreme of G_i. Then for some $x_1' \in X_1^i$ and $x_2' \in X_2^i$, x_1' and x_2' are both simplicial extremes of G_i. But then x_1' and x_2' are both of degree 2 in G_i, and hence $|X_1^i| = |X_2^i| = 1$. This completes the proof of Claim 1.

Claim 2: *For some $i \in \{1, 2\}$, G_i induces a 5-hole.*

Proof of Claim 2: Suppose that G_1 does not induce a 5-hole. We now show that C_2 contains a simplicial extreme of G^*. Assume it does not. Then by Claim 1, $|X_1^2| = |X_2^2| = 1$ and the two nodes of $X_1^2 \cup X_2^2$ are both of degree 2 in G_2. Let G_2' be the graph that consists of $G^*[C_2 \cup X_1^2 \cup X_2^2 \cup \{x, x_1, x_2\}]$ and a chordless path $P_1 = x_1, a, b, x_2$ so that no node of $\{a, b\}$ has a neighbor in $G_2' \setminus \{x_1, x_2\}$. Since G_1 does not induce a 5-hole, $|V(G_2')| < |V(G^*)|$. By the construction of G_2', since G^* is diamond-free, so is G_2'. Suppose G_2' contains an even hole H'. Since H' cannot be an even hole of G^*, it must contain P_1. Since $|X_1^2| = |X_2^2| = 1$, $H' \cap S = H \cap S$. But then $(H' \setminus P_1) \cup S_1$ induces an even hole of G^*, a contradiction. Therefore, G_2' is (even-hole, diamond)-free, and hence property $*$ holds for G_2'. So G_2' contains two nonadjacent simplicial extremes. Let H' be the hole of G_2' induced by $(H \setminus S_1) \cup P_1$. Then (H', x) is a wheel, and hence no node of $X_1^2 \cup X_2^2 \cup \{x, x_1, x_2\}$ can be a simplicial extreme of G_2'. Therefore there exists $c_2 \in C_2$ that is a simplicial extreme of G_2'. But then c_2 is also a simplicial extreme of G^*, contradicting our assumption. Therefore, C_2 must contain a simplicial extreme of G^*.

Since G^* cannot contain two nonadjacent simplicial extremes, C_1 cannot contain a simplicial extreme of G^*. Then by Claim 1, $|X_1^1| = |X_2^1| = 1$, *i.e.*, $X_1^1 = \{x_1\}$ and $X_2^1 = \{x_2\}$. Now suppose that G_2 does not induce a 5-hole. If (H, x) is a bug, then by symmetry it would follow that C_1 contains a simplicial extreme of G^*, contradicting our assumption. Therefore (H, x) is a proper wheel. So (H, x) must contain at least three long sectors, and hence C_2 must contain at least 5 nodes. We now construct G_1' as follows: G_1' consists of G_1 together with a chordless path $P_2 = x_1, a, b, c, x_2$ such that the only adjacencies between $\{a, b, c\}$ and G_1 are the three edges ax_1, ax, cx_2. Note that since C_2 contains at least 5 nodes, $|V(G_1')| < |V(G^*)|$. We now show that G_1' is (even-hole, diamond)-free. Suppose G_1' contains a diamond D. Then since G^* does not contain a diamond, $D = \{a, x, x_1, u\}$, where u is adjacent to both x and x_1. But then $u \in X_1^1$, contradicting the assumption that $|X_1^1| = 1$. So G_1' does not contain a diamond. Now suppose that G_1' contains an even hole H'. Since H' is not an even hole of G^*, H' must contain P_2. Let H_2 be the path obtained from H by removing the interior nodes of S_1. Since H is an odd hole and S_1 is of odd length, H_2 must be of even length. So P_2 and H_2 have the same parity. Since $|X_1^1| = |X_2^1| = 1$, $H' \cap S = \{x_1, x_2\}$ and no node of $S \setminus \{x, x_1, x_2\}$ has a neighbor in H'. But then $(H' \setminus \{a, b, c\}) \cup H_2$ induces an even hole of G^*, a contradiction. Therefore, G_1' is (even-hole, diamond)-free.

So G_1' satisfies property $*$, and hence G_1' must contain two nonadjacent simplicial extremes. Since no node of $\{x, x_1, x_2, a\}$ is a simplicial extreme of G_1', it follows that a node $c_1 \in C_1$ is a simplicial extreme of G_1'. But then c_1 is also a simplicial extreme of G^*, a contradiction.

This completes the proof of Claim 2.

The lemma now follows from Claim 2. □

Lemma 11.6 *G^* does not contain a proper wheel.*

Proof: Suppose G^* contains a proper wheel (H, x). By Theorem 3.2, for some two distinct long sectors S_i and S_j of (H, x), there exists a bisimplicial cutset with center x that separates S_i from $H \setminus S_i$, and there exists a bisimplicial cutset with center x that separates S_j from $H \setminus S_j$. By Lemma 11.5, the interior nodes of both S_i and S_j are all of degree 2 in G^*. So G^* contains two nonadjacent simplicial extremes, a contradiction. □

Lemma 11.7 *G^* does not have a bisimplicial cutset.*

Proof: Suppose G^* does have a bisimplicial cutset S' with center x. Then for some wheel (H', x) and for some long sector S^*, S' separates S^* from $H' \setminus S^*$. By Lemma 11.6, G^* does not contain a proper wheel, and hence (H', x) is a bug. Let x_1, x_2, c be the neighbors of x in H' such that x_2c is an edge. Then $S' = X_1 \cup X_2 \cup x$, where $X_1 = N[x_1] \cap N(x)$ and $X_2 = N[x_2] \cap N(x) = N[c] \cap N(x)$. By Lemma 11.5, without loss of generality the long sector S^* of (H', x) with endnodes x_1 and c is of length 3 and its interior nodes are both of degree 2 in G^*. Let $S^* = x_1, a, b, c$.

Let C be the connected component of $G^* \setminus S'$ that contains the interior nodes of sector S_1 of (H', x) (*i.e.,* the sector with endnodes x_1 and x_2). Let X_1^C (resp. X_2^C) be the nodes of X_1 (resp. X_2) that have a neighbor in C. Let $G = G^*[C \cup X_1^C \cup X_2^C \cup x]$. Then G satisfies property $*$, and hence G must contain two nonadjacent simplicial extremes. If a node $u \in C$ is a simplicial extreme of G, then it is also a simplicial extreme of G^*. But then u and a are two nonadjacent simplicial extremes of G^*, a contradiction. So no node of C is a simplicial extreme of G. Hence $|X_1^C| = |X_2^C| = 1$ (*i.e.,* $X_1^C = \{x_1\}$ and $X_2^C = \{x_2\}$) and x_1 and x_2 are both simplicial extremes in G. In particular, c has no neighbor in C, *i.e.,* x and x_2 are the only neighbors of c in $V(G)$.

Note that $S_1 \cup \{x, x_1, x_2\}$ induces a hole of G. So far we have shown that G satisfies the following:

(1) $d_G(x_1) = d_G(x_2) = 2$.

(2) $G \setminus \{x, x_1, x_2\}$ does not contain a simplicial extreme of G.

(3) x_1, x, x_2 are contained in a hole of G.

Claim 1: *G does not contain a clique cutset.*

Proof of Claim 1: Suppose S is a clique cutset of G. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$. By (3), without loss of generality $\{x, x_1, x_2\} \subseteq C_1 \cup S$. By Lemma 11.2, C_2 contains a simplicial extreme c_2 of G. But then $c_2 \in G \setminus \{x, x_1, x_2\}$, contradicting (2). This completes the proof of Claim 1.

Claim 2: *G contains a* 3PC$(\triangle, x)$ *that contains x_1 and x_2.*

Proof of Claim 2: By (3) let H be a hole of G that contains x_1, x, x_2. We first show that $d_G(x) \geq 3$. Suppose that $d_G(x) = 2$. Let x_1' (resp. x_2') be the neighbor of x_1 (resp. x_2) in

$H \setminus x$. Note that since G is 4-hole-free, $x'_1 \neq x'_2$. Let $A_1 = \{x_1\}$, $A_2 = \{x'_1\}$, $B_1 = \{x_2\}$, $B_2 = \{x'_2\}$, $V_1 = \{x, x_1, x_2, a, b, c\}$ and $V_2 = G \setminus \{x, x_1, x_2\}$. By (2), $G \neq H$ and hence $V_1|V_2$ is a 2-join of $G' = G^*[V(G) \cup \{a, b, c\}]$. By Claim 1, G does not have a clique cutset, and hence neither does G'. By Lemma 11.4, V_2 contains a simplicial extreme of G' that contradicts (2). Hence $d_G(x) \geq 3$.

Let U be the set of nodes of $G \setminus H$ that are adjacent to x. Since $d_G(x) \geq 3$, $U \neq \varnothing$. By (1) and the fact that G^* does not contain a proper wheel nor a 3PC$(\cdot,\cdot)$, if $u \in U$, then either $N(u) \cap H = \{x\}$ or (H, u) is a bug. If some $u \in U$ is such that (H, u) is a bug, then (H, u) is the desired 3PC$(\triangle, x)$. So, we may assume that for every $u \in U$, $N(u) \cap H = \{x\}$. By Claim 1, $\{x\}$ cannot be a clique cutset of G separating U from $H \setminus x$. Let $P = p_1, \ldots, p_k$ be a shortest path of $G \setminus x$ such that $p_1 \in U$ and p_k is adjacent to a node of H. Since G^* does not contain a proper wheel nor a 3PC$(\cdot,\cdot)$, P is an appendix of H (Definition 2.1), and $H \cup P$ induces the desired 3PC$(\triangle, x)$. This completes the proof of Claim 2.

In the following claims we will use some terminology that was introduced in Section 4: in particular, the types of nodes adjacent to a 3PC$(\triangle,\cdot)$ referred to in Lemma 4.1 and right after that lemma, and the other definitions in that section.

Claim 3: *Let Σ be any* 3PC$(\triangle, x)$ *contained in G that contains x_1 and x_2. If Σ has a crossing Q in G, then Σ is a bug and Q its hat. In particular, if Σ is not a bug, then it has no crossing, and consequently no node is of type b w.r.t. Σ.*

Proof of Claim 3: Assume that $\Sigma =$ 3PC$(y_1y_2y_3, x)$ contained in G is such that path P_{y_1x} (resp. P_{y_2x}) of Σ contains x_1 (resp. x_2). Let x_3 be the neighbor of x in path P_{y_3x}. Suppose that Σ has a crossing $Q = q_1, \ldots, q_l$ in G, and assume that if Σ is a bug then Q is not its hat.

First suppose that Q is a crosspath of Σ. By (1), Q is an x_3-crosspath of Σ. Without loss of generality q_1 is adjacent to x_3. But then either $G^*[(\Sigma \setminus y_1) \cup Q \cup \{a, b, c\}]$ (if q_l has a neighbor in P_{y_1x}) or $G^*[(\Sigma \setminus y_2) \cup Q \cup \{a, b, c\}]$ (if q_l has a neighbor in P_{y_2x}) contains an even wheel with center x. So Q cannot be a crosspath.

Now suppose that Q is a hat of Σ. Then by our assumption Σ is not a bug, so by Lemma 5.3, G has a clique cutset, contradicting Claim 1. So Q cannot be a hat.

By (1) Q cannot satisfy (iv) of Lemma 4.7, and hence Q must satisfy (iii) of Lemma 4.7. Without loss of generality q_1 is of type pb w.r.t. Σ and q_l is of type p2 w.r.t. Σ. Suppose that the neighbors of q_1 in Σ are in P_{y_3x}. Then $G^*[(\Sigma \setminus x) \cup Q \cup \{a, b, c\}]$ contains a 3PC$(y_1y_2y_3, \triangle)$ (when q_l is not adjacent to y_1 nor y_2) or an even wheel with center y_1 or y_2 (otherwise). So we may assume without loss of generality that the neighbors of q_1 in Σ are in P_{y_1x}. If q_1 is adjacent to x, then $G^*[(\Sigma \setminus y_1) \cup Q \cup \{a, b, c\}]$ contains a proper wheel with center x. So q_1 is not adjacent to x. But then either $G[(\Sigma \setminus y_3) \cup Q]$ (if the neighbors of q_l in Σ are in P_{y_3x}) or $G[(\Sigma \setminus y_2) \cup Q]$ (if the neighbors of q_l in Σ are in P_{y_2x}) contains a 3PC(x, q_1). This completes the proof of Claim 3.

Claim 4: *Let Σ be any* 3PC$(\triangle, x)$ *contained in G that contains x_1 and x_2. If Σ is not a bug, then there does not exist a path $Q = q_1, \ldots, q_l$ in $G \setminus \Sigma$ such that q_1 and q_l are both of type p w.r.t. Σ, they both have neighbors in $\Sigma \setminus x$, they have neighbors in different paths of $\Sigma \setminus x$, and no node of $Q \setminus \{q_1, q_l\}$ has a neighbor in $\Sigma \setminus x$.*

Proof of Claim 4: Assume that $\Sigma =$ 3PC$(y_1y_2y_3, x)$ contained in G is such that path P_{y_1x} (resp. P_{y_2x}) of Σ contains x_1 (resp. x_2). Let x_3 be the neighbor of x in path P_{y_3x}. Assume that Σ is not a bug and path Q exists. If no node of $Q \setminus \{q_1, q_l\}$ has a neighbor in Σ, then Q is a crossing of Σ, contradicting Claim 3. Therefore, x has a neighbor in $Q \setminus \{q_1, q_l\}$.

We now show that q_1 has a neighbor in $\Sigma \setminus \{y_1, y_2, y_3, x\}$. Assume not. Then q_1 is adjacent to a node of $\{y_1, y_2, y_3\}$. By Lemma 4.1, and since Σ is not a bug, q_1 is of type p1 w.r.t. Σ adjacent to a node of $\{y_1, y_2, y_3\}$. If q_1 is adjacent to y_3, then $(Q \setminus q_l) \cup P_{y_1x} \cup P_{y_3x}$ induces a 3PC(y_3, x). So without loss of generality q_1 is adjacent to y_1, and hence $(Q \setminus q_l) \cup P_{y_1x} \cup P_{y_3x}$ induces a 3PC(y_1, x). Therefore, q_1 has a neighbor in $\Sigma \setminus \{y_1, y_2, y_3, x\}$, and by symmetry so does q_l.

If the neighbors of q_1 and q_l in Σ are contained in $P_{y_1x} \cup P_{y_2x}$, then $G^*[(P_{y_1x} \setminus y_1) \cup (P_{y_2x} \setminus y_2) \cup Q \cup \{a, b, c\}]$ contains a proper wheel with center x. So without loss of generality q_1 has a neighbor in P_{y_3x}. If q_l has a neighbor in $P_{y_1x} \setminus x$, then $G^*[(\Sigma \setminus y_1) \cup Q \cup \{a, b, c\}]$ contains a proper wheel with center x. So q_l has a neighbor in $P_{y_2x} \setminus x$. But then $G^*[(\Sigma \setminus y_2) \cup Q \cup \{a, b, c\}]$ contains a proper wheel with center x. This completes the proof of Claim 4.

We say that a $\Sigma = $ 3PC$(\triangle, x)$ contained in G is *simple* if it contains x_1 and x_2, it is not a bug, and no node is of type t3b w.r.t. Σ adjacent to x.

Claim 5: *G contains a simple* 3PC$(\triangle, x)$.

Proof of Claim 5: Let $\Sigma = $ 3PC$(y_1y_2y_3, x)$ be a 3PC$(\triangle, x)$ contained in G such that it contains x_1 and x_2, the path of Σ that contains x_1 is shortest possible, and with respect to all these conditions, the path of Σ that does not contain x_1 nor x_2 is shortest possible. Note that by Claim 2 such a Σ exists. Assume without loss of generality that path P_{y_1x} (resp. P_{y_2x}) of Σ contains x_1 (resp. x_2), and let x_3 be the neighbor of x on path P_{y_3x} of Σ. Note that by (1), $x_1 \neq y_1$ and $x_2 \neq y_2$. We now show that Σ is simple. Assume it is not. Then either $x_3 = y_3$ or there exists a type t3b node w.r.t. Σ adjacent to x. In fact, by our choice of Σ, $x_3 = y_3$, *i.e.,* Σ is a bug with center y_3.

We now show that $S = Y_1 \cup Y_2 \cup y_3$, where $Y_1 = N[x] \cap N(y_3)$ and $Y_2 = N[y_1] \cap N(y_3) = N[y_2] \cap N(y_3)$, is a bisimplicial cutset of G separating P_{y_1x} from P_{y_2x}. Assume not and let $Q = q_1, \ldots, q_l$ be a direct connection from P_{y_1x} to P_{y_2x} in $G \setminus S$. By (1), Claim 3 and Lemma 4.1, $l > 1$, q_1 is of type p w.r.t. Σ with a neighbor in $P_{y_1x} \setminus \{x, x_1, y_1\}$, q_l is of type p w.r.t. Σ with a neighbor in $P_{y_2x} \setminus \{x, x_2, y_2\}$, and no node of $Q \setminus \{q_1, q_l\}$ has a neighbor in $\Sigma \setminus \{x, y_1, y_2, y_3\}$. If x has a neighbor in Q, then $G^*[(\Sigma \setminus \{y_1, y_2, y_3\}) \cup Q \cup \{a, b, c\}]$ contains a proper wheel with center x, a contradiction. So x does not have a neighbor in Q. If no node of $\{y_1, y_2, y_3\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$, then Q is a crossing of Σ that contradicts Claim 3. So a node of $\{y_1, y_2, y_3\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$. Note that by definition of S and since there is no diamond, a node of $Q \setminus \{q_1, q_l\}$ cannot be adjacent to more than one node of $\{y_1, y_2, y_3\}$. Let q_i be the node of $Q \setminus \{q_1, q_l\}$ with lowest index that has a neighbor in $\{y_1, y_2, y_3\}$. If q_i is adjacent to y_2 or y_3, then $q_1, \ldots, q_i$ is a crossing of Σ that contradicts Claim 3. So q_i is adjacent to y_1. By analogous argument applied to the node of $Q \setminus \{q_1, q_l\}$ with highest index adjacent to a node of $\{y_1, y_2, y_3\}$, y_2 also has a neighbor in $Q \setminus \{q_1, q_l\}$. Let q_j be the node of Q with lowest index adjacent to y_2. Let u_1 (resp. u_2) be the neighbor of q_1 (resp. q_l) in P_{y_1x} (resp. P_{y_2x}) that is closest to x. Let H' be the hole induced by xu_1-subpath of P_{y_1x}, P_{y_2x} and $q_1, \ldots, q_j$. Then (H', y_1) must be a bug, *i.e.,* u_1y_1 is not an edge and and one of the following holds: (a) $N(y_1) \cap \{q_1, \ldots, q_j\} = \{q_i, q_{i+1}\}$ or (b) $i = 2$ and $N(y_1) \cap \{q_1, \ldots, q_j\} = \{q_1, q_2\}$. First suppose that (a) holds. Suppose y_3 has no neighbor in $q_{i+2}, \ldots, q_{j-1}$. Then y_3 has no neighbor in $q_1, \ldots, q_j$. Let H'' be the hole induced by

xu_1-subpath of P_{y_1x} and $\{q_1, \dots, q_j, y_2, y_3\}$. Then (H'', y_1) is an even wheel. So y_3 must have a neighbor in $q_{i+2}, \dots, q_{j-1}$. But then $\{q_{i+1}, \dots, q_j, y_1, y_2, y_3\}$ must induce a bug, *i.e.*, y_3 has a unique neighbor in $q_{i+2}, \dots, q_{j-1}$, and so it also has a unique neighbor in $q_1, \dots, q_j$. Hence u_1u_2-subpath of $P_{y_1x} \cup P_{y_2x}$ that contains x, together with Q and y_3 induces either a 3PC$(\cdot, \cdot)$ or a proper wheel, a contradiction. When (b) holds, contradiction is obtained by analogous argument. Therefore $S = Y_1 \cup Y_2 \cup y_3$ is a bisimplicial cutset that separates P_{y_1x} from P_{y_2x}.

Let C_1 be the connected component of $G \setminus S$ that contains x_1. Let Y_1^1 (resp. Y_2^1) be the nodes of Y_1 (resp. Y_2) that have a neighbor in C_1. Let $G_1 = G[C_1 \cup Y_1^1 \cup Y_2^1 \cup y_3]$. Then property $*$ holds for G_1. Note that if a node of $C_1 \setminus x_1$ is a simplicial extreme of G_1, then it is also a simplicial extreme of G, contradicting (2). So no node of $C_1 \setminus x_1$ is a simplicial extreme of G_1. Since G_1 must contain two nonadjacent simplicial extremes, a node of $(Y_1^1 \setminus x) \cup Y_2^1 \cup y_3$ must be a simplicial extreme of G_1. Since every node of $Y_1^1 \setminus x$ is adjacent to x and y_3 and it has a neighbor in C_1, it follows that no node of $Y_1^1 \setminus x$ can be a simplicial extreme of G_1. So a node of $Y_2^1 \cup y_3$ must be a simplicial extreme of G_1. Hence $|Y_2^1| = 1$, *i.e.*, $Y_2^1 = \{y_1\}$.

Next we show that $|Y_1^1| = 1$, *i.e.*, $Y_1^1 = \{x\}$. Assume not. Then there exists u that is adjacent to x and y_3 and has a neighbor in C_1. By Lemma 4.1, u is of type p2 w.r.t. Σ. Since u has a neighbor in C_1, there exists a path $Q = q_1, \dots, q_l$ such that $q_1 = u$, $Q \setminus q_1 \subseteq C_1$, q_l has a neighbor in $P_{y_1x} \setminus \{y_1, x\}$, and no intermediate node of Q has a neighbor in $P_{y_1x} \setminus \{y_1, x\}$. But then $G^*[(\Sigma \setminus y_1) \cup Q \cup \{y_3, a, b, c\}]$ contains a proper wheel with center x, a contradiction. Therefore $Y_1^1 = \{x\}$.

Note that since $\{x, x_1, y_1, y_3\}$ cannot induce a 4-hole, x_1y_1 is not an edge. We now show that $\{x_1, y_1\}$ is a cutset of G separating $P_{y_1x} \setminus \{x, x_1, y_1\}$ from $P_{y_2x} \cup y_3$. Assume not. Since $Y_1^1 = \{x\}$ and $Y_2^1 = \{y_1\}$, $\{x, y_1, y_3\}$ is a cutset of G separating $P_{y_1x} \setminus \{x, y_1\}$ from P_{y_2x}. So there exists a path $Q = q_1, \dots, q_l \subseteq C_1$ such that q_1 is adjacent to x or y_3, q_l is adjacent to a node of $P_{y_1x} \setminus \{x, x_1, y_1\}$, and no intermediate node of Q has a neighbor in $\Sigma \setminus \{x_1, y_1\}$. Actually, by (1), no intermediate node of Q has a neighbor in $\Sigma \setminus y_1$. If q_1 is adjacent to y_3, then y_1 must be of degree 2 in G_1 (recall that a node of $Y_2^1 \cup y_3$ must be a simplicial extreme of G_1), so y_1 cannot have a neighbor in Q, and hence Q is a crossing of Σ that contradicts Claim 3. So q_1 is not adjacent to y_3, and hence it is adjacent to x. If y_1 has a neighbor in $Q \setminus q_l$, then $P_{y_1x} \cup (Q \setminus q_l) \cup y_3$ contains a 3PC(y_1, x). So y_1 does not have a neighbor in $Q \setminus q_l$. By the choice of Σ, Q cannot be an appendix of the hole induced by $P_{y_1x} \cup P_{y_2x}$. In particular, by Lemma 4.1, $l > 1$, q_1 is of type p1 w.r.t. Σ and q_l is of type p1 or pb w.r.t. Σ. Note that if q_l is of type pb w.r.t. Σ, then by (1) and our choice of Σ, q_l is not adjacent to x. In both cases $P_{y_1x} \cup Q \cup y_3$ contains a 3PC$(x, \cdot)$. Therefore $\{x_1, y_1\}$ is a cutset of G that separates $P_{y_1x} \setminus \{x, x_1, y_1\}$ from $P_{y_2x} \cup y_3$. But then $\{x_1, y_1\}$ is a cutset of $G^*[G \cup \{a, b, c\}]$. By Lemma 11.3, $C_1 \setminus x_1$ contains a simplicial extreme of $G^*[G \cup \{a, b, c\}]$, and hence of G as well, contradicting (2). This completes the proof of Claim 5.

By Claim 5, let $\Sigma = $ 3PC$(y_1y_2y_3, x)$ be a simple 3PC$(\triangle, x)$ contained in G. We assume that x_1 is on path P_{y_1x} of Σ, x_2 is on path P_{y_2x} of Σ, and x_3 is the neighbor of x on path P_{y_3x} of Σ. We say that a node $u \in G \setminus \Sigma$ is a *pendant* of Σ if one of the following holds:

(i) u is of type t3 w.r.t. Σ and every attachment of u to Σ ends in a type p node w.r.t. Σ whose neighbors are contained in P_{y_3x}.

(ii) u is of type t3b w.r.t. Σ and it has a neighbor in $P_{y_3x} \setminus \{y_3, x\}$.

(iii) u is of type p w.r.t. Σ and it has a neighbor in $P_{y_3x} \setminus x$.

We say that $\Sigma \cup u$ is an *extension* of a simple 3PC($\triangle, \cdot$).

We now show that every simple $\Sigma = 3PC(y_1y_2y_3, x)$ has a pendant. Since Σ is not a bug, $x_3 \neq y_3$. By (2), the intermediate nodes of P_{y_3x} cannot be of degree 2 in G. So there exists $u \in G \setminus \Sigma$ that is adjacent to a node of $P_{y_3x} \setminus \{y_3, x\}$. By Claim 3, no node is of type b w.r.t. Σ, so by Lemma 4.1, u is of type p or t3b w.r.t. Σ, *i.e.,* it is a pendant of Σ.

Let $H = \Sigma \cup u$ be an extension of a simple $\Sigma = 3PC(y_1y_2y_3, x)$. Let $H_1 = P_{y_1x} \cup P_{y_2x}$ and $H_2 = H \setminus H_1$. Let $A_1 = \{y_1, y_2\}$ and $B_1 = \{x\}$. If u is of type t3 or t3b w.r.t. Σ, then let $A_2 = \{y_3, u\}$, and otherwise let $A_2 = \{y_3\}$. Let B_2 contain x_3 and possibly u (if u is of type p w.r.t. Σ adjacent to x). Then $H_1|H_2$ is a 2-join of H with special sets (A_1, A_2, B_1, B_2).

We now show that it is not possible that the 2-join $H_1|H_2$ of H extends to a 2-join of G. Assume it does. Then there exists a 2-join $V_1|V_2$ of G such that $P_{y_3x} \setminus x \subseteq V_2$ and $P_{y_1x} \cup P_{y_2x} \subseteq V_1$. By Lemma 11.4, V_2 contains a simplicial extreme v_2 of G. But then since $\{x, x_1, x_2\} \subseteq V_1$, $v_2 \in G \setminus \{x, x_1, x_2\}$, contradicting (2). So the 2-join $H_1|H_2$ of H does not extend to a 2-join of G. By Theorem 7.4 there exists a blocking sequence $S = p_1, \ldots, p_n$ in G for $H_1|H_2$. Without loss of generality we assume that $H = \Sigma \cup u$ and S are chosen so that the size of S is minimized. Let p_j be the node of S with lowest index that is adjacent to a node of H_2. By Lemma 7.5, $p_1, \ldots, p_j$ is a chordless path.

Claim 6: *Let v be a type t3 node w.r.t. Σ. Then v is attached to Σ. Let $Q = q_1, \ldots, q_k$ be an attachment of v to Σ. Then q_k is of type p w.r.t. Σ, with neighbors contained in say P_{y_ix}. Furthermore, no node of $\Sigma \setminus y_i$ has a neighbor in $Q \setminus q_k$, and if q_k is adjacent to x, then q_k has no neighbor in $(P_{y_1x} \cup P_{y_2x}) \setminus x$ and y_1 and y_2 have no neighbor in Q.*

Proof of Claim 6: Let Y be the set comprised of y_1, y_2, y_3 and all type t nodes w.r.t. Σ. Since G is diamond-free, Y induces a clique. By Claim 1, G does not contain a clique cutset, and hence there exists a direct connection $Q = q_1, \ldots, q_k$ from v to Σ in $G \setminus (Y \setminus \{v\})$, *i.e.,* v is attached. By definition of Q and Lemma 4.1, no node of Q has more than one neighbor in $\{y_1, y_2, y_3\}$. The only nodes of Σ that may have a neighbor in $Q \setminus q_k$ are y_1, y_2, y_3. If at least two nodes of $\{y_1, y_2, y_3\}$ have a neighbor in $Q \setminus q_k$, then a subpath of $Q \setminus q_k$ is a hat of Σ, contradicting Claim 3 (since Σ is simple). So without loss of generality y_2 and y_3 do not have neighbors in $Q \setminus q_k$. If y_1 has a neighbor in $Q \setminus q_k$, let q_i be such a neighbor with highest index.

Since Σ is simple, by Claim 3 no node is of type b w.r.t. Σ. So by Lemma 4.1 and definition of Q, q_k is of type p w.r.t. Σ and it has a neighbor in $\Sigma \setminus \{y_1, y_2, y_3\}$. Suppose that y_1 has a neighbor in $Q \setminus q_k$, and q_k has a neighbor in $\Sigma \setminus P_{y_1x}$. Then $q_i, \ldots, q_k$ is a crossing of Σ, contradicting Claim 3.

Suppose that q_k is adjacent to x and it has a neighbor in $P_{y_1x} \setminus x$. Then by (1), q_k is of type pb w.r.t. Σ. But then $G^*[(P_{y_1x} \setminus y_1) \cup P_{y_2x} \cup Q \cup \{v, a, b, c\}]$ contains a 4-wheel with center x. So q_k does not have a neighbor in $P_{y_1x} \setminus x$ and similarly it does not have a neighbor in $P_{y_2x} \setminus x$. So the neighbors of q_k in Σ are contained in P_{y_3x}. Suppose that y_1 has a neighbor in Q, and let q_t be such a neighbor with highest index. Note that $t < k$, *i.e.,* $t = i$. Then y_2 and y_3 do not have neighbors in Q and $N(q_k) \cap \Sigma = x$ (else there is a crossing that contradicts Claim 3). But then $P_{y_1x} \cup P_{y_2x} \cup \{q_i, \ldots, q_k\}$ induces a 3PC(x, y_1). Hence y_1 does not have a neighbor in Q, and similarly neither does y_2. This completes the proof of Claim 6.

Claim 7: *Node p_j cannot be of type t3b w.r.t. Σ.*

Proof of Claim 7: Assume it is. Since Σ is simple, p_j is not adjacent to x. Suppose that

p_j has a neighbor in $P_{y_3x} \setminus \{y_3, x\}$. Then $\Sigma \cup p_j$ is an extension of a simple 3PC($\triangle, x$). Let $H' = \Sigma \cup p_j$ and $H'_2 = H' \setminus H_1$. Then $H_1|H'_2$ is a 2-join of H' with special sets $A'_1 = A_1$, $A'_2 = \{y_3, p_j\}$, $B'_1 = B_1$, $B'_2 = \{x_3\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H'_2$ of H', contradicting our choice of H and S.

Therefore, without loss of generality p_j has a neighbor in $P_{y_1x} \setminus \{y_1, x\}$. Let $\Sigma' =$ 3PC($p_jy_2y_3, x$) obtained by substituting p_j into Σ. Clearly Σ' is not a bug and it contains x_1 and x_2. If a node v is of type t3b w.r.t. Σ' adjacent to x, then by Lemma 4.1, v is also of type t3b w.r.t. Σ adjacent to x. Hence, since Σ is a simple 3PC($\triangle, x$), so is Σ'. If u is of type t3 (resp. t3b) w.r.t. Σ, then by Lemma 4.1, u is of the same type w.r.t. Σ'. Since p_j is not adjacent to x, and by Lemma 4.1, if u is of type p w.r.t. Σ, then it is also of type p w.r.t. Σ'. Suppose that u is of type t3 w.r.t. Σ (and Σ'). Since every attachment of u to Σ ends in a type p node w.r.t. Σ with neighbors in P_{y_3x}, the same is true of the attachments of u to Σ'.

Therefore, $H' = \Sigma \cup u$ is an extension of a simple 3PC($\triangle, x$). Let $H'_1 = H' \setminus H_2$. Then $H'_1|H_2$ is a 2-join of H' with special sets $A'_1 = \{p_j, y_2\}$, $A'_2 = A_2$, $B'_1 = B_1$, $B'_2 = B_2$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H'_1|H_2$ of H', contradicting our choice of H and S. This completes the proof of Claim 7.

Claim 8: *Node p_j cannot be of type t3 w.r.t. Σ.*

Proof of Claim 8: Assume it is. By Claim 6, p_j is attached to Σ. Suppose that every attachment of p_j to Σ ends in a type p node w.r.t. Σ whose neighbors are contained in P_{y_3x}. Then $H' = \Sigma \cup p_j$ is an extension of a simple 3PC($\triangle, x$). Let $H'_2 = H' \setminus H_1$. Then $H_1|H'_2$ is a 2-join of H' with special sets $A'_1 = A_1$, $A'_2 = \{y_3, p_j\}$, $B'_1 = B_1$, $B'_2 = \{x_3\}$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H'_2$ of H', contradicting our choice of H and S.

So by Claim 6 we may assume without loss of generality that p_j has an attachment $Q = q_1, \ldots, q_k$ to Σ such that q_k is of type p w.r.t. Σ adjacent to a node of $P_{y_1x} \setminus \{y_1, x\}$. By Claim 6, q_k is not adjacent to x, and no node of $Q \setminus q_k$ has a neighbor in $\Sigma \setminus y_1$. Let Σ' be the 3PC($p_jy_2y_3, x$) contained in $\Sigma \cup Q \cup p_j$. Clearly Σ' is not a bug and it contains x_1 and x_2. If node v is of type t3b w.r.t. Σ' adjacent to x, then by Lemma 4.1, it is of the same type w.r.t. Σ. Hence, since Σ is simple, so is Σ'.

Since Σ' is simple, by Claim 3 no node is of type b w.r.t. Σ'. Hence, by Lemma 4.1, if u is of type p w.r.t. Σ, then it is of type p w.r.t. Σ'. Similarly, if u is of type t3b w.r.t. Σ, then it is of the same type w.r.t. Σ' (with a neighbor in $P_{y_3x} \setminus \{y_3, x\}$). So suppose that u is of type t3 w.r.t. Σ. By Lemma 4.1, u is of type t w.r.t. Σ'. Since all attachments of u to Σ end in type p node w.r.t. Σ whose neighbors are contained in P_{y_3x}, u cannot have a neighbor in Q. Hence u is of type t3 w.r.t. Σ'. If u has an attachment to Σ' that ends in a type p node whose neighbors are not contained in P_{y_3x}, then so does Σ. So every attachment of u to Σ' ends in a type p node whose neighbors are contained in P_{y_3x}. Hence, u is a pendant of Σ'.

But then $H' = \Sigma' \cup u$ is an extension of a simple 3PC($\triangle, x$). Let $H'_1 = H' \setminus H_2$. Then $H'_1|H_2$ is a 2-join of H' with special sets $A'_1 = \{y_2, p_j\}$, $A'_2 = A_2$, $B'_1 = B_1$, $B'_2 = B_2$. By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H'_1|H_2$ of H', contradicting our choice of H and S. This completes the proof of Claim 8.

Claim 9: *Node p_j does not have a neighbor in $\Sigma \setminus x$, it is adjacent to u, and u is of type t3 or p w.r.t. Σ.*

Proof of Claim 9: First suppose that p_j has a neighbor in $\Sigma \setminus x$. So by Lemma 4.1 and Claims

3, 7 and 8, p_j is of type p w.r.t. Σ. Suppose that the neighbors of p_j in Σ are contained in P_{y_1x}. Since p_j has a neighbor in H_2, it must be adjacent to u. If u is of type p w.r.t. Σ, then path u, p_j is a crossing of Σ, contradicting Claim 3. If u is of type t3 w.r.t. Σ, then since G is diamond-free, p_j is not adjacent to y_1, and hence p_j is an attachment of u that has a neighbor in $P_{y_1x} \setminus x$, contradicting the assumption that all attachment of u to Σ end in P_{y_3x}. So u is of type t3b w.r.t. Σ. Let $\Sigma' = 3\text{PC}(y_1y_2u, x)$ contained in $(\Sigma \setminus y_3) \cup u$. Clearly Σ' contains x_1 and x_2. Since u is not adjacent to x, Σ' is not a bug. If a node is of type t3b w.r.t. Σ' adjacent to x, then by Lemma 4.1, it is also of type t3b w.r.t. Σ adjacent to x, contradicting the assumption that Σ is simple. Hence Σ' is simple. But then by Lemma 4.1, p_j is of type b w.r.t. Σ', contradicting Claim 3 applied to Σ'. Therefore, the neighbors of p_j in Σ cannot be contained in P_{y_1x}, and by symmetry they cannot be contained in P_{y_2x}.

So p_j is of type p w.r.t. Σ and it has a neighbor in $P_{y_3x} \setminus x$. But then $H' = \Sigma \cup p_j$ is an extension of a simple $3\text{PC}(\triangle, x)$. Let $H_2' = H' \setminus H_1$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A_1' = A_1$, $A_2' = \{y_3\}$, $B_1' = B_1$, B_2' contains x_3 and possibly p_j (if p_j is adjacent to x). By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S.

Therefore, p_j does not have a neighbor in $\Sigma \setminus x$. Since p_j has a neighbor in H_2, it must be adjacent to u. Suppose that u is of type t3b w.r.t. Σ. Let $\Sigma' = 3\text{PC}(y_1y_2u, x)$ contained in $(\Sigma \setminus y_3) \cup u$. As above, Σ' is simple, and hence $H' = \Sigma' \cup p_j$ is an extension of a simple $3\text{PC}(\triangle, x)$. Let $H_2' = H' \setminus H_1$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A_1' = A_1$, $A_2' = \{u\}$, $B_1' = B_1$, B_2' contains x_3 and possibly p_j (if p_j is adjacent to x). By Theorem 7.6, a proper subset of S is a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H and S. Therefore, u is of type t3 or p w.r.t. Σ. This completes the proof of Claim 9.

Claim 10: *Node p_1 is of type p w.r.t. Σ with a neighbor in $(P_{y_1x} \cup P_{y_2x}) \setminus x$.*

Proof of Claim 10: By Lemma 4.1 and Claims 3, 7 and 8, p_1 is of type p w.r.t. Σ. Since $H_1|H_2 \cup p_1$ is not a 2-join of $H \cup p_1$ (by definition of a blocking sequence), p_1 must have a neighbor in H_1. By Remark 7.2, p_1 must have a neighbor in $(P_{y_1x} \cup P_{y_2x}) \setminus x$. This completes the proof of Claim 10.

Claim 11: *$j > 1$ and nodes $p_2, \dots, p_{j-1}$ are either not adjacent to any node of H or are of type p1 w.r.t. Σ adjacent to x.*

Proof of Claim 11: By Claims 9 and 10, $j > 1$. Let $i \in \{2, \dots, j-1\}$. By definition of p_j, $N(p_i) \cap H_2 = \emptyset$. The result now follows from Lemma 4.1 and Lemma 7.3. This completes the proof of Claim 11.

By Claims 9, 10 and 11, $p_1, \dots, p_j, u$ is a chordless path such that p_1 is of type p w.r.t. Σ with a neighbor in $(P_{y_1x} \cup P_{y_2x}) \setminus x$, u is of type p or t3 w.r.t. Σ, and no node of $p_2, \dots, p_j$ has a neighbor in $\Sigma \setminus x$. If u is of type p w.r.t. Σ, then path $p_1, \dots, p_j, u$ contradicts Claim 4. So u is of type t3 w.r.t. Σ.

First suppose that x has a neighbor in $p_2, \dots, p_j$, and let p_i be such a neighbor with highest index. Let Σ' be the $3\text{PC}(y_1y_2u, x)$ induced by $P_{y_1x} \cup P_{y_2x} \cup \{u, p_i, \dots, p_j\}$. Clearly Σ' contains x_1 and x_2, and it is not a bug. If $i = 2$, then by Lemma 4.1, p_1 is of type b w.r.t. Σ', contradicting Claim 3. So $i > 2$ and hence path $p_1, \dots, p_{i-1}$ contradicts Claim 4.

Therefore, no node of $p_2, \dots, p_j$ has a neighbor in Σ. Since all attachments of u to Σ end in a type p node w.r.t. Σ whose neighbors are contained in P_{y_3x}, $p_1, \dots, p_j$ cannot be an attachment of u to Σ. Hence without loss of generality $N(p_1) \cap \Sigma = y_1$.

Let $Q = q_1, \dots, q_l$ be an attachment of u to Σ. Note that by Claim 6, no node of $\Sigma \setminus y_3$ has a neighbor in $Q \setminus q_l$. Let Σ' be the 3PC(y_1y_2u, x) contained in $(\Sigma \setminus y_3) \cup Q \cup u$. Clearly Σ is not a bug and it contains x_1 and x_2. Let p_i be the node of $p_1, \dots, p_j$ with highest index adjacent to a node of $Q \cup u$. Then by Lemma 4.1, $i > 1$, and hence path $p_1, \dots, p_i$ is a crossing of Σ', contradicting Claim 3.

Therefore, G^* does not have a bisimplicial cutset. □

References

[1] L. Addario-Berry, M. Chudnovsky, F. Havet, B.Reed, and P. Seymour, *Bisimplicial vertices in even-hole-free graphs*, manuscript, 2006.

[2] L.W. Beineke, *Characterizations of derived graphs*, J. Comb. Th. 9 (1970) 129–135.

[3] C. Berge, *Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung)*, Wiss. Z. Martin-Luther Univ. Halle-Wittenberg, Math.-Natur. Reihe 10 (1961) 114–115.

[4] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Annals of Math. 164 (2006) 51–229.

[5] M. Chudnovsky and P. Seymour, *The three-in-a-tree problem*, preprint (2006), submitted.

[6] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Even and odd holes in cap-free graphs*, Journal of Graph Theory 30 (1999) 289–308.

[7] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Even-hole-free graphs, Part I: Decomposition theorem*, Journal of Graph Theory 39 (2002) 6–49.

[8] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Even-hole-free graphs, Part II: Recognition algorithm*, Journal of Graph Theory 40 (2002) 238–266.

[9] H. Everett, C.M.H. de Figueiredo, C. Linhares Sales, F. Maffray, O. Porto and B.A. Reed, *Even Pairs* in *Perfect Graphs*, J.L. Ramírez-Alfonsín and B.A. Reed eds., Wiley Interscience (2001) 67–92.

[10] P. Erdős, *Graph theory and probability*, Canad. J. Math. 11 (1959) 34–38.

[11] M. Farber, *On diameters and radii of bridged graphs*, Discrete Mathematics 73 (1989) 249–260.

[12] C.M.H. de Figueiredo and K. Vušković, *A class of β-perfect graphs*, Discrete Mathematics 216 (2000) 169–193.

[13] A. Gyárfás, *Problems from the world surrounding perfect graphs*, Zastos. Mat. 19 (1987) 413–431.

[14] J. Keijsper and M. Tewes, *Conditions for β-perfectness*, Discuss. Math. Graph Theory 22 (2002), 123–148.

[15] H. Kierstead, and J.H. Schmerl, *The chromatic number of graphs which induce neither $K_{1,3}$ nor $K_5 - e$*, Discrete Mathematics 58 (1986) 253–262.

[16] S.E. Markossian, G.S. Gasparian and B.A. Reed, *β-perfect graphs*, Journal of Combinatorial Theory B 67 (1996) 1–11.

[17] B. Randerath, and I. Schiermeyer, *Vertex coloring and forbidden subgraphs—a survey*, Graphs and Combinatorics 20 (1) (2004) 1–40.

[18] M.V.G. da Silva and K. Vušković, *Triangulated neighborhoods in even-hole-free graphs*, Discrete Mathematics 307 (2007) 1065–1073.

[19] M.V.G. da Silva and K. Vušković, *Decomposition of even-hole-free graphs by 2-joins and star cutsets*, in preparation.

[20] K. Truemper, *Alpha-balanced graphs and matrices and GF(3)-representability of matroids*, Journal of Combinatorial Theory B 32 (1982) 112–139.

[21] V.G. Vizing, *On an estimate of the chromatic class of p-graphs*, Diskret. Analiz. 3 (1964) 25–30 [in Russian].