Even-Hole-Free Graphs
Part I: Decomposition Theorem

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Abstract

We prove a decomposition theorem for even-hole-free graphs. The decompositions used are 2-joins and star, double-star and triple-star cutsets. This theorem is used in the second part of this paper to obtain a polytime recognition algorithm for even-hole-free graphs.

1 Introduction

In this paper, all graphs are simple. A cycle is *even* if it contains an even number of nodes, and is *odd* otherwise. A *hole* is a chordless cycle with at least four nodes. We say that a graph G *contains* a graph H if H is an induced subgraph of G, and a graph is H*-free* if it does not contain H. In this paper we study *even-hole-free graphs.* The main result is a structural characterization of even-hole-free graphs in terms of a decomposition theorem. It is used in Part II [5] to construct a polytime recognition algorithm for this class of graphs.

1.1 Related Results

Bienstock [1] shows that it is NP-complete to recognize whether a graph contains an even hole containing a specified node. Porto [13] gives a linear time recognition algorithm for planar even-hole-free graphs and Markossian, Gasparian and Reed [12] show how to recognize in polynomial time even-hole-free graphs that are diamond-and-cap-free. A *diamond* is a cycle of length four with a single chord. A *cap* is a cycle of length greater than four with a single chord that forms a triangle with two edges of the cycle. In [6], we decompose every cap-free graph into triangle-free graphs and hole-free graphs (triangulated graphs). This decomposition is obtained using 1-amalgams, a well-studied structure [2]. It reduces the problem of recognizing cap-free graphs that are even-hole-free to recognizing triangle-free graphs that are even-hole-free. This question is solved in [7].

In [12], Markossian, Gasparian and Reed introduce β-perfect graphs. $\beta(G) = \max\{\delta_H + 1 : H \text{ is an induced subgraph of } G\}$, where δ_H is the minimum vertex degree in H. Consider the following ordering of the vertices of a graph G: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order gives an upper bound for the chromatic number of G: $\chi(G) \leq \beta(G)$. A graph is β-*perfect* if, for every induced subgraph H of G, $\chi(H) = \beta(H)$. β-perfect graphs are a subclass of even-hole-free graphs. The complexity of their recognition remains open. Markossian, Gasparian and Reed [12] show that both G and its complement are β-perfect if and only if both G and its complement are even-hole-free. In [12], it is also shown that if G is an even-hole-free graph then $\chi(G) \geq \frac{\beta(G)}{2} + 1$. Thus, if G is an even-hole-free graph, then the greedy algorithm can be used to color G using at most $2(\chi(G) - 1)$ colors.

Another motivation for this research is indirect. Odd-hole-free graphs are interesting because of the strong perfect graph conjecture due to Berge, stating that "a graph is perfect if and only if the graph and its complement are odd-hole-free". Odd-hole-free graphs contain the class of perfect graphs and one suspects that understanding their structure will lead to insight that may help settle the strong perfect graph conjecture. So, part of the motivation for this research is to develop techniques that may then be used to study odd-hole-free graphs.

It is also worth pointing out that decompositions similar to the ones used here led to the recognition algorithm for balanced matrices [8], [4].

1.2 Notation and Background

In this paper we use standard graph theory notation (see for example [15]).

Given a node set S and a graph G, $G \setminus S$ denotes the subgraph of G obtained by removing the node set S and the edges with at least one node in S. $S \subseteq V(G)$ is a *node cutset* of a connected graph G if the graph $G \setminus S$ is disconnected. Similarly a subset S of the edges of a connected graph G is an *edge cutset* if the graph obtained from G by removing the edges of S is disconnected. Let H be an induced subgraph of G. We say that a cutset S of G *separates* H if there are nodes of H in different components of $G \setminus S$.

Where clear from context we write H to mean $V(H)$. To denote the singleton set $\{x\}$ we sometimes write x. Also we write $H \cup x$ to mean the graph induced by the nodes of H together with node x.

A *path* P is a sequence of distinct nodes $x_1, x_2, \dots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. These are called the edges of the path P. If $n > 1$ then nodes x_1 and x_n are the *endnodes* of the path. The nodes of $V(P)$ that are not endnodes are called *intermediate* nodes of P. Let x_i and x_l be two nodes of P, where $l \geq i$. The path $x_i, x_{i+1}, \dots, x_l$ is called the $x_i x_l$-subpath of P and is denoted by $P_{x_i x_l}$. We write $P = x_1, \dots, x_{i-1}, P_{x_i x_l}, x_{l+1}, \dots, x_n$ or $P = x_1, \dots, x_i, P_{x_i x_l}, x_l, \dots, x_n$. A cycle C is a sequence of nodes $x_1, x_2, \dots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, x_2, \dots, x_n$ form a path and $x_1 x_n$ is an edge. The edges of the path $x_1, \dots, x_n$ together with the edge $x_1 x_n$ are called the edges of the cycle C. The length of a path P is the number of edges in P and is denoted by $|P|$. Similarly the length of a cycle C is the number of edges in C and is denoted by $|C|$.

Given a path or a cycle Q in a graph G, any edge of G between nodes of Q that is not an edge of Q is called a *chord* of Q. Q is *chordless* if no edge of G is a chord of Q. As mentioned before a chordless cycle of length at least four is called a *hole*. It is called a k-hole if it has k edges. A hole is even if k is even and odd otherwise.

Let A, B be two disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, x_2, \dots, x_n$ *connects* A *and* B if either $n = 1$ and x_1 has neighbors in A and B or $n > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between* A *and* B if, in the subgraph induced by the node set $V(P) \cup A \cup B$, no path connecting A and B is shorter than P. The direct connection P is said to be *from* A *to* B if x_1 is adjacent to some node in A and x_n to some node in B.

For $x \in V(G)$, $N(x)$ denotes the set of nodes adjacent to x. A node $v \notin V(H)$ is *strongly adjacent* to H, if $|N(v) \cap V(H)| \geq 2$. We say that a node v is a *twin* of a node $x \in V(H)$ with respect to H, if $N(v) \cap V(H) = N(x) \cap V(H)$ and vx is an edge.

For $S \subseteq V(G)$, $N(S)$ denotes the set of nodes in $V(G) \setminus S$ that are adjacent to at least one node in S.

In figures, a solid line represents an edge and a dotted line represents a chordless path of length at least 1.

1.3 The Decomposition Theorem

The cutsets we use to decompose even-hole-free graphs are an edge cutset called 2-join and node cutsets called star, double-star and triple-star cutsets.

A *k-star* is a graph comprised of a clique C of size k and a subset of the nodes having at least one neighbor in C. Note that a k-star may have edges not incident with C. We refer to 1-star as a *star*, to 2-star as a *double-star* and to 3-star as a *triple-star*. In a connected graph G, a *k-star cutset* is a node set $S \subseteq V(G)$ that induces a k-star and whose removal disconnects G.

A connected graph G has a *2-join*, denoted by $H_1|H_2$, with special sets A, B, C, D that are nonempty and disjoint, if the nodes of G can be partitioned into sets H_1 and H_2 so that $A, C \subseteq H_1$, $B, D \subseteq H_2$, all nodes of A are adjacent to all nodes of B, all nodes of C are adjacent to all nodes of D and these are the only adjacencies between H_1 and H_2. Also, for $i = 1, 2$, $|H_i| > 2$ and if A and C (resp. B and D) are both of cardinality 1, then the graph induced by H_1 (resp. H_2) is not a chordless path.

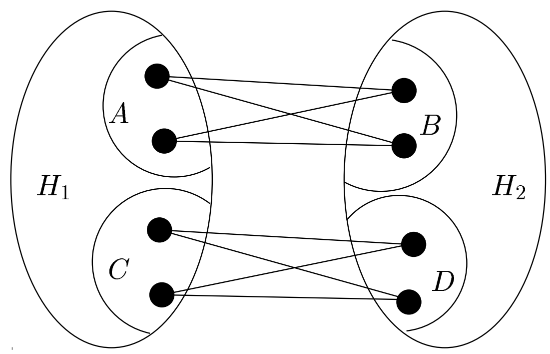


Figure 1: 2-join

Star cutsets were introduced by Chvátal [3] and 2-joins by Cornuéjols and Cunningham [10]. In [8] and [4], 2-joins, star and double-star cutsets are used for recognizing balanced 0, 1 matrices and, together with another edge cutset, the 6-join, for recognizing balanced $0, \pm 1$ matrices.

We now introduce two classes of graphs that have no 2-join and no star, double-star or triple-star cutset.

Given a triangle $\{x_1, x_2, x_3\}$ and a node y adjacent to at most one node in $\{x_1, x_2, x_3\}$, a $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y$, $P_2 = x_2, \ldots, y$ and $P_3 = x_3, \ldots, y$, having no common nodes other than y and such that the only adjacencies between the nodes of $P_1 \setminus y$, $P_2 \setminus y$ and $P_3 \setminus y$ are the edges of the triangle $\{x_1, x_2, x_3\}$. A $3PC(x_1x_2x_3, y)$ is also referred to as a $3PC(\Delta, \cdot)$.

Another class of graphs, which we call nontrivial basic graphs, can be built as follows: Let L be the line graph of a tree. Note that every edge of L belongs to exactly one maximal clique and that every node of L belongs to at most two maximal cliques. The nodes of L that belong to exactly one maximal clique are called *leaf nodes*. A clique of L is *big* if it has size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an *internal segment* if it has its endnodes in distinct big cliques (when P is of length 0, it is called an internal segment when the node of P belongs to two big cliques). The other paths P are called *leaf segments*. Note that one of the endnodes of a leaf segment is a leaf node.

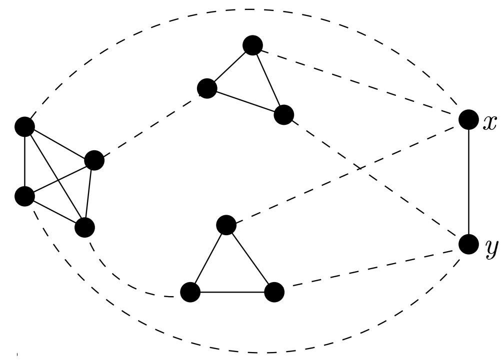


Figure 2: Nontrivial basic graph

Define now a *nontrivial basic* graph R as follows: R contains two adjacent nodes x and y, called the *special* nodes. The graph L induced by $R \setminus \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R, each leaf node of L is adjacent to exactly one of the two special nodes, and no other node of L is adjacent to special nodes. The last condition for R is that no two leaf segments of L with leaf nodes adjacent to the same special node have their other endnode in the same big clique. The *internal segments* of R are the internal segment of L, and the *leaf segments* of R are the leaf segments of L together with the node in $\{x, y\}$ to which the leaf segment is adjacent to.

We define a *basic graph* to be either a $3PC(\triangle, \cdot)$ or a nontrivial basic graph.

We now state the decomposition theorem for even-hole-free graphs.

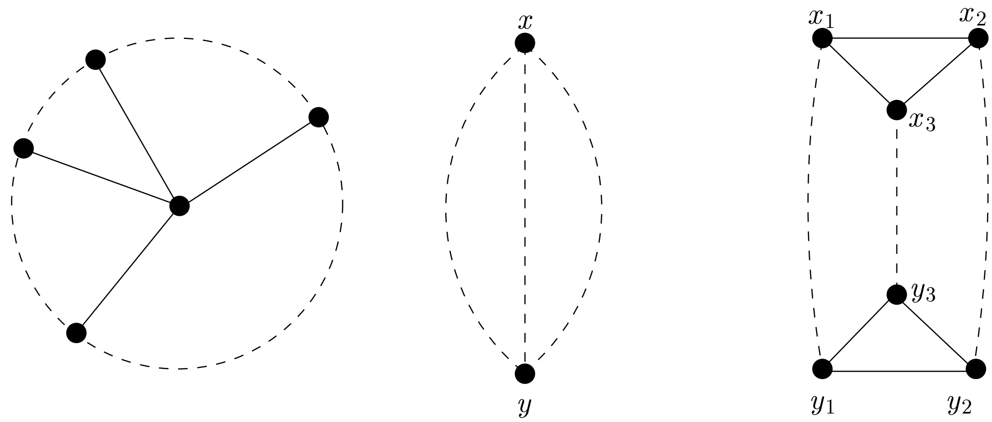
Theorem 1.1 *A connected even-hole-free graph is either basic or cap-free, or it has a 2-join, or a star, double-star or triple-star cutset.*

1.4 Odd-Signable Graphs

We *sign* a graph by assigning 0,1 weights to its edges in such a way that, for every triangle in the graph, the sum of the weights of its edges is odd. A graph G is *odd-signable* if there is a signing of its edges so that, for every hole in G, the sum of the weights of its edges is odd. Every even-hole-free graph is odd-signable, since we can get a correct signing by assigning a weight of 1 to every edge of the graph.

So Theorem 1.1 is implied by the following result, which we find more convenient to prove.

Theorem 1.2 (Main Theorem) *Let G be a connected odd-signable graph that does not contain a 4-hole. Then either G is basic or cap-free, or it has a 2-join or a star, double-star or triple-star cutset.*



$3PC(x,y)$ $\quad$ $3PC(x_1x_2x_3, y_1y_2y_3)$

Figure 3: An even wheel, a $3PC(\cdot,\cdot)$ and a $3PC(\Delta,\Delta)$

Now we introduce some graphs that are not odd-signable.

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \dots, x_n$. Node x is the *center* of the wheel. The hole H is called the *rim* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. A *short sector* is a sector of length 1 (i.e. it consists of one edge), and a *long sector* is a sector of length at least 2. A wheel is *even* if it contains an even number of sectors. A wheel with k sectors is called a *k-wheel*.

Given nonadjacent nodes x and y, a $3PC(x,y)$ is a graph induced by three chordless paths with endnodes x and y, having no common or adjacent intermediate nodes. A $3PC(x,y)$ is also referred to as a $3PC(\cdot,\cdot)$.

Given node disjoint triangles $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$, a $3PC(x_1x_2x_3, y_1y_2y_3)$, is a graph induced by three chordless paths, $P_1 = x_1, \dots, y_1$, $P_2 = x_2, \dots, y_2$ and $P_3 = x_3, \dots, y_3$, having no common nodes and such that the only adjacencies between the nodes of distinct paths are the edges of the two triangles. A $3PC(x_1x_2x_3, y_1y_2y_3)$ is also referred to as a $3PC(\Delta,\Delta)$.

Let P_1, P_2 and P_3 be the three paths of a $3PC(\cdot,\cdot)$. Every pair of these paths induces a hole. No matter how we sign the edges of the three paths, two of them will have the sum of the weights of their edges congruent modulo 2, so one of the holes will have even weight. Therefore $3PC(\cdot,\cdot)$'s are not odd-signable. Similarly, it can be shown that even wheels and $3PC(\Delta,\Delta)$'s are not odd-signable. So graphs that are odd-signable do not contain even wheels, $3PC(\cdot,\cdot)$'s and $3PC(\Delta,\Delta)$'s. The following theorem is an easy consequence of a theorem of Truemper [14], see also [9], and states that the converse is also true.

Theorem 1.3 *A graph is odd-signable if and only if it does not contain an even wheel, a $3PC(\cdot,\cdot)$ or a $3PC(\Delta,\Delta)$.*

The fact that odd-signable graphs do not contain even wheels, $3PC(\cdot,\cdot)$'s and $3PC(\Delta,\Delta)$'s will be used throughout the paper.

2 Proof of the Main Theorem

The first step of the proof is to show that when G contains one of three structures called gem, Mickey Mouse and proper wheel, then G has a star, double-star or triple-star cutset.

In the second step of the proof, we assume that G does not have a star, double-star or triple-star cutset (and therefore G does not contain a gem, a Mickey Mouse or a proper wheel). We show that, if G contains any of three structures called connected diamond, decomposable $3PC(\Delta, \cdot)$ and decomposable connected triangles, then G has a 2-join.

In the last step, we show that if G contains a cap but no 2-join, star, double-star or triple-star cutset, then G must be basic.

To help readability, some of the intermediate results are stated without proof in this section. The missing proofs are provided in later sections.

2.1 Node Cutset Decompositions

A *gem* is a graph on five nodes, such that four of the nodes induce a chordless path of length three and the fifth node is adjacent to all of the nodes of this path.

Theorem 2.1 *If an odd-signable graph G contains a gem, then G has a triple-star cutset.*

Proof: Suppose that the node set $\{x_1, \dots, x_5\}$ induces a gem, such that $P = x_1, x_2, x_3, x_4$ is a chordless path. Let $S = (N(x_2) \cup N(x_3) \cup N(x_5)) \setminus \{x_1, x_4\}$. If S is not a triple-star cutset separating x_1 from x_4, then there is a chordless path P' that connects x_1 to x_4 in $G \setminus S$, and the node set $V(P) \cup V(P') \cup \{x_5\}$ induces a 4-wheel with center x_5, contradicting the assumption that G is odd-signable. □

The following theorems are proved in Section 3.

Definition 2.2 *A* Mickey Mouse*, denoted by $M(xyz, H_1, H_2)$, is a graph induced by the node set $V(H_1) \cup V(H_2)$ that satisfies the following:*

- *the node set $\{x, y, z\}$ induces a clique,*
- *H_1 is a hole that contains edge xy but does not contain node z,*
- *H_2 is a hole that contains edge xz but does not contain node y, and*
- *the node set $V(H_1) \cup V(H_2)$ induces a cycle with exactly 2 chords, xy and xz.*

Theorem 2.3 *Let G be an odd-signable graph containing no 4-hole. If G contains a Mickey Mouse, then G has a triple-star cutset.*

A *bug* is a 3-wheel with exactly two long sectors.

Theorem 2.4 *Let G be an odd-signable graph containing no 4-hole. If G contains a bug, then G has a double-star cutset.*

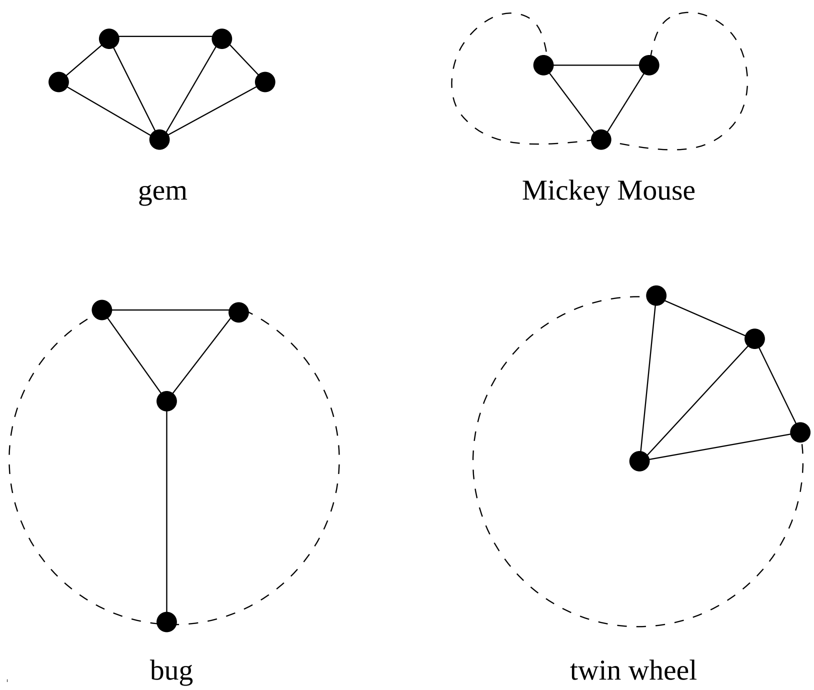


Figure 4: A gem, a Mickey Mouse, a bug and a twin wheel

A *twin wheel* is a 3-wheel with exactly two short sectors. A wheel is said to be *proper* if it is not a twin wheel.

Theorem 2.5 *Let G be an odd-signable graph that does not contain a 4-hole, a gem, a Mickey Mouse or a bug. If G contains a proper wheel, then G has a star cutset.*

2.2 Nodes Adjacent to a $3PC(\Delta, \cdot)$ and their Attachments

Throughout this section, we assume that G is an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. Consequently, by Theorems 2.1, 2.3, 2.4 and 2.5, G does not contain a gem, a Mickey Mouse or a proper wheel.

Lemma 2.6 *If H is a hole of G, then any node $u \notin V(H)$ has at most three neighbors in H. Furthermore, they are consecutive nodes of H.*

Proof: Let u have exactly two neighbors in H, say a and b. If ab is not an edge, the node set $V(H) \cup \{u\}$ induces a $3PC(a, b)$. If u has more than two neighbors in H and it is not a twin of a node in H, then (H, u) is a proper wheel. $\Box$

Throughout the rest of the section, Σ denotes a $3PC(a_1a_2a_3, a_4)$. The three paths of Σ are denoted by $P_{a_1a_4}$, $P_{a_2a_4}$ and $P_{a_3a_4}$ (where $P_{a_ia_4}$ is the path that contains a_i). Note that all three paths of Σ are of length greater than one, since G does not contain a proper wheel

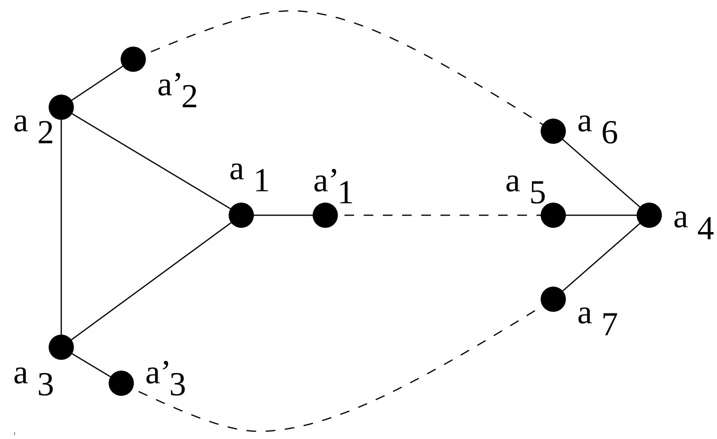


Figure 5: $\Sigma = 3PC(a_1a_2a_3, a_4)$

and a twin wheel is not a $3PC(\Delta, \cdot)$. For $i = 1, 2, 3$, we denote the neighbor of a_i in $P_{a_ia_4}$ by a'_i. Also, a_{i+4} is the neighbor of a_4 in $P_{a_ia_4}$. See Figure 5.

Applying Lemma 2.6 to the three holes of Σ, we get the following result. See Figure 6.

Lemma 2.7 *If u is a strongly adjacent node to Σ, then u is one of the following types:*

Type 1: u is a twin of a_1, a_2 or a_3.

Type 2: u is a twin of a_4.

Type 3: u is adjacent to a_1, a_2, a_3 and to no other node of Σ.

Type 4: u has exactly three neighbors in Σ, it is adjacent to a_4 and two of the nodes in $\{a_5, a_6, a_7\}$.

Type 5: u is a twin of a node of Σ, that is distinct from a_1, a_2, a_3 and a_4.

Type 6: u has exactly two neighbors in Σ, they are adjacent and they do not both belong to the set $\{a_1, a_2, a_3\}$.

Type 7: u has exactly two neighbors in Σ and they belong to the set $\{a_1, a_2, a_3\}$.

Proof: Let u be a strongly adjacent node to Σ. Suppose that u is not one of the Types 1 through 7. It is easy to check that by applying Lemma 2.6 to the three holes induced by the nodes of Σ, w.l.o.g. u is adjacent to a_2, a_3 and a'_3. But then the node set $\{a_1, a_2, a_3, a'_3, u\}$ induces a gem. $\square$

Nodes adjacent to Σ are further classified as follows.

Type 5a: A Type 5 node that is not adjacent to a_4,

Type 5b: A Type 5 node adjacent to a_4,

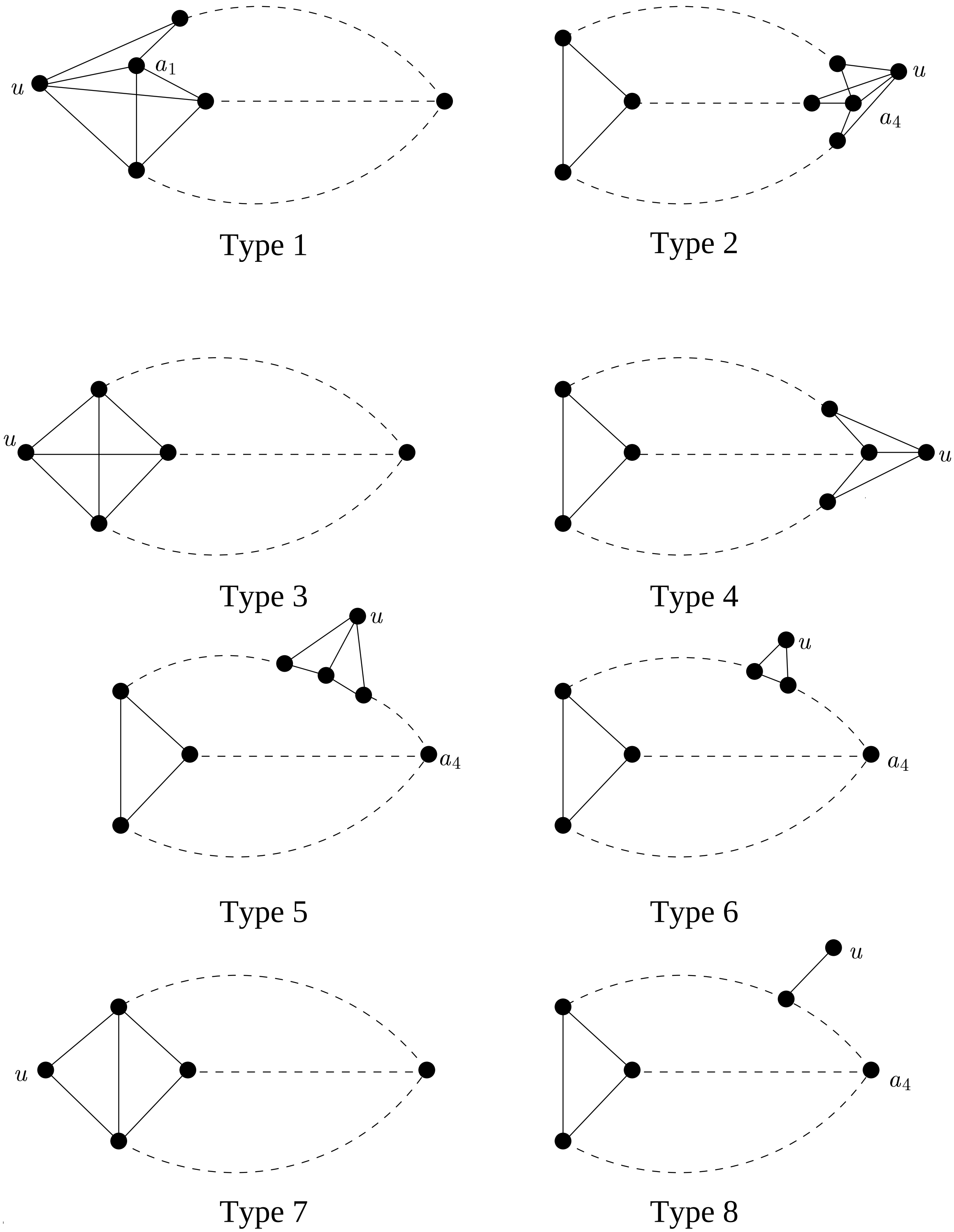


Figure 6: Nodes adjacent to a $3PC(\Delta, \cdot)$

Type 6a: A Type 6 node that is not adjacent to a_4,

Type 6b: A Type 6 node adjacent to a_4,

Type 8: A node that is adjacent to Σ, but not strongly adjacent,

Type 8a: A Type 8 node that is not adjacent to a_4, and

Type 8b: A Type 8 node adjacent to a_4.

Lemma 2.8 *Let u be a Type 3 node w.r.t. Σ. Let $S = N(a_1) \cup N(a_2) \cup N(a_3) \setminus \{u, a_1', a_2', a_3'\}$. Then, in every direct connection $P = u_1, \dots, u_n$ from u to $\Sigma \setminus S$ in $G \setminus S$, the node u_n is of Type 2, 5 or 8 w.r.t. Σ. Furthermore, for some $i \in \{1, 2, 3\}$, there exists $R \subseteq P_{a_i a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(P) \cup \{u\}$ is a $3PC(\Delta, \cdot)$.*

Proof: Since $u_n \notin S$, it cannot be of Type 1, 3 or 7 w.r.t. Σ. If u_n is of Type 4 w.r.t. Σ, say adjacent to a_4, a_5 and a_7, then there exists a $3PC(a_1 a_2 u, a_5 a_4 u_n)$. If u_n is of Type 6 w.r.t. Σ, say with neighbors r and s in path $P_{a_1 a_4}$, with r contained in the $a_1 s$-subpath of $P_{a_1 a_4}$, then since r cannot be coincident with a_1, there exists a $3PC(a_1 a_2 u, r s u_n)$. So u_n is of Type 2, 5 or 8 w.r.t. Σ and the lemma follows. □

Lemma 2.9 *Let u be a Type 7 node w.r.t. Σ, adjacent to say a_1 and a_3. Let $S = N(a_1) \cup N(a_2) \cup N(a_3) \setminus \{u, a_1', a_2', a_3'\}$. Then, in every direct connection $P = u_1, \dots, u_n$ from u to $\Sigma \setminus S$ in $G \setminus S$, the node u_n is either of Type 4 w.r.t. Σ, adjacent to a_4, a_5 and a_7, or it is of Type 6 w.r.t. Σ, with both neighbors in $P_{a_2 a_4} \setminus a_2$. Furthermore, there exists $R \subseteq P_{a_2 a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(P) \cup \{u\}$ is a $3PC(\Delta, \cdot)$.*

Proof: First we show that u_n must be strongly adjacent to Σ. Suppose not and assume that the unique neighbor of u_n in Σ is node s. If node s is not contained in $V(P_{a_3 a_4}) \setminus \{a_4\}$, then if $s \neq a_1'$, there exists a $3PC(a_1, s)$ and otherwise there exists an even wheel with center a_1. Similarly, if $s \in V(P_{a_3 a_4}) \setminus \{a_4\}$, then if $s \neq a_3'$ there exists a $3PC(a_3, s)$ and otherwise there exists an even wheel with center a_3. Hence u_n must be strongly adjacent to Σ.

Node u_n cannot be of Type 1, 3 or 7 w.r.t. Σ. Suppose u_n is of Type 2 or 5 w.r.t. Σ, and let Σ' be a $3PC(a_1 a_2 a_3, \cdot)$ obtained from Σ by substituting u_n for its twin in Σ. If $n = 1$, then u and Σ' contradict Lemma 2.7. Otherwise, $u_1, \dots, u_{n-1}$ is a direct connection from u to $\Sigma' \setminus S$ in $G \setminus S$, contradicting the first paragraph of the proof, since u_{n-1} is not strongly adjacent to Σ'. Therefore, u_n cannot be of Type 2 or 5 w.r.t. Σ. Hence u_n is of Type 4 or 6 w.r.t. Σ.

Suppose that u_n is of Type 4 w.r.t. Σ, but is not adjacent to both a_5 and a_7. W.l.o.g. assume that u_n is not adjacent to a_7. Then there is a $3PC(u_n, a_1)$. Hence, if u_n is of Type 4 w.r.t. Σ then it is adjacent to a_4, a_5 and a_7. Finally suppose that u_n is of Type 6 w.r.t. Σ, but its neighbors in Σ are not contained in $P_{a_2 a_4}$. Let r and s be the neighbors of u_n in Σ and w.l.o.g. assume they are contained in $P_{a_1 a_4}$. Let r be contained in the $a_1 s$-subpath of $P_{a_1 a_4}$. Since r cannot be coincident with a_1, there is a $3PC(a_1 a_3 u, r s u_n)$. This completes the proof of the lemma. □

Lemma 2.10 *Let u be a node of Type 4 w.r.t. Σ, adjacent to a_5 and a_7. Let $S = (N(a_4) \cup a_4) \setminus u$. Then, in every direct connection $P = u_1, \ldots, u_n$ from u to $\Sigma \setminus S$ in $G \setminus S$, the node u_n is a twin of a_2, or it is of Type 5a or 8a with neighbors in $P_{a_2a_4}$ or of Type 7 adjacent to a_1 and a_3. Furthermore, there exists $R \subseteq P_{a_2a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(P) \cup \{u\}$ is a $3PC(\Delta, \cdot)$.*

Proof: If there exists one, let u_i be the node of lowest index adjacent to a_5, a_6 or a_7. If u_i is adjacent to more than one node in a_5, a_6 and a_7 then u_i contradicts Lemma 2.7 since it is not adjacent to a_4. First assume that $i < n$. If u_i is adjacent to a_5 or a_7, then either there exists a Mickey Mouse (when $i > 1$), or there exists a gem (when $i = 1$). If u_i is adjacent to a_6 there is a proper wheel with center a_4. Now we consider the case when u_n is the only node in P that may have a neighbor in $\{a_5, a_6, a_7\}$. Note that u_n cannot be of Type 2, 4, 5b, 6b or 8b since it is not adjacent to a_4. Let u_n be of Type 5a, 6a or 8a, with $N(u_n) \cap \Sigma \subseteq P_{a_1a_4}$ or $P_{a_3a_4}$. Assume w.l.o.g. that $N(u_n) \cap \Sigma \subseteq P_{a_1a_4}$. Now there exists a $3PC(ua_4a_7, a_1a_2a_3)$. If u_n is adjacent to a_1 and a_2 and no other node of $P_{a_1a_4} \cup P_{a_2a_4}$ there exists a $3PC(ua_4a_5, u_na_2a_1)$. By symmetry u_n cannot be adjacent to a_3 and a_2 and no other node of $P_{a_3a_4} \cup P_{a_2a_4}$. So u_n must be of Type 7 adjacent to a_1 and a_3, or u_n is a twin of node a_2, or $N(u_n) \cap \Sigma \subseteq P_{a_2a_4}$. In the last case, if u_n is of Type 6a there exists a $3PC(ua_4a_5, u_nrs)$ where r and s are the neighbors of u_n in $P_{a_2a_4}$ with r contained in the sa_4-subpath of $P_{a_2a_4}$. □

Lemma 2.11 *Let u be a Type 6b node w.r.t. Σ, say adjacent to a_4 and a_5. Let $S = (N(a_4) \cup N(a_5)) \setminus \{u, a_6, a_7, a_5'\}$, where a_5' is the neighbor of a_5 in $P_{a_1a_4}$ distinct from a_4. Then, in every direct connection $P = u_1, \ldots, u_n$ from u to $\Sigma \setminus S$ in $G \setminus S$, the node u_n is one of the following types:*

- *(i) a Type 8a node w.r.t. Σ, with a neighbor in $V(P_{a_1a_4}) \setminus \{a_4, a_5, a_5'\}$,*
- *(ii) a Type 5a node w.r.t. Σ, with neighbors in $P_{a_1a_4}$,*
- *(iii) a Type 1 node w.r.t. Σ, that is a twin of a_1,*
- *(iv) a Type 7 node w.r.t. Σ, that is adjacent to a_2 and a_3.*

Furthermore, there exists $R \subseteq P_{a_1a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(P) \cup \{u\}$ is a $3PC(\Delta, \cdot)$.

Proof: Node u_n is not of Type 2, 4, 5b, 6b or 8b w.r.t. Σ. If u_n is of Type 5a, 6a or 8a w.r.t. Σ, with a neighbor in $V(P_{a_2a_4}) \setminus \{a_4, a_6\}$ or in $V(P_{a_3a_4}) \setminus \{a_4, a_7\}$, assume w.l.o.g. the former, then there is a $3PC(a_1a_2a_3, a_5ua_4)$. If u_n is of Type 8a w.r.t. Σ and it is adjacent to node a_6 then there exists a proper wheel with center a_4. Similarly if u_n is of Type 8a and adjacent to a_5' there exists a proper wheel with center a_5. So if u_n is of Type 8a it satisfies (i). If u_n is of Type 5a w.r.t. Σ, it satisfies (ii). If u_n is of Type 6a w.r.t. Σ, with neighbors r and s in Σ that are contained in $P_{a_1a_4}$ with r contained in the sa_4-subpath of $P_{a_1a_4}$, then since r cannot be coincident with a_5 there is a $3PC(sru_n, a_4a_5u)$. Hence u_n cannot be of Type 6 w.r.t. Σ. If u_n is adjacent to a_1 and a_3 but not to any other node of $V(P_{a_1a_4}) \cup V(P_{a_3a_4})$, then there is a $3PC(a_1u_na_3, a_5ua_4)$. So u_n cannot be of Type 3. In addition, if u_n is of Type 1, then it must be a twin of a_1, and if u_n is of Type 7, then it must be adjacent to a_2 and a_3. This completes the proof of the lemma. □

Lemma 2.12 *Let u be a Type 8b node w.r.t. Σ. Let $S = (N(a_4) \cup a_4) \setminus u$. Then, in every direct connection $P = u_1, \ldots, u_n$ from u to $\Sigma \setminus S$ in $G \setminus S$, the node u_n is of Type 3 or 6a w.r.t. Σ. Furthermore, for some $i \in \{1,2,3\}$, there exists $R \subseteq P_{a_i a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(P) \cup \{u\}$ is a $3PC(\Delta, \cdot)$.*

Proof: Let $P = u_1, \ldots, u_n$ be a direct connection from u to $\Sigma \setminus S$ in $G \setminus S$. P may have neighbors in $\{a_5, a_6, a_7\}$. No node of P is adjacent to more than one node in $\{a_5, a_6, a_7\}$ since otherwise by Lemma 2.6 it is adjacent to a_4 contradicting the assumption that P avoids nodes in S. If it contains neighbors of all three, P contains a path $u_i, \ldots, u_j$, $j \neq n$, with u_i adjacent to say a_5, u_j adjacent to a_6 and no intermediate node adjacent to a_5, a_6 or a_7. But then there exists a $3PC(a_5, a_6)$. If exactly two of a_5, a_6, a_7 have a neighbor in P, say a_5 and a_6, then there exists a $3PC(a_5, a_6)$ unless the unique neighbor of say a_5, in P is u_n and u_n is strongly adjacent to Σ with another neighbor in $P_{a_1 a_4}$. In this case, if a_6 has more than one neighbor in P there exists a proper wheel with center a_6, and otherwise there exists a $3PC(a_4, u_i)$ where u_i is the unique neighbor of a_6 in P. If exactly one of a_5, a_6, a_7, say a_5, has a neighbor in P then, if a_5 has more than one neighbor in P, there exists a proper wheel with center a_5. Let a_5 have exactly one neighbor in P, say u_i. Either there exists a proper wheel with center a_5, or there exists a $3PC(a_4, u_i)$. So P does not contain a neighbor of a_5, a_6 or a_7.

First we show that u_n must be strongly adjacent to Σ. Suppose not and let r be the unique neighbor of u_n in Σ. Note that $r \notin \{a_4, a_5, a_6, a_7\}$. W.l.o.g. assume that r does not belong to $P_{a_3 a_4}$. But then the node set $V(P) \cup V(P_{a_1 a_4}) \cup V(P_{a_2 a_4}) \cup \{u\}$ induces a $3PC(r, a_4)$.

Node u_n cannot be of Type 2, 4, 5b or 6b w.r.t. Σ. If u_n is of Type 7 w.r.t. Σ, say adjacent to a_1 and a_3, then the node set $V(P) \cup V(P_{a_1 a_4}) \cup V(P_{a_2 a_4}) \cup \{u\}$ induces a $3PC(a_1, a_4)$. If u_n is of Type 1 or 5a w.r.t. Σ, then there is a $3PC(u_n, a_4)$. □

Definition 2.13 *For any node u and path P described in Lemmas 2.8-2.12, we say that the path P is an* attachment *of node u to Σ.*

Corollary 2.14 *Let Σ be a $3PC(a_1 a_2 a_3, a_4)$. Every node u of Type 3, 4, 6b, 7 and 8b w.r.t. Σ has an attachment Q to Σ. Furthermore, for some $i \in \{1,2,3\}$, there exists $R \subseteq P_{a_i a_4}$ such that the graph induced by $V(\Sigma \setminus R) \cup V(Q) \cup \{u\}$ is a $3PC(\Delta, \cdot)$ Σ'.*

Proof: Since G contains no k-star cutset, $k = 1, 2, 3$, the graphs $G \setminus S$ defined in Lemmas 2.8-2.12 contain a direct connection from u to $\Sigma \setminus S$. By definition, these direct connections are attachments of u to Σ and, in each case, Σ' exists. □

Definition 2.15 *A graph Σ' as described in Corollary 2.14 is said to be a $3PC(\Delta, \cdot)$ obtained from Σ by* substituting *u and its attachment Q in Σ.*

2.3 Crosspaths

Throughout this section we assume that G is an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. Consequently, by Theorems 2.1, 2.3, 2.4 and 2.5, G does not contain a gem, Mickey Mouse or a proper wheel.

In this section we study certain paths that connect nodes in different paths $P_{a_i a_4}, P_{a_j a_4}$, $i \neq j$ of a $3PC(a_1 a_2 a_3, a_4)$.

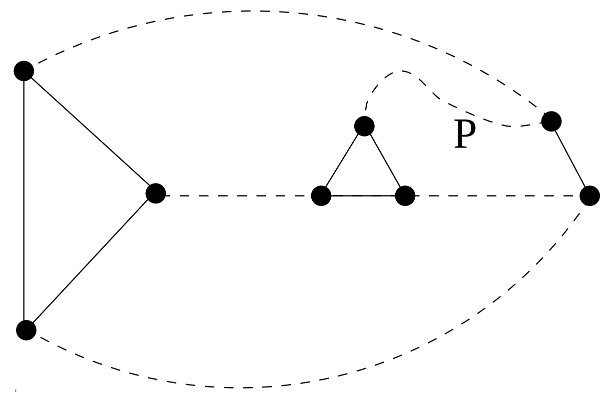


Figure 7: Crosspath

Definition 2.16 *Let $P = u_1, \ldots, u_n$, $n \geq 2$, be a chordless path in $G \setminus \Sigma$ such that u_1 is of Type 8a w.r.t. Σ adjacent to a_5, u_n is of Type 6a w.r.t. Σ with neighbors in $P_{a_2a_4}$ or $P_{a_3a_4}$, and no node u_i, for $2 \leq i \leq n-1$, is adjacent to a node of Σ. Such a path P is called an a_5-crosspath w.r.t. Σ. Similarly we define a_6-crosspaths and a_7-crosspaths. For $i \in \{5,6,7\}$, if there exists an a_i-crosspath we say that a_i has a crosspath. If $P = u_1, \ldots, u_n$ is an a_i-crosspath such that u_n has neighbors in $P_{a_ja_4}$, $j \in \{1,2,3\} \setminus \{i-4\}$, then we say that P is a crosspath from a_i to $P_{a_ja_4}$.*

Lemma 2.17 *Let $P = u_1, \ldots, u_n$, $n \geq 2$, be a chordless path in $G \setminus \Sigma$ with $N(u_k) \cap \Sigma = \emptyset$, for all $k \in \{2, \ldots, n-1\}$, $N(u_1) \cap \Sigma \subseteq P_{a_ia_4}$, $N(u_1) \cap \Sigma \neq a_4$, and $N(u_n) \cap \Sigma \subseteq P_{a_ja_4}$, $N(u_n) \cap \Sigma \neq a_4$, where $i \neq j$. Then either P is a crosspath w.r.t. Σ or one of u_1 or u_n is of Type 5b w.r.t. Σ, say u_1, and $u_2, \ldots, u_n$ is a crosspath w.r.t. Σ' obtained by substituting u_1 for its twin in Σ.*

Proof: Let Σ and P be a counterexample to the lemma, chosen to minimize $|P|$. By Lemma 2.7, nodes u_1 and u_n must be of Type 5, 6 or 8a w.r.t. Σ. If one of u_1 or u_n, say u_1, is of Type 5 we substitute it for its twin in Σ to obtain Σ' and a path $P' = P \setminus u_1$, that has one less node than P. If $n = 2$, i.e. P' contains only node u_n, by Lemma 2.7, u_n is of Type 6b in Σ and of Type 4 in Σ'. Now nodes u_n, u_1 and the neighbors of u_1 in Σ induce a gem. So $n \geq 3$, P' contains at least two nodes, so P' is a crosspath for Σ'. But then P and Σ satisfy the lemma as well. Thus w.l.o.g. we only need to consider the case where neither u_1 nor u_n is of Type 5.

If both u_1 and u_n are of Type 8a, let their neighbors in Σ be r and s respectively. If rs is not an edge, then there exists a $3PC(r,s)$. Hence rs is an edge. Since r and s are not contained in any one path of Σ, this implies $r = a_i$ and $s = a_j$. But now there exists a Mickey Mouse in G. If both u_1 and u_n are of Type 6, then if they are both adjacent to a_4 there exists a proper wheel with center a_4 and otherwise there exists a $3PC(\Delta, \Delta)$. So, w.l.o.g., node u_1 is of Type 8a and u_n is of Type 6. If u_1 is not adjacent to a_{i+4}, there exists a $3PC(a_1a_2a_3, T)$ where T is the triangle induced by u_n and its neighbors in Σ, or a proper wheel with center a_j. So u_1 is of Type 8a adjacent to a_{i+4}. If u_n is of Type 6b, there exists a proper wheel with center a_4. Therefore P is a crosspath from a_{i+4} to $P_{a_ja_4}$. □

Lemma 2.18 *At most one node in $\{a_5, a_6, a_7\}$ has a crosspath.*

Proof: Suppose not and let $P = u_1, \dots, u_n$ be an a_5-crosspath and $Q = v_1, \dots, v_m$ an a_6-crosspath. Let r_1 and r_2 be the neighbors of u_n in Σ, and let s_1 and s_2 be the neighbors of v_m in Σ. If paths P and Q do not have adjacent nodes (note that in that case, they also cannot have coincident nodes), then it is straightforward to check that there is a $3PC(a_5, a_6)$ or a proper wheel. So P and Q have adjacent nodes. Then a subset of $P \cup Q \cup \{a_5, a_6\}$ induces a chordless path P' from a_5 to a_6. Nodes r_1, r_2, s_1, s_2 cannot all be contained in $P_{a_3a_4}$, since otherwise the node set $S = P_{a_1a_4} \cup P_{a_2a_4} \cup P'$ induces a $3PC(a_5, a_6)$. We now show that u_n cannot have a neighbor in Q. Note that u_n is not adjacent to a_4 (by the definition of a crosspath) and it is not adjacent to a_6 (since otherwise $\Sigma \cup P$ contains a bug). Let Σ_1 and Σ_2 denote, respectively, the $3PC(v_m s_1 s_2, a_4)$ and $3PC(v_m s_1 s_2, a_6)$ contained in $\Sigma \cup Q$. For some $i \in \{1, 2\}$, u_n has neighbors in $\Sigma_i \setminus Q$. By Lemma 2.7 applied to u_n and Σ_i, it follows that u_n cannot have a neighbor in Q. Similarly, v_m cannot have a neighbor in P. Hence, P' does not contain u_n and v_m, so the node set S induces a $3PC(a_5, a_6)$. □

2.4 2-Join Decompositions

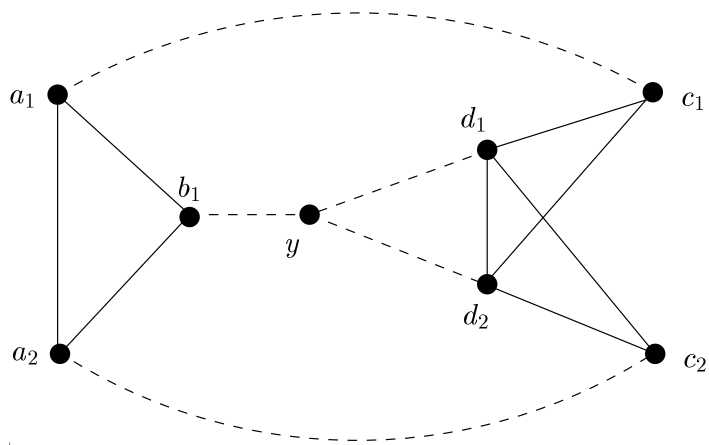


Figure 8: A connected diamond

Definition 2.19 *A* connected diamond *is a $3PC(d_1 d_2 c_1, y)$ together with a Type 7 node c_2 adjacent to d_1, d_2 and an attachment of c_2.*

In Section 5.1, we prove the following theorem.

Theorem 2.20 *Let G be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. If G contains a connected diamond, then G has a 2-join.*

Lemma 2.21 *Let G be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. If G does not contain a connected diamond, then G does not contain a wheel.*

Proof: Assume that G does not contain a connected diamond. Then, by Lemma 2.10, G cannot contain a $3PC(\Delta, \cdot)$ with a Type 4 node.

Suppose that G contains a wheel (H, u). By the assumption that G contains no proper wheel, (H, u) is a twin wheel. Let v be the common endnode of the two short sectors of (H, u). Let $S = v \cup N(v) \setminus u$ and let $P = y_1, \ldots, y_m$ be a direct connection in $G \setminus S$ from u to $H \setminus S$. Let v_1 and v_2 be the neighbors of v in H. If P contains no node adjacent to v_1 or v_2, then the neighbors of y_m in H are two adjacent nodes of H, since otherwise G contains a proper wheel or a $3PC(\cdot, \cdot)$. But then $H \cup P \cup u$ induces a $3PC(\Delta, \cdot)$ with a Type 4 node v.

Node y_1 is adjacent to neither v_1 nor v_2, since otherwise there exists a gem or a 4-hole. Let y_i be a node with lowest index that is adjacent to v_1 or v_2. W.l.o.g. y_i is adjacent to v_1. Let H' be a hole in the graph induced by $P \cup H \cup u$ that contains y_1, u and v_2. Node v_1 is adjacent to at least two nodes in H' (u and y_i). However, v_1 is not adjacent to v_2 and y_1. This contradicts Lemma 2.6. □

Lemma 2.22 *Let G be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. If G does not contain a connected diamond, then the only strongly adjacent nodes to a $3PC(\Delta, \cdot)$ are of Type 3 or Type 6.*

Proof: Assume that G does not contain a connected diamond. Then, G cannot contain a $3PC(\Delta, \cdot)$ with a Type 1, 2, 4 or 5 node else G contains a twin wheel, a contradiction to Lemma 2.21. If a Type 7 node exists, it must be attached, contradicting the assumption that G contains no connected diamond. □

Definition 2.23 *Let Σ be a $3PC(a_1a_2b_1, c_1)$, with the neighbors of c_1 on the paths $P_{a_1c_1}$, $P_{a_2c_1}$ and $P_{b_1c_1}$ being node e_1, e_2 and d_1 respectively. Σ is a* decomposable $3PC(\Delta, \cdot)$ *if the following two properties hold:*

1. *If G contains a $3PC(\Delta, \cdot)$ with a crosspath, then Σ has an e_1-crosspath and all crosspaths of Σ are from e_1 to $P_{a_2c_1}$.*
2. *One of the following holds:*
   - *(i) There exists a node u_H of Type 3 w.r.t. Σ such that every attachment of u_H to Σ ends in $P_{b_1c_1}$.*
   - *(ii) There exists a node u_H of Type 8a or 6 w.r.t. Σ adjacent to a node in $P_{b_1c_1}$.*

Let $H_1 = P_{a_1c_1} \cup P_{a_2c_1}$, $H_2 = P_{b_1d_1} \cup u_H$ and $H = \Sigma \cup u_H$. H is called an extension *of the decomposable $3PC(a_1a_2b_1, c_1)$. Let $A = \{a_1, a_2\}$ and $C = \{c_1\}$. If (i) holds, let $B = \{b_1, u_H\}$ and $D = \{d_1\}$. If (ii) holds, let $B = \{b_1\}$ and let set D contain node d_1 and possibly node u_H, if u_H is of Type 6b. The 2-join of H induced by the partition $H_1|H_2$ has special sets A, B, C, D.*

In Section 5.2 we prove the following theorem.

Theorem 2.24 *Let G be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. If G contains a decomposable $3PC(\Delta, \cdot)$, then G has a 2-join.*

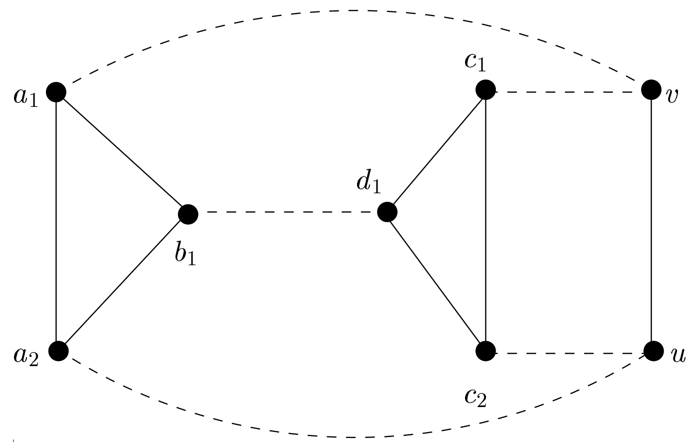


Figure 9: Connected triangles

Definition 2.25 Connected triangles $T(a_1a_2b_1, c_1c_2d_1, u, v)$ *consist of a* $3PC(a_1a_2b_1, u)$, Σ_1, *with node* $v \in P_{a_1u}$ *adjacent to node* u, *together with a* v*-crosspath* P *with endnode* c_1 *of Type 6 w.r.t.* Σ_1 *adjacent to* $c_2, d_1 \in P_{b_1u}$, *where* d_1 *lies on the* b_1c_2*-subpath of* P_{b_1u}. *The* $3PC(a_1a_2b_1, v)$ *is denoted by* Σ_2, *the* $3PC(c_1c_2d_1, u)$ *is denoted* Σ_3 *and* $3PC(c_1c_2d_1, v)$ *is denoted* Σ_4. *Note that* $b_1 = d_1$ *is allowed in this definition. All other nodes must be distinct. When* $b_1 = d_1$, *we say that the connected triangles are* degenerate.

In Section 5.3, we prove the following two theorems.

Theorem 2.26 *Let* G *be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. Let* T *be a degenerate connected triangles. Then there exists no node* $w \notin T$ *such that* $b_1 = d_1$ *is the unique neighbor of* w *in* T.

Definition 2.27 *Connected triangles* $T(a_1a_2b_1, c_1c_2d_1, u, v)$ *are* decomposable *if they are nondegenerate, there exists no* v*-crosspath w.r.t.* Σ_1 *(nor* u*-crosspath w.r.t.* Σ_2*)* $P' = y_1, \ldots, y_m$ *with* y_m *adjacent to an intermediate node of* $P_{b_1d_1}$. *Furthermore, there exists* $w \notin T$ *whose neighbors in* T *are two adjacent nodes of* $P_{b_1d_1}$ *or* w *is not strongly adjacent to* T *and its unique neighbor in* T *is in* $P_{b_1d_1}$. *The graph* $H = T \cup w$ *is an* extension *of* T. *Let* $H_2 = P_{b_1d_1} \cup w$ *and* $H_1 = H \setminus H_2$. *The 2-join of* H *with partition* $H_1|H_2$ *has special sets* A, B, C, D *containing the correspondingly labeled nodes.*

Theorem 2.28 *Let* G *be an odd-signable graph that does not contain a 4-hole, and does not have a star, double-star or triple-star cutset. If* G *contains a decomposable connected triangles, then* G *has a 2-join.*

2.5 Basic Graphs

Lemma 2.29 *Let* K *be a big clique of a nontrivial basic graph* R *with special nodes* x, y *and* u, v *two distinct nodes of* K. *Then* R *contains a hole* H, *that contains nodes* u, v, x *and* y *and no other node of* K.

Proof: By the definition of nontrivial basic graph, R contains two node disjoint paths, say P_u, P_v, between u, v and x, y such that the only edges between P_u, P_v are uv and xy. So H is induced by the nodes of these two paths. $\Box$

A nontrivial basic graph that plays an important role in the proof is connected triangles. Let $T(a_1a_2b_1, c_1c_2d_1, u, v)$ be connected triangles. The path $P_{b_1d_1}$ is the *internal segment* of T and paths P_{a_1v}, P_{a_2u}, P_{c_1v} and P_{c_2u} are the *leaf segments* of T.

Lemma 2.30 *Every leaf (internal) segment of a nontrivial basic graph R is the leaf (internal) segment of connected triangles $T(\Delta, \Delta, x, y)$ contained in R.*

Proof: Let P be an internal segment of R and K_1, K_2 be the big cliques that contain the endnodes of P, say u_1, u_2. Let $v_i, w_i \in K_i \setminus u_i$, $i = 1, 2$. For $i = 1, 2$, by Lemma 2.29, R contains a hole H_i that contains v_i, w_i, x and y and no other node of K_i. Since R is basic $H_1 \cup H_2 \cup P$ induces the desired connected triangles $T(\Delta, \Delta, x, y)$.

Now let P be a leaf segment of R and K_1 be the big clique containing the endnode of P, say w_1, distinct from x, y. Let $u_1 \in K_1 \setminus w_1$ where u_1 is an endnode of an internal segment Q. (Such a node u_1 exists since, from the definition of nontrivial basic graphs, R contains at least two big cliques.) Let the other endnode of Q be $u_2 \in V(K_2)$. By the previous argument, Q belongs to a connected triangles containing w_1 and therefore P. Furthermore P is a leaf segment of this connected triangles. $\Box$

Lemma 2.31 *For any pair of segments P and Q of a nontrivial basic graph R, R contains a $3PC(\Delta, z)$, for some $z \in \{x, y\}$, that contains $P \cup Q \cup \{x, y\}$ such that P and Q belong to distinct paths of Σ. Furthermore, R contains a z'-crosspath w.r.t. Σ, where $z' = \{x, y\} \setminus z$.*

Proof: First we show that R contains a $\Sigma = 3PC(\Delta, z)$, for some $z \in \{x, y\}$, that contains $P \cup Q \cup \{x, y\}$ such that paths P and Q belong to distinct paths of Σ. In $R \setminus \{x, y\}$, there exists a chordless path from an endnode of P to an endnode of Q, that does not contain any intermediate node of P or Q. Let u be the endnode of P contained in this chordless path, let v be the neighbor of u in this path. Let K be the big clique of R that contains u and v, and let $w \in K \setminus \{u, v\}$. By Lemma 2.29, R contains a hole H that contains nodes u, w, x and y and no other node of K. Note that P must be contained in H. In $R \setminus (K \setminus v)$ there is a path P_v from v to x or y, that contains Q. Since R is basic, no node of Q is adjacent to a node of $H \setminus \{x, y\}$. Then $H \cup P_v$ induces the desired $\Sigma = 3PC(\Delta, z)$.

W.l.o.g. assume that $z = x$. Now we show that R contains a y-crosspath w.r.t. Σ. Since R is nontrivial, the two paths of Σ that do not contain y cannot both be leaf segments. Let P_{tx} be a path of Σ that does not contain y and is not a leaf segment of R. Let s be the node of P_{tx} closest to x that belongs to a big clique K'. Note that P_{sx} is a leaf segment of R. The neighbor s' of s in P_{ts} also belongs to K'. Let $r \in K' \setminus \{s, s'\}$. By Lemma 2.29, R contains a hole H' that contains r, s, x and y and no other node of K'. Note that P_{sx} must be contained in H'. Since R is basic no node of $H' \setminus P_{sx}$ can be adjacent to a node of $\Sigma \setminus \{s', s, y\}$. Hence $H' \setminus P_{sx}$ is the desired y-crosspath of Σ. $\Box$

A graph R contained in G is a *maximum basic graph of* G, if it is basic and G does not contain a basic graph that has a larger number of segments than R.

Lemma 2.32 *Let G be an odd-signable graph that does not contain a 4-hole and does not have a 2-join, a star, double-star or triple-star cutset. Let R be a maximum basic graph of G. Assume R is nontrivial and has special nodes x and y.*

(1) If P is a leaf segment of R containing x, then R contains a $\Sigma = 3PC(\Delta, x)$ in which P is one of the paths and y is contained in one of the other two paths. Furthermore, R contains a y-crosspath w.r.t. Σ and all crosspaths of Σ in G are y-crosspaths that do not end in P.

(2) If P is an internal segment of R, then R contains connected triangles $T(a_1a_2b_1, c_1c_2d_1, x, y)$ such that P is the internal segment of T and there is neither a y-crosspath in G w.r.t. the $3PC(a_1a_2b_1, x)$ contained in T nor an x-crosspath in G w.r.t. the $3PC(a_1a_2b_1, y)$ contained in T, that is adjacent to an intermediate node of P.

Proof: Since G contains no 2-join, by Theorem 2.20, G contains no connected diamonds.

We first prove (1). Let P be a leaf segment of R containing x. By Lemma 2.30, R contains a connected triangles $T(\Delta, \Delta, x, y)$ with P being a leaf segment of T. So T contains a $\Sigma = 3PC(\Delta, x)$ in which P is one of the paths and y is contained in one of the other two paths. Also T contains a y-crosspath w.r.t. Σ. By Lemma 2.18, all crosspaths of Σ are y-crosspaths. Suppose there exists a y-crosspath $P' = y_1, \ldots, y_m$ such that y_m has neighbors r and s in P. Note that since P is a segment of R, $y_m \notin R$. If no node of P' is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$, then $R' = R \cup P'$ is a basic graph. (Note that in this case, $R' \setminus \{x, y\}$ is a line graph of a tree in which P' is a leaf segment and it is easy to check that all conditions for R' to be basic are satisfied.) Since this would contradict the maximality of R, we may assume that some node of P' is adjacent to or coincident with a node of $R \setminus \{r, s, y\}$. Let y_j be the node of P' with highest index that is adjacent to a node, say u, of $R \setminus \{r, s, y\}$. Node u belongs to some segment Q ($\neq P$) of R. By Lemma 2.31, R contains a $\Sigma' = 3PC(\Delta, z)$, where $z = x$ or y, that contains both x and y and such that P and Q belong to distinct paths of Σ'. So by Lemma 2.22, $j < n$. Furthermore, R contains a z'-crosspath w.r.t. Σ', where $z' = \{x, y\} \setminus z$. Node y_j cannot be of Type 3 w.r.t. Σ', since otherwise path $y_j, \ldots, y_m$ contradicts Lemma 2.8. So by Lemma 2.17 and Lemma 2.22, $y_j, \ldots, y_m$ is a u-crosspath w.r.t. Σ'. By Lemma 2.18, $u = z'$ and hence $j = 1$ and $u = y$, which contradicts our choice of u.

We now prove (2). Let P be an internal segment of R. By Lemma 2.30, R contains a connected triangles $T(a_1a_2b_1, c_1c_2d_1, x, y)$ such that P is the internal segment of T. Suppose w.l.o.g. that there is a y-crosspath w.r.t. the $3PC(a_1a_2b_1, x)$ contained in T, $P' = y_1, \ldots, y_m$ such that y_m has neighbors r and s in P. A contradiction is now obtained as in proof of (1). □

Proof of the Main Theorem: Assume G contains a cap but no 2-join, star, double-star or triple-star cutset. By Theorem 2.20, G contains no connected diamonds and by Lemma 2.21, G contains no wheel. We will show that G is a basic graph.

Claim 1: G contains a basic graph.

Proof of Claim 1: Let H be a hole that together with node w induces a cap. Let the neighbors of w in H be u and v. Let u' (resp. v') be the neighbor of u (resp. v) in H that is distinct from v (resp. u). Since $S = (N(u) \cup N(v)) \setminus \{u', v', w\}$ is not a cutset separating w from H, in $G \setminus S$ there exists a direct connection $P = x_1, \ldots, x_n$ from w to $H \setminus S$. Since G contains no wheel, x_n is either not strongly adjacent to H or has exactly two neighbors in H. In the latter case either $H \cup P \cup w$ induces a $3PC(\Delta, \Delta)$ (if the neighbors of x_n in H are adjacent) or $H \cup x_n$ induces a $3PC(\cdot, \cdot)$ (if the neighbors of x_n in H are not adjacent). Hence x_n is not strongly adjacent to H, so $H \cup P \cup w$ induces a $3PC(\Delta, \cdot)$. This completes the proof of Claim 1.

Case 1: Every maximum basic graph of G is a $3PC(\Delta, \cdot)$.

Then no $3PC(\Delta, \cdot)$ has a crosspath. Let R be any $3PC(\Delta, \cdot)$ in G. If there exists no node $w \notin R$ adjacent to a node in R then $G = R$, proving the theorem. So let $w \in G \setminus R$ be adjacent to R. By Lemma 2.22, w is of Type 3, 6 or 8 w.r.t.R. If w is of Type 6 or 8a, R is a decomposable $3PC(\Delta, \cdot)$ satisfying Condition (ii) of Definition 2.23. If all adjacent nodes to R are of Type 3 or 8b, then by Lemma 2.12, there is a node w of Type 3 w.r.t. R. By Lemma 2.8, all attachments of w to R end in a Type 8b node. Hence R is a decomposable $3PC(\Delta, \cdot)$ satisfying Condition (i) of Definition 2.23. So by Theorem 2.24, G has a 2-join, contradicting our assumption.

Case 2: G contains a nontrivial maximum basic graph R.

Let x, y be the special nodes of R and suppose that $G \neq R$. Then there exists a node $w \in G \setminus R$ that is adjacent to a node of R.

Claim 2: If w is strongly adjacent to R, then the neighbors of w in R are either a big clique, or a pair of adjacent nodes in a segment of R.

Proof of Claim 2: If $N(w) \cap R \subseteq P$, where P is a segment of R, then by Lemma 2.31 and Lemma 2.22, the neighbors of w in P are a pair of adjacent nodes. So assume that w has neighbors in distinct segments of R.

We first show that $N(w) \cap R \subseteq K$, for some big clique K of R. Assume not and let w have neighbors in segments P and Q of R such that the node set $N(w) \cap (P \cup Q)$ is not contained in a big clique of R. By Lemma 2.31, R contains a $\Sigma = 3PC(a_1a_2a_3, x)$ that contains $P \cup Q \cup \{x, y\}$ and such that P and Q belong to distinct paths of Σ. Now it follows from Lemma 2.22 that w is of Type 3 w.r.t. Σ or that $N(w) \cap R = \{x, y\}$. The case where w is of Type 3 w.r.t. Σ cannot occur since, by assumption, $N(w) \cap (P \cup Q)$ is not contained in a big clique of R. We show next that $N(w) \cap R = \{x, y\}$ cannot occur either. Assume otherwise. W.l.o.g. y is contained in P_{a_3x}. By Lemmas 2.11 and 2.22, node w is attached by a path $W = w_1, \ldots, w_m$ where w_m has a unique neighbor in $P_{a_3y} \setminus \{y, y'\}$ where y' is the neighbor of y distinct from x in Σ. Also Σ contains a y-crosspath $Y = y_1, \ldots, y_n$ where y_n is adjacent to r, s in, say, P_{a_2x}. Let Σ' be the $3PC(y_nrs, y)$ in $\Sigma \cup Y$. By Lemma 2.22, w is of Type 6b w.r.t. Σ' and has a direct connection to Σ' ending with node a_3 which is of Type 6 in Σ', a contradiction to Lemma 2.11.

Hence $N(w) \cap R \subseteq K$, for some big clique K of R. Suppose that there is a node $t \in K$ that w is not adjacent to. Let r and s be distinct nodes of K that w is adjacent to. By Lemma 2.29, R contains a hole H that contains r, s, x, y and no other node of K. If t is an endnode of a leaf segment P, then $H \cup P$ induces a $3PC(rst, \cdot)$. Otherwise, by Lemma 2.29, R contains a chordless path P from t to x that does not contain any node of K as an intermediate node. Since R is basic no node of P is adjacent to a node of $H \setminus \{x, y\}$, and hence $H \cup P$ induces a

$3PC(rst, \cdot)$. But then w and Σ contradict Lemma 2.22. So $N(w) \cap R = K$. This completes the proof of Claim 2.

By Claim 2 and symmetry, we need only consider the following four cases.

Case 2.1: $N(w) \cap R \subseteq P$, where P is an internal segment of R.

By Lemma 2.32, R contains connected triangles $T(a_1a_2b_1, c_1c_2d_1, x, y)$ such that P is the internal segment of T and there is neither a y-crosspath w.r.t. the $3PC(a_1a_2b_1, x)$ contained in T nor an x-crosspath w.r.t. the $3PC(a_1a_2b_1, y)$ contained in T, adjacent to an intermediate node of P. By Theorem 2.26, T is not degenerate. Hence T is decomposable with extension $T \cup w$. So by Theorem 2.28, G has a 2-join, contradicting our assumption.

Case 2.2: $N(w) \cap R \subseteq P$, where P is a leaf segment of R and $N(w) \cap R \not\subseteq \{x, y\}$.

W.l.o.g. P contains x. By Lemma 2.32 R contains a $\Sigma = 3PC(\Delta, x)$ in which P is one of the paths and y is contained in one of the other two paths. Also Σ has a y-crosspath and all crosspaths of Σ are y-crosspaths that do not end in P. Hence Σ is decomposable with extension $\Sigma \cup w$. So by Theorem 2.24, G has a 2-join, contradicting our assumption.

Case 2.3: $N(w) \cap R = x$.

Let $S = (N(x) \cup N(y)) \setminus ((R \setminus \{x, y\}) \cup w)$ and let $P = u_1, \ldots, u_n$ be a direct connection from w to $R \setminus S$ in $G \setminus S$. By Claim 2 and Cases 2.1 and 2.2 (with u_n playing the role of w), $N(u_n) \cap R = K$ where K is a big clique of R.

If K does not contain an endnode of a leaf segment whose other endnode is x, then $R \cup P$ is a basic graph, contradicting the assumption that R is a maximum basic graph of G. So there exists a leaf segment Q of R with endnodes x and $r \in K$. By Lemma 2.32, R contains a $\Sigma = 3PC(\Delta, x)$ in which one of the paths is Q and y is contained in one of the other two paths. Also Σ has a y-crosspath and all crosspaths of Σ are y-crosspaths that do not end in Q. Note that, by Lemma 2.22, u_n is of Type 3 w.r.t. Σ. By Lemma 2.8 and Lemma 2.22, all attachments of u_n to Σ end in Type 8 node w.r.t. Σ. If all attachments of u_n to Σ end in Type 8 nodes with a neighbor in Q, then Σ is decomposable, and by Theorem 2.24 G has a 2-join, contradicting our assumption. So there is an attachment $P' = x_1, \ldots, x_k$ of u_n to Σ such that x_k is of Type 8a w.r.t. Σ with a neighbor in $\Sigma \setminus Q$. Since x_k is not strongly adjacent to Σ, $N(x_k) \cap R$ is not a big clique, so by Claim 2 and Cases 2.1 and 2.2, x_k must be adjacent to y.

Next we show that no node of P' is adjacent to or coincident with a node in $R \setminus y$. Suppose not and let x_i be the node of P' with lowest index that is adjacent to or coincident with a node of $R \setminus y$ and let such a node be u. Node u is contained in some segment Q' ($\neq Q$) of R. By Lemma 2.31, R contains a $\Sigma'' = 3PC(\Delta, \cdot)$ that contains x and y and such that Q and Q' belong to different paths of Σ''. Note that by the choice of P', $u \notin K$. Then u_n is of Type 3 w.r.t. Σ'' and $x_1, \ldots, x_i$ is an attachment of u_n to Σ''. By Lemma 2.8 and Lemma 2.22, x_i is of Type 8 w.r.t. Σ''. Since x_i is not strongly adjacent to Σ'', $N(x_i) \cap R$ cannot be a big clique, so by Claim 2 and Cases 2.1 and 2.2, u must be x or y. Since no node of P' can be adjacent to x, $u = y$ which contradicts our choice of u.

If no node of $P \cup w \setminus u_n$ is adjacent to or coincident with a node of P', then the node set $P \cup Q \cup P' \cup \{w, x, y\}$ induces a $3PC(u_n, x)$. So let x_j be the node of P' with highest index that is adjacent to a node of $P \cup w \setminus u_n$. Let Σ' be a $3PC(\Delta, x)$ obtained from Σ by substituting P, w, x for Q. Then by Lemma 2.17, $x_j, \ldots, x_k$ is a y-crosspath w.r.t. Σ', and hence x_j is adjacent to two adjacent nodes of P, w say r and s. Let R' be a graph obtained from R by replacing Q with the path $x, w, u_1, \ldots, u_n$. It is easy to see that R' is basic. But then so is $R' \cup \{x_j, \ldots, x_k\}$, which contradicts the choice of R.

Case 2.4: $N(w) \cap R = K$, where K is a big clique of R.

By Cases 2.1, 2.2 and 2.3, we may assume w.l.o.g. that all nodes $u \in G \setminus R$ that have a neighbor in R have the property that $N(u) \cap R$ is a big clique of R. Let Q and Q' be distinct segments of R with endnodes in K. By Lemma 2.31, R contains a $\Sigma = 3PC(\Delta, \cdot)$ such that Q and Q' belong to different paths of Σ. The triangle of Σ consists of nodes of K, and hence w is of Type 3 w.r.t. Σ. By Corollary 2.14 there is an attachment $u_1, \ldots, u_n$ of w to Σ. By Lemma 2.8 and Lemma 2.22, u_n is of Type 8 w.r.t. Σ. Since u_n is not strongly adjacent to Σ, $u_n \notin R$ and $N(u_n) \cap R$ is not a big clique, a contradiction. □

3 Node Cutset Decompositions

3.1 Mickey Mouse

In this section we prove Theorem 2.3, stating that if G is an odd-signable graph containing a a Mickey Mouse but no 4-hole, then G has a triple-star cutset.

Given a Mickey Mouse $M(xyz, H_1, H_2)$, we let x_1 and x_2 be the neighbors of x in H_1 and H_2 that are distinct from y and z. We also let y_1 and z_2 be the neighbors of y and z in H_1 and H_2 that are distinct from x.

Remark 3.1 *If we add to a Mickey Mouse $M(xyz, H_1, H_2)$ an arbitrary nonempty set of edges connecting a node u in $V(H_1) \setminus N(x)$ and nodes in $V(H_2) \setminus \{x, z\}$, the resulting graph is not odd-signable.*

Proof: Indeed such a graph contains a $3PC(u, x)$. □

Lemma 3.2 *Let G be a graph obtained from a Mickey Mouse $M(xyz, H_1, H_2)$ by adding a direct connection $P = p_1, \ldots, p_n$ (possibly $n = 1$), between $V(H_1) \setminus \{y, x, x_1\}$ and $V(H_2) \setminus \{z, x, x_2\}$ avoiding $(N(x) \cup N(y) \cup N(z)) \setminus \{y_1, z_2\}$. Then either G contains a 4-hole or G is not odd-signable.*

Proof: Let G be a counterexample to the above lemma, with a minimal number of nodes. Note that an intermediate node of P may be adjacent to x_1, x_2 and no other node of M.

Claim 1: Node p_1 is of one of the following types:

-Type 1: Node p_1 has a unique neighbor in M, say p_1', and p_1' is in $V(H_1) \setminus \{y, x, x_1\}$.

-Type 2: Node p_1 has exactly two neighbors in M, say p_1', p_1'', and p_1', p_1'' are adjacent nodes of H_1.

-Type 3: Node p_1 has exactly three neighbors in M, namely x_2 in H_2 and two adjacent nodes, say p_1', p_1'' in H_1. Furthermore p_1 is not adjacent to x_1.

Proof of Claim 1: If p_1 has neighbors in H_1 that are nonadjacent and p_1 also has at least one neighbor in H_2, then by Remark 3.1, the graph induced by $V(M) \cup \{p_1\}$ is not odd-signable. If p_1 has neighbors in H_1 that are nonadjacent and p_1 has no neighbor in H_2, then the minimality of G is contradicted. So p_1 has either a unique neighbor or two adjacent neighbors in H_1. If p_1 has a unique neighbor in H_1, say p'_1, then p'_1 is in $V(H_1) \setminus \{y, x, x_1\}$ and p_1 has no neighbor in H_2, else we have a $3PC(p'_1, x)$, so p_1 is of Type 1. If p_1 has exactly two neighbors in H_1, say p'_1, p''_1, then p_1 has no neighbor in $V(H_2) \setminus \{x_2\}$, else there is a $3PC(p_1p'_1p''_1, xyz)$. If G contains no 4-hole, p_1 cannot be adjacent to both x_1 and x_2. So p_1 is of Type 2 or 3.

Claim 2: At least one of p_1, p_n is of Type 1 or 2.

Proof of Claim 2: Assume both p_1, p_n are of Type 3. So p_1 is not adjacent to x_1 and has exactly three neighbors in M, namely x_2 in H_2 and two adjacent nodes p'_1, p''_1 in H_1. Node p_n is not adjacent to x_2 and has exactly three neighbors in M, namely x_1 in H_1 and two adjacent nodes p'_n, p''_n in H_2. Now if p_1, p_n are nonadjacent, there is a $3PC(p_1p'_1p''_1, p_np'_np''_n)$ and if they are adjacent, there is a $3PC(p_n, x_2)$.

Claim 3: At least one of x_1, x_2 has no neighbor in $V(P) \setminus \{p_1, p_n\}$

Proof of Claim 3: Assume not and let p_i, p_j in $V(P) \setminus \{p_1, p_n\}$ adjacent to x_1 and x_2, such that the subpath P' of P between them is shortest. Then $V(M) \cup V(P')$ induces an even wheel with center x.

By Claim 2 and symmetry, we can assume that p_1 is of Type 1 or 2. Assume x_2 has no neighbors in $V(P) \setminus \{p_1, p_n\}$. If p_n has two neighbors in H_2, say p'_n, p''_n, we have a $3PC(xyz, p_np'_np''_n)$ and if p_n has a unique neighbor in H_2, say p'_n we have a $3PC(x, p'_n)$.

Assume finally that x_2 is adjacent to some node in $V(P) \setminus \{p_1, p_n\}$. By Claim 3, x_1 has no neighbor in $V(P) \setminus \{p_1, p_n\}$. Now by symmetry, the above argument shows that p_n is of Type 3 and we have a $3PC(p_n, x_2)$. $\square$

Proof of Theorem 2.3 Assume G is an odd-signable graph that contains no 4-hole but contains a Mickey Mouse $M(xyz, H_1, H_2)$. It is enough to show that $(N(x) \cup N(y) \cup N(z)) \setminus \{y_1, z_2\}$ is a cutset of G, separating $V(H_1) \setminus \{y, x, x_1\}$ and $V(H_2) \setminus \{z, x, x_2\}$.

Assume not: Then G contains a subgraph G' that is obtained from $M(xyz, H_1, H_2)$ by adding a direct connection $P = p_1, \ldots, p_n$ between $V(H_1) \setminus \{y, x, x_1\}$ and $V(H_2) \setminus \{z, x, x_2\}$ avoiding $(N(x) \cup N(y) \cup N(z)) \setminus \{y_1, z_2\}$. Since G' contains no 4-hole, by Lemma 3.2, G' (and hence G) is not odd-signable. $\square$

3.2 Bugs

In this section we prove Theorem 2.4 which states that if G is an odd-signable graph that contains a bug but no 4-hole, then G contains a double-star cutset.

Given a bug (H, x), let y, x_1 and x_2 be the neighbors of x in H where x_1 and x_2 are adjacent, while y is not adjacent to x_1 or x_2. Let H_1, H_2 be the holes containing x_1, x, y and x_2, x, y respectively. Finally let y_1, y_2 be the neighbors of y in H_1, H_2, distinct from x.

Remark 3.3 *If we add to a bug (H, x) an arbitrary nonempty set of edges connecting a node u in $V(H_1) \setminus (N(x) \cup N(y))$ and nodes in $V(H_2) \setminus \{x, y\}$, the resulting graph is not odd-signable.*

Proof: Indeed this graph contains a $3PC(u, y)$ or a $3PC(x_2, y)$. □

Lemma 3.4 *Let G be a graph obtained from a bug (H, x) by adding a direct connection $P = p_1, \dots, p_n$ (possibly $n = 1$), between $V(H_1) \setminus \{y_1, y, x, x_1\}$ and $V(H_2) \setminus \{y_2, y, x, x_2\}$ avoiding $N(x) \cup N(y)$. Then either G contains a 4-hole or G is not odd-signable.*

Proof: Let G be a counterexample to the above lemma with minimal number of nodes. Note that intermediate nodes of P may be adjacent to x_1, x_2, y_1, y_2 but to no other node of (H, x).

Claim 1: Node p_1 is of one of the following types:

-Type 1: Node p_1 has a unique neighbor in (H, x), say p'_1 and p'_1 is in $V(H_1) \setminus \{x_1, x, y, y_1\}$.

-Type 2: Node p_1 has exactly two neighbors in (H, x), say p'_1, p''_1 and p'_1, p''_1 are adjacent nodes of H_1.

-Type 3: Node p_1 has exactly three neighbors in (H, x), namely y_2 in H_2 and two adjacent nodes, say p'_1, p''_1 in H_1. Furthermore p_1 is not adjacent to y_1.

Proof of Claim 1: Assume that p_1 has a unique neighbor, say p'_1, in H_1. If p_1 has x_2 as unique neighbor in H_2, we have a $3PC(p'_1, x_2)$ and if p_1 has a neighbor in $V(H_2) \setminus \{x_2\}$ we have a $3PC(p'_1, y)$. So p_1 is of Type 1 in this case.

Assume that p_1 has exactly two neighbors, say p'_1 and p''_1, in H_1 and p'_1, p''_1 are adjacent. If p_1 has a neighbor in $V(H_2) \setminus \{y_2\}$, we have a $3PC(p_1p'_1p''_1, xx_1x_2)$ if p_1 is not adjacent to x_1, and an even wheel with center x_1 otherwise. So y_2 is the only node of H_2 that may be adjacent to p_1. Since G contains no 4-hole, p_1 cannot be adjacent to both y_1 and y_2. So p_1 is of Type 2 or 3 in this case.

Assume finally that p_1 has two nonadjacent neighbors in H_1. If p_1 has no neighbor in H_2, the minimality of G is contradicted. If p_1 has a neighbor in H_2, by Remark 3.3 the graph induced by $V(H) \cup \{x, p_1\}$ is not odd-signable and this completes the proof of Claim 1.

Claim 2: No node in $V(P) \setminus \{p_1, p_n\}$ is adjacent to x_1, x_2, y_1 or y_2.

Proof of Claim 2: We first show that no node in $V(P) \setminus \{p_1, p_n\}$ is adjacent to x_1 or x_2.

Assume that p_i, $2 \le i \le n-1$, is the node of highest index adjacent to x_1. Let p_j be the node of lowest index $j \ge i$ adjacent to a node in $\{y_1\} \cup V(H_2) \setminus \{x_2\}$. If p_j is adjacent to y_1, there is a $3PC(x_1, y_1)$ and if p_j is not adjacent to y_1 there is a $3PC(x_1, y)$. By symmetry, this shows that no node in $V(P) \setminus \{p_1, p_n\}$ is adjacent to x_1 or x_2.

If both y_1, y_2 have a neighbor in $V(P) \setminus \{p_1, p_n\}$, there is a $3PC(y_1, y_2)$.

Assume that y_2 has a neighbor in $V(P) \setminus \{p_1, p_n\}$ but that y_1 does not. If p_1 is of Type 1, there is a $3PC(y, p'_1)$ and if p_1 is of Type 2 or 3, there is a $3PC(x_1x_2x, p_1p'_1p'_2)$ when p_1 is not adjacent to x_1 and an even wheel with center x_1 otherwise. By symmetry, this completes the proof of Claim 2.

Now if both p_1, p_n are of Type 3, there is a $3PC(p_1, y_1)$. So assume w.l.o.g. that p_n is not of Type 3. Now if p_1 is of Type 2 or 3 there is a $3PC(x_1x_2x, p_1p'_1p''_1)$ or an even wheel with center x_1, and if p_1 is of Type 1 there is a $3PC(p'_1, y)$. □

Proof of Theorem 2.4 Assume G is an odd-signable graph containing a bug (H, x) but no 4-hole. If $N(x) \cup N(y)$ is not a cutset of G, separating $V(H_1) \setminus \{y_1, y, x, x_1\}$ and $V(H_2) \setminus$

$\{y_2, y, x, x_2\}$, then G contains an induced subgraph G' that satisfies the conditions of Lemma 3.4. Since G' contains no 4-hole, by Lemma 3.4, G' (and hence G) is not odd-signable. □

3.3 Wheels

In this section we prove Theorem 2.5, which states that if G is an odd-signable graph that contains a proper wheel but no 4-hole, gem, Mickey Mouse or bug, then G has a star cutset.

Remark 3.5 *Let (H, x) be an odd-signable proper wheel that is not a bug and let u be an intermediate node in some long sector S_u of (H, x). Let x_1, x_2 be the endnodes of S_u and let v_1, v_2 be the neighbors of x_1, x_2 in H that are not in S_u. The only way of adding to (H, x) a nonempty set of edges connecting u and nodes of $V(H) \setminus V(S_u)$ to obtain an odd-signable graph is to add both edges uv_1 and uv_2.*

Proof: Let G be an odd-signable graph obtained from an odd-signable proper wheel (H, x) by adding a nonempty set of edges connecting u and nodes of $V(H) \setminus V(S_u)$. Since (H, x) is proper and is neither a bug nor an even wheel, x has a neighbor on H distinct from x_1, x_2, v_1, v_2. Since G contains no $3PC(u, x)$, $\{x_1, x_2, v_1, v_2\}$ is a cutset of G, separating the intermediate nodes of S_u from the rest of the wheel. So the only possible edges are uv_1 and uv_2. If only one of them exists and (H, x) has more that three spokes, there is an even wheel, otherwise if (H, x) has three spokes, all the three sectors must be long and there is a $3PC(\cdot, \cdot)$. □

Theorem 3.6 *Let (H, x) be a proper wheel with the smallest number of spokes in an odd-signable graph G. If G contains no gem, Mickey Mouse or bug as induced subgraph, then (H, x) contains at least three long sectors and no connected component of $G \setminus N(x)$ contains the intermediate nodes of two distinct long sectors.*

Proof: In an odd-signable graph G containing no gem, Mickey Mouse or bug as induced subgraph, each short sector of a wheel (C, u) is adjacent to exactly one other short sector. Since (C, u) is not an even wheel, this shows that the number of long sectors of (C, u) is odd and greater than 1.

Now assume that the theorem is false. Then G contains a direct connection $P = p_1, \ldots, p_n$ (possibly $n = 1$), between two long sectors of (H, x) and avoiding $N(x)$.

Claim 1: Every long sector of (H, x) contains an intermediate node that is adjacent to p_1 or p_n.

Proof of Claim 1: Let S be a long sector of (H, x) that does not contain an intermediate node adjacent to p_1 or p_n. Let x_i, x_{i+1} be the endnodes of S and let Q be a shortest $x_i x_{i+1}$-path in $P \cup H$ that misses x, all intermediate nodes of S and at least one neighbor of x in H. If no intermediate node of Q is adjacent to x, we have a $3PC(x_i, x_{i+1})$. Otherwise we have a proper wheel that has less spokes that (H, x), a contradiction to your choice.

Note that Claim 1 and the fact that (H, x) contains at least 3 long sectors implies that $n = 1$.

Assume first that some long sector, say S with endnodes x_i, x_{i+1} has a unique node, say p'_1, that is adjacent to p_1. By Claim 1, p'_1 is intermediate in S. Let v_i, v_{i+1} be the neighbors of x_i, x_{i+1} in $H \setminus S$. Then all the neighbors of p_1 in H are contained in $V(S) \cup \{v_i, v_{i+1}\}$, else there is a $3PC(p'_1, x)$. Now p_1 is adjacent to both v_i, v_{i+1} by Claim 1 and the fact that (H, x) has at least three long sectors. Since G contains no 4-hole, we can assume w.l.o.g. that p'_1 and x_i are nonadjacent and we have a $3PC(p'_1, x_i)$.

So every long sector of (H, x) has at least two neighbors of p_1. Since G has no 4-hole, p_1 is adjacent to at most one neighbor of x on H and therefore there is some long sector S with endnodes x_i, x_{i+1} such that p_1 is adjacent to neither x_i nor x_{i+1}. Since short sectors of (H, p_1) come in pairs, p_1 has nonadjacent neighbors in S. By Remark 3.5 applied to the graph obtained from (H, x) by adding p_1 and removing the intermediate nodes of $S_{p_i p_{i+1}}$, we have that p_1 is adjacent to v_i, v_{i+1} and no other node of $H \setminus S$. But now some long sector of (H, x) has at most one neighbor of p_1, a contradiction. □

Proof of Theorem 2.5: This now follows immediately from Theorem 3.6 by considering a wheel (H, x) in G with a minimum number of spokes. □

4 2-Joins and Blocking Sequences

In this section, we consider an induced subgraph H of G that contains a 2-join $H_1|H_2$. We say that a 2-join $H_1|H_2$ *extends* to G if there exists a 2-join of G, $H'_1|H'_2$ with $H_1 \subseteq H'_1$ and $H_2 \subseteq H'_2$. We characterize the situation in which the 2-join of H does not extend to a 2-join of G.

Definition 4.1 *A* blocking sequence *for a 2-join $H_1|H_2$ of a subgraph H of G is a sequence of distinct nodes $x_1, \ldots, x_n$ in $G \setminus H$ with the following properties:*

1. *i) $H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$,*
   ii) $H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, and
   iii) if $n > 1$ then, for $i = 1, \ldots, n-1$, $H_1 \cup x_i|H_2 \cup x_{i+1}$ is not a 2-join of $H \cup \{x_i, x_{i+1}\}$.
2. *$x_1, \ldots, x_n$ is minimal with respect to Property 1, in the sense that no sequence $x_{j_1}, \ldots, x_{j_k}$ with $\{x_{j_1}, \ldots, x_{j_k}\} \subset \{x_1, \ldots, x_n\}$, satisfies Property 1.*

Blocking sequences were introduced and studied by Geelen in [11]. Many of the results we show here were first proved in a different setting in [11].

Let H be an induced subgraph of G with 2-join $H_1|H_2$ and special sets A, B, C, D.

In the following remarks and lemmas, we let $S = x_1, \ldots, x_n$ be a blocking sequence for the 2-join $H_1|H_2$ of a subgraph H of G.

Remark 4.2 *$H_1|H_2 \cup u$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_1 = \emptyset, A$ or C. Similarly $H_1 \cup u|H_2$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_2 = \emptyset, B$ or D.*

Proof: Follows from the definition of a 2-join. □

Lemma 4.3 *If $n > 1$ then, for every node x_j, $j \in \{1, \dots, n-1\}$, $N(x_j) \cap H_2 = \emptyset, B$ or D, and for every node x_j, $j \in \{2, \dots, n\}$, $N(x_j) \cap H_1 = \emptyset, A$ or C.*

Proof: If for some $j = 1, \dots, n-1$, $H_1 \cup x_j | H_2$ is not a 2-join then $x_1, \dots, x_j$ satisfies Property 1 of Definition 4.1 and contradicts the minimality of $x_1, \dots, x_n$. Similarly if for some $j = 2, \dots, n$, $H_1 | H_2 \cup x_j$ is not a 2-join then $x_j, \dots, x_n$ contradicts the minimality of $x_1, \dots, x_n$. So the result follows from Remark 4.2. □

Lemma 4.4 *Assume $n > 1$. Nodes x_i, x_{i+1}, $1 \leq i \leq n-1$, are not adjacent if and only if $N(x_i) \cap H_2 = B$ and $N(x_{i+1}) \cap H_1 = A$, or $N(x_i) \cap H_2 = D$ and $N(x_{i+1}) \cap H_1 = C$.*

Proof: By Lemma 4.3, $N(x_i) \cap H_2 = \emptyset, B$ or D, and $N(x_{i+1}) \cap H_1 = \emptyset, A$ or C. Since $x_i x_{i+1}$ is not an edge, and $H_1 \cup x_i | H_2 \cup x_{i+1}$ is not a 2-join in $H \cup \{x_i, x_{i+1}\}$, the lemma follows. □

Theorem 4.5 *Let H be an induced subgraph of graph G that contains a 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G if and only if there exists no blocking sequence for $H_1|H_2$ in G.*

Proof: If a blocking sequence exists it is clearly not possible to extend the 2-join $H_1|H_2$ to a 2-join of G. To prove the converse, assume that there is no blocking sequence for $H_1|H_2$ in G. Let the directed graph G' be constructed as follows. G' contains two special nodes h_1 and h_2, together with all nodes in $G \setminus H$. If for a node $u \in G \setminus H$, $H_1|H_2 \cup u$ is not a 2-join in $H \cup u$, then add directed edge $h_1 u$. Similarly, if $H_1 \cup u|H_2$ is not a 2-join in $H \cup u$, add directed edge $u h_2$. For every pair of nodes u, v in $G \setminus H$, if $H_1 \cup u | H_2 \cup v$ is not a 2-join in $H \cup \{u, v\}$, add directed edge uv. By construction, since G contains no blocking sequence for $H_1|H_2$, G' contains no directed path from h_1 to h_2. We now prove the following:

Let X be the set of nodes reachable from h_1 by directed paths in G'. Then $H_1 \cup X | G \setminus (H_1 \cup X)$ is a 2-join in G.

Claim 1: For every node $u \in G \setminus (H_1 \cup X)$, $N(u) \cap H_1 = \emptyset, A$ or C.

Proof of Claim 1: The claim is certainly true for every node $u \in H_2$. For all other nodes in $G \setminus (H_1 \cup X)$ if the claim were false there would exist an edge from node h_1 to u, contradicting the maximality of X.

Claim 2: For every node $u \in H_1 \cup X$, $N(u) \cap H_2 = \emptyset, B$ or D.

Proof of Claim 2: If $u \in H_1$, then the claim clearly holds. Assume $u \in X$. Since there is no direct path in G' from h_1 to h_2, there is no edge from u to h_2. Hence, $H_1 \cup u | H_2$ is a 2-join and so the claim follows.

Let $H_1' = H_1 \cup X$ and $H_2' = G \setminus H_1'$. Let A' (resp. C') be the set of nodes $u \in H_1'$ such that $N(u) \cap H_2 = B$ (resp. D). Let B' (resp. D') be the set of nodes $u \in H_2'$ such that $N(u) \cap H_1 = A$ (resp. C). Note that, by definition, $A' \cap C' = \emptyset$ and $B' \cap D' = \emptyset$.

Claim 3: $H_1'|H_2'$ is a 2-join of G.

Proof of Claim 3: Let $u \in H_1'$ and $v \in H_2'$. We show that uv is an edge if and only if either $u \in A'$ and $v \in B'$, or $u \in C'$ and $v \in D'$.

First, assume that uv is an edge. If $u \notin A' \cup C'$ then, by Claim 2, $N(u) \cap H_2 = \emptyset$ and consequently $v \notin H_2$. If $u \in H_1$, then by Claim 1, $N(v) \cap H_1 = A$ or C and hence $u \in A' \cup C'$, which contradicts the assumption. So, $u \notin H_1$ and $v \notin H_2$. But $H_1 \cup u|H_2 \cup v$ is not a 2-join, hence uv is a directed edge in G', contradicting the assumption that $v \in H_2'$. Hence $u \in A' \cup C'$. W.l.o.g. assume that $u \in A'$. Suppose that $v \notin B'$. Since $N(u) \cap H_2 = B$, node $v \notin H_2$. Also $u \notin H_1$, since otherwise, by Claim 1, $N(v) \cap H_1 = A$ and hence $v \in B'$, a contradiction. So $u \notin H_1$ and $v \notin H_2$. But $H_1 \cup u|H_2 \cup v$ is not a 2-join, hence uv is a directed edge in G', contradicting the assumption that $v \in H_2'$.

To prove the converse, suppose that uv is not an edge and, w.l.o.g., $u \in A'$ and $v \in B'$. Then $N(u) \cap H_2 = B$ and $N(v) \cap H_1 = A$, so $u \notin H_1$ and $v \notin H_2$. But $H_1 \cup u|H_2 \cup v$ is not a 2-join, so uv is a directed edge in G' which contradicts the assumption that $v \in H_2'$. □

Lemma 4.6 *For* $1 \leq i \leq n$, $H_1 \cup \{x_1, \ldots, x_{i-1}\}|H_2 \cup \{x_{i+1}, \ldots, x_n\}$ *is a 2-join in* $H \cup (S \setminus \{x_i\})$.

Proof: By the minimality of S, the set $S \setminus \{x_i\}$ does not contain a blocking sequence for $H_1|H_2$. So it follows from Properties 1. i) and ii) of Definition 4.1 that $H_1 \cup \{x_1, \ldots, x_{i-1}\}|H_2 \cup \{x_{i+1}, \ldots, x_n\}$ is the only possible extension of $H_1|H_2$. □

Lemma 4.7 *If* $x_i x_k$, $n \geq k > i+1 \geq 2$, *is an edge then either* $N(x_i) \cap H_2 = B$ *and* $N(x_k) \cap H_1 = A$, *or* $N(x_i) \cap H_2 = D$ *and* $N(x_k) \cap H_1 = C$.

Proof: By Lemma 4.6, $H_1 \cup \{x_1, \ldots, x_i\}|H_2 \cup \{x_{i+2}, \ldots, x_n\}$ is a 2-join in $H \cup S \setminus x_{i+1}$. Let the 2-join have special sets A', B', C', D'. Since $x_i x_k$ is an edge, either $x_i \in A'$ and $x_k \in B'$, or $x_i \in C'$ and $x_k \in D'$. Since $A \subseteq A'$, $B \subseteq B'$, $C \subseteq C'$ and $D \subseteq D'$, the lemma follows. □

Lemma 4.8 *Let* x_j *be the node of lowest index adjacent to a node in* H_2. *Then* $x_1, \ldots, x_j$ *is a chordless path.*

Proof: If $j = 1$ then the claim holds. Suppose now $j > 1$. If $x_i x_{i+1}$, $i \in \{1, \ldots, j-1\}$ is not an edge, then by Lemma 4.4, x_i is adjacent to a node in H_2, contradicting the choice of x_j. If $x_i x_k$ is an edge, $2 \leq i+1 < k \leq j$ then by Lemma 4.7, x_i must be universal for B or D, contradicting the choice of x_j. Thus $x_1, \ldots, x_j$ is a chordless path. □

Theorem 4.9 *Let* G *be a graph and* H *an induced subgraph of* G *with 2-join* $H_1|H_2$ *and special sets* A, B, C, D. *Let* H' *be an induced subgraph of* G *with 2-join* $H_1'|H_2$ *and special sets* A', B, C', D *such that* $A' \cap A \neq \emptyset$ *and* $C' \cap C \neq \emptyset$. *If* S *is a blocking sequence for* $H_1|H_2$ *and* $H_1' \cap S \neq \emptyset$, *then a proper subset of* S *is a blocking sequence for* $H_1'|H_2$.

Proof: Let $S = x_1, \ldots, x_n$ be a blocking sequence for $H_1|H_2$ such that $H_1' \cap S \neq \emptyset$. Let $x_j \in S$ be the node of highest index that belongs to H'. Note that $j \neq n$ since otherwise by Remark 4.2 $N(x_n) \cap H_2 \neq \emptyset, B$ or D, and consequently $H_1'|H_2$ is not a 2-join with special sets B and D in H_2. The proof of the theorem follows from the following two claims.

Claim 1: $H_1'|H_2 \cup x_{j+1}$ is not a 2-join in the graph $H' \cup x_{j+1}$.

Proof of Claim 1: Assume the contrary. By the definition of a blocking sequence $H_1 \cup x_j | H_2 \cup x_{j+1}$ is not a 2-join in $H \cup \{x_j, x_{j+1}\}$.

If $x_j x_{j+1}$ is not an edge then, by Lemma 4.4, $N(x_j) \cap H_2 = B$ (or D) and $N(x_{j+1}) \cap H_1 = A$ (or C resp.). Thus $x_j \in A'$ or C', assume w.l.o.g. $x_j \in A'$. Since $A \cap A' \neq \emptyset$, x_{j+1} is adjacent to a node in $A' \subseteq H'_1$ but not universal for A', contradicting the assumption that $H'_1 | H_2 \cup x_{j+1}$ is a 2-join.

If $x_j x_{j+1}$ is an edge, our assumption that $H'_1 | H_2 \cup x_{j+1}$ is a 2-join implies that $x_j \in A'$ (or C') and $N(x_{j+1}) \cap H'_1 = A'$ (or C' resp.). Assume w.l.o.g. that $x_j \in A'$ and $N(x_{j+1}) \cap H'_1 = A'$. Since $A' \cap A \neq \emptyset$, x_{j+1} is adjacent to a node in A. By Lemma 4.3, $N(x_{j+1}) \cap H_1 = A$. But now $H_1 \cup x_j | H_2 \cup x_{j+1}$ is a 2-join in $H \cup \{x_j, x_{j+1}\}$, contradicting the definition of a blocking sequence. This completes the proof of Claim 1.

Claim 2: Let x_l be the node of highest index such that $H'_1 | H_2 \cup x_l$ is not a 2-join in $H' \cup x_l$ (note that by Claim 1 such an x_l must exist). Then $x_l, \ldots, x_n$ contains a blocking sequence for $H'_1 | H_2$.

Proof of Claim 2: To show that $x_l, \ldots, x_n$ contains a blocking sequence for $H'_1 | H_2$ it is sufficient to show that it satisfies the properties 1. i), ii) and iii) of Definition 4.1. By assumption $H'_1 | H_2 \cup x_l$ is not a 2-join in $H' \cup x_l$, giving 1. i). We next show 1. ii). Assume $H'_1 \cup x_n | H_2$ is a 2-join in $H' \cup x_n$. Then $N(x_n) \cap H_2 = \emptyset$ or B or D. But since $H_1 \cup x_n | H_2$ is not a 2-join this is not possible.

We now show 1. iii). For all $l < i \leq n$, $N(x_i) \cap H'_1 = \emptyset$, A' or C', since otherwise x_i contradicts the choice of x_l. Since $A \cap A' \neq \emptyset$ and $C \cap C' \neq \emptyset$ and $N(x_i) \cap H_1 = \emptyset$, A or C (by Lemma 4.3), we have that $N(x_i) \cap H'_1 = \emptyset$ (resp. A' or C') if and only if $N(x_i) \cap H_1 = \emptyset$ (resp. A or C). But then $H'_1 \cup x_{i-1} | H_2 \cup x_i$ is a 2-join in $H' \cup \{x_{i-1}, x_i\}$ if and only if $H_1 \cup x_{i-1} | H_2 \cup x_i$ is a 2-join in $H \cup \{x_{i-1}, x_i\}$. So 1. iii) holds. This completes the proof of Claim 2. □

5 2-Join Decompositions

In this section, we decompose connected diamonds, decomposable $3PC(\Delta, \cdot)$'s and decomposable connected triangles by 2-joins. Throughout the section, we assume that G is an odd-signable graph that does not contain a 4-hole and does not have a star, double-star or triple-star cutset (and therefore does not contain a gem, a Mickey Mouse or a proper wheel).

5.1 Connected Diamond

Recall (Definition 2.19) that a connected diamond is a $\Sigma = 3PC(d_1 d_2 c_1, y)$ together with a Type 7 node c_2 adjacent to d_1, d_2 and an attachment $Y = y_1, \ldots, y_m$. By Lemma 2.9, y_m is of Type 4 or 6 with respect to Σ. We introduce some additional notation. Let $a_2 = y_m$ and let a_1 be the closest neighbor of a_2 to c_1 in $P_{c_1 y}$. Now let $A = \{a_1, a_2\}$, $B = V(\Sigma) \cap N(a_2) \setminus \{a_1\}$, $C = \{c_1, c_2\}$, $D = \{d_1, d_2\}$. The connected diamond $\Sigma \cup Y$ is denoted by $H(A, B, C, D)$. When y_m is of Type 4, B has cardinality 2 and we let $B = \{b_1, b_2\}$, whereas when y_m is of Type 6, B has cardinality 1 and we let $b_1 = b_2$ denote its unique node. Let $H_1 = P_{a_1 c_1} \cup P_{a_2 c_2}$ and $H_2 = H(A, B, C, D) \setminus H_1$. When $|B| = 2$, H_2 consists of two node-disjoint paths, say

$P_{b_1d_1}$ and $P_{b_2d_2}$. When $|B| = 1$, these two paths are identical between $b_1 = b_2$ and y. Note that since G does not contain proper wheels, $P_{b_1d_1}$ and $P_{b_2d_2}$ have length greater than 1 and, if $|B| = 2$, both $P_{a_1c_1}$, $P_{a_2c_2}$ have length greater than 1.

We denote by Σ_1 the $3PC(a_1a_2b_1, d_1)$ induced by $H_1 \cup P_{b_1d_1}$ and by Σ_2 the $3PC(a_1a_2b_2, d_2)$ induced by $H_1 \cup P_{b_2d_2}$. Σ' denotes the $3PC(d_1d_2c_2, y)$ when $|B| = 1$ and the $3PC(d_1d_2c_2, a_2)$ when $|B| = 2$. We denote by a_1' the neighbor of a_1 in $P_{a_1c_1}$, and we define a_2', b_1', b_2', c_1', c_2', d_1' and d_2' similarly. Finally, when $|B| = 1$, we let y_1', y_2' be the neighbors of y in P_{d_1y} and P_{d_2y} respectively and, if $y \neq b_1$, we let y' denote the neighbor of y in P_{b_1y}.

Lemma 5.1 *A node u strongly adjacent to a connected diamond $H(A, B, C, D)$ is one of the following types:*

Type a: $N(u) \cap H = A$

Type b: $N(u) \cap H = A \cup B$

Type c: $N(u) \cap H = C \cup D$

Type d: $N(u) \cap H = D$

Type e: $N(u) \cap H$ consists of two adjacent nodes of $P_{a_1c_1}$ or $P_{a_2c_2}$ or $P_{b_1d_1}$ or $P_{b_2d_2}$.

Type f: Node u is a twin of a node in H.

Type g: Node u has three neighbors in H, either the two nodes of D and one node in C or, if $|B| = 2$, the two nodes of A and one node in B.

Type h: $|B| = 1$ and u has two neighbors in H, the node of B and one node of A.

Type i: $|B| = 1$ and u has three neighbors in H: y and two nodes among y_1', y_2' and y'. If u is adjacent to y', then $y \neq b_1$.

Type j: $|B| = 1$, $y = b_1$, and u has four neighbors in H: a_1, a_2, b_1 and either y_1' or y_2'.

Proof: Let u be a node that is strongly adjacent to $H(A, B, C, D)$. Assume first that u is not strongly adjacent to Σ or Σ'. Then u has exactly one neighbor in $P_{a_1c_1}$ and one in $P_{a_2c_2}$. By Lemma 2.7 applied to Σ_1, u is of Type 7 for Σ_1 and therefore u is of Type a in H. By symmetry between Σ and Σ', we now assume w.l.o.g. that u is strongly adjacent to Σ. We examine all the possibilities of Lemma 2.7.

Assume u is of Type 1 in Σ. If u is a twin of d_1 or d_2 in Σ, then by Lemma 2.7 applied to Σ', u is adjacent to c_2 and no other node of H. So u is of Type f in this case. If u is a twin of c_1 in Σ, then u must be of Type 5 relative to Σ_1, so u is a twin of c_1 in H, i.e. u is of Type f.

Assume u is of Type 2 in Σ. If $|B| = 2$ or if $|B| = 1$ and $b_1 = y$, by Lemma 2.7 applied to Σ_1, u is adjacent to a_2 and to no other node of H and u is of Type f. If $|B| = 1$ and $b_1 \neq y$ by Lemma 2.7 applied to Σ', u has no other neighbor in H and u is of Type f.

Assume u is of Type 3 in Σ. By Lemma 2.7 applied to Σ_1, we have that u is of Type c or g in H.

Assume u is of Type 4. If $|B| = 2$ then, by Lemma 2.7 applied to Σ_1, u is adjacent to a_2 and therefore by Lemma 2.7 applied to Σ_2, u is not adjacent to a'_1. So u is adjacent to a_1, b_1, b_2 and, by Lemma 2.7 applied to Σ', u is also adjacent to a_2. So u is of Type b or f. If $|B| = 1$ and $y \neq b_1$, by Lemma 2.7 applied to Σ', u has no other neighbor in H and u is of Type i. If $|B| = 1$ and $y = b_1$, we distinguish two cases. First, if u is adjacent to y'_1 and y'_2, then by Lemma 2.7 applied to Σ', u has no other neighbor in H and u is of Type i. Now consider the case where u is adjacent to y'_1 and a_1. Then u is also adjacent to a_2, since otherwise there is a gem. By Lemma 2.7 applied to Σ', u has no other neighbor in H and u is of Type j.

Assume u is of Type 5 in Σ. If u is not a twin of a_1 or b_1 or b_2 then, by Lemma 2.7 applied to Σ_1, u has no other neighbor in H, and if u is a twin of a_1, b_1 or b_2, then u must also be adjacent to a_2 and no other node. So u is of Type f.

Assume u is of Type 6 in Σ. If $|B| = 2$, by Lemma 2.7 applied to Σ', Σ_1 and Σ_2, u has no other neighbor in H (and u is of Type e), except when u is adjacent to a_1 and either b_1 or b_2. Now u must be adjacent to a_2 (else there is a gem) and to no other node. So u is of Type g. If $|B| = 1$, by Lemma 2.7 applied to Σ', Σ_1 and Σ_2, u has no other neighbor in H, except when u is adjacent to a_1 and b_1. Then u may be adjacent to only a_2 (and u is of Type b) or to a_2 and y_{m-1} (and u is of Type f). If u has no other neighbors in H, u is of Type h.

Assume u is of Type 7 in Σ. Since G has no gem, by Lemma 2.7 applied to Σ', Σ_1 and Σ_2, u is of Type d, g or f in H. □

Lemma 5.2 *If a node u is of Type g or h w.r.t. a connected diamond H, then there exists a connected diamond H' with $H_2 \subseteq H'$ and $u \in H'_1 = H' \setminus H_2$. Furthermore, $H'_1|H_2$ is a 2-join of H' with special sets A', B, C', D such that $A \cap A' \neq \emptyset$ and $C \cap C' \neq \emptyset$.*

Proof: First assume that u is of Type g, w.l.o.g. adjacent to d_1, d_2 and c_1. Then u is of Type 6b w.r.t. both Σ_1 and Σ_2. Let $S = (N(c_1) \cup N(d_1) \cup N(d_2)) \setminus \{u, c'_1, c_2, d'_1, d'_2\}$ and let $P = p_1, \ldots, p_k$ be a direct connection from u to $H \setminus S$ in $G \setminus S$. W.l.o.g. p_k has a neighbor in Σ_1. By Lemma 2.11, p_k is of Type 7 (adjacent to a_2 and b_1), Type 1 (with a neighbor in $P_{a_1c_1} \setminus a_1$), Type 5a or 8a (with neighbors in $P_{a_1c_1}$) w.r.t. Σ_1. By substituting u, P for a subpath of $P_{a_1c_1}$, we obtain the desired H'.

Now assume that u is of Type h, w.l.o.g. adjacent to a_1 and b_1. Then u is of Type 7 w.r.t. both Σ_1 and Σ_2. Let $S = (N(a_1) \cup N(a_2) \cup N(b_1)) \setminus \{u, a'_1, a'_2, b'_1, b'_2\}$ and let $P = p_1, \ldots, p_k$ be a direct connection from u to $H \setminus S$ in $G \setminus S$. W.l.o.g. p_k has a neighbor in Σ_1. By Lemma 2.9, p_k is of Type 4 or 6 w.r.t. Σ_1.

By Lemma 5.1, if p_k is of Type 4 w.r.t. Σ_1 (i.e. p_k is adjacent to c_1, c_2 and d_1), then P_k is of Type c in H (i.e. p_k is also adjacent to d_2). If p_k is of Type 6 w.r.t. Σ_1 adjacent to d_1 and c_2, then p_k is of Type g w.r.t. H. So p_k is of Type 6 with both neighbors in $P_{a_2c_2} \setminus a_2$. In both cases, by substituting u, P into Σ_1, we obtain the desired H'. □

Lemma 5.3 *If a node u is of Type a, b, c, d with respect to a connected diamond H with $|B| = 2$, then there exist a connected diamond H' with $H_i \subseteq H'$ for some $i \in \{1, 2\}$, and $u \in H' \setminus H_i$. W.l.o.g. assume $i = 1$ and let $H'_2 = H' \setminus H_1$. Then $H_1|H'_2$ is a 2-join in H' with special sets A, B', C, D' where $|B'| = 2$, $B \cap B' \neq \emptyset$ and $D \cap D' \neq \emptyset$.*

Proof: We consider the following cases:

Case 1: u is of Type a or d.

By symmetry, we assume w.l.o.g. that u is of Type a. Node u is of Type 7 w.r.t. both Σ_1 and Σ_2. By Corollary 2.14, u has an attachment to both Σ_1 and Σ_2. Amongst all attachments of u to Σ_1 or Σ_2, let $P = p_1, \dots, p_k$ be the shortest. Assume w.l.o.g. that P is an attachment to Σ_1. By Lemma 2.9, p_k is of Type 4 (adjacent to c_1, c_2 and d_1) or of Type 6 (with neighbors in $P_{b_1 d_1}$) w.r.t. Σ_1.

If p_k is of Type 4 w.r.t. Σ_1, then by Lemma 5.1, p_k is either of Type c w.r.t. H or is a twin of d_2. But then by replacing $P_{b_2 d_2}$ with u, P we obtain the desired H'.

So we may assume that p_k is of Type 6 w.r.t. Σ_1. Then by Lemma 5.1, p_k is of Type e w.r.t. H, and so p_k is not adjacent to any node of $P_{b_2 d_2}$. Let C be the hole contained in $P_{b_1 d_1} \cup P \cup \{u, a_1\}$. If b_2 has a neighbor in P, then $C \cup b_2$ induces either a proper wheel (if b_2 has at least two neighbors in P) or a $3PC(a_1, \cdot)$. So b_2 does not have a neighbor in P. If a node of P is adjacent to or coincident with a node of $P_{b_2 d_2}$, then a proper subpath of P is an attachment of u to Σ_2, contradicting our choice of P. Hence no node of P is adjacent to or coincident with a node of $P_{b_2 d_2}$. But then by substituting u, P into Σ_1 and keeping Σ_2 the same, we obtain the desired H'.

Case 2: u is of Type b or c.

By symmetry, we assume w.l.o.g. that u is of Type c. Then u is of Type 4 w.r.t. both Σ_1 and Σ_2. Let $S = (N(d_1) \cup N(d_2)) \setminus u$ and let $P_u = p_1, \dots, p_k$ be a direct connection from u to $H \setminus S$ in $G \setminus S$. W.l.o.g. p_k has a neighbor in Σ_1. By Lemma 2.10 and Lemma 5.1, p_k is either of Type 7 w.r.t. both Σ_1 and Σ_2, or p_k is of Type 5a or 8a w.r.t. Σ_1 with all neighbors of p_k in H contained in $P_{b_1 d_1}$. So by substituting u, P for an appropriate subpath of $P_{b_1 d_1}$ we obtain the desired H'. □

Proof of Theorem 2.20: We prove that for some a connected diamond H, the 2-join $H_1|H_2$ of H extends to a 2-join of G. Assume not. Then, by Theorem 4.5, every connected diamond H has a blocking sequence for $H_1|H_2$. Consider all H such that $P_{b_1 d_1}$ and $P_{b_2 d_2}$ have as few common nodes as possible, and amongst them choose an H with a shortest blocking sequence $S = x_1, \dots, x_n$ for $H_1|H_2$.

First note that, if node x_i is of Type f w.r.t. H, then $|B| = 1$ and x_i is a twin of b_1, since otherwise by substituting x_i into H we obtain a connected diamond H' that satisfies the conditions of Theorem 4.9, and hence our choice of H is contradicted. Similarly, by Lemma 5.2, Theorem 4.9 and our choice of H, no node of S is of Type g or h.

By Lemma 5.1 and Remark 4.2, $n > 1$. Since $H_1|H_2 \cup x_1$ is not a 2-join, node x_1 cannot be of Type a, b, c, d, i or a twin of b_1. So, by Lemma 5.1, since x_1 has a neighbor in H_1, it is either not strongly adjacent to H or is of Type e. Similarly, x_n has a neighbor in H_2 and is either not strongly adjacent to H or of Type e or, in case $|B| = 1$, x_n could be a twin of b_1 or of Type i or j.

Claim 1: An intermediate node of S has a neighbor in H.

Proof of Claim 1: Assume not. Then, by Lemmas 4.4 and 4.7, S is a chordless path. W.l.o.g. assume that x_1 is adjacent to a node in $P_{a_1 c_1}$, x_n is adjacent to a node in $P_{b_1 d_1}$, and if x_n is of Type j it is adjacent to b_1'.

First, assume that r is the unique neighbor of x_1 in H. Node x_n is not a twin of b_1 or of Type j, since otherwise if $r \neq a_1$ then there is a $3PC(r, x_n)$ that contains $P_{a_1c_1}$ and S, and else either $S \cup H_1 \cup d_1$ induces a Mickey Mouse (if $n > 2$) or $\{x_1, x_n, a_1, b_1, b'_1\}$ induces a gem (if $n = 2$). Also, d_1 cannot be the unique neighbor of x_n in H, since otherwise there in a Mickey Mouse when $r = c_1$ and a $3PC(r, d_1)$ when $r \neq c_1$. Hence, by Lemma 2.17, S is a crosspath w.r.t. Σ_1. Hence by Lemma 2.17, $r = c_1$ and x_n has two neighbors in $P_{b_1d_1}$ that are adjacent. So if $|B| = 2$ or if $|B| = 1$ and x_n has two adjacent neighbors in the d_1d_2-path of H_2, then $S \cup H_2 \cup \{a_1, c_1\}$ induces a $3PC(\Delta, \Delta)$. So either x_n has two neighbors in P_{b_1y} or x_n is of Type i, adjacent to y, y'_1 and y'_2. In both cases, the choice of H is contradicted.

Now, assume that x_1 is of Type e. If x_n is a twin of b_1 or of Type j, then $H_1 \cup S \cup d_1$ induces either a $3PC(\Delta, \Delta)$ or an even wheel with center a_1. If d_1 is the unique neighbor of x_n in H, then $P_{a_1c_1} \cup P_{b_2d_2} \cup S$ induces either a $3PC(\Delta, \Delta)$ or an even wheel with center c_1. Hence, by Lemma 2.17, S is a crosspath w.r.t. Σ_1. So d'_1 is the unique neighbor of x_n in H. If $|B| = 2$, then $d_1, d'_1, x_n, \ldots, x_1$ contradicts Lemma 2.10 applied to Σ_2. Otherwise, S and a subpath of $P_{b_1d_1} \setminus d_1$ contradict Lemma 2.17 applied to Σ_2. This completes the proof of Claim 1.

By Claim 1, let x_i be the node of lowest index in $S \setminus \{x_1, x_n\}$ that is adjacent to a node in H. By Lemmas 4.3 and 5.1, x_i is either of Type a, b, c or d w.r.t. H, or $|B| = 1$ and b_1 is the unique neighbor of x_i in H. Then, by Lemma 5.3 and Theorem 4.9, the case $|B| = 2$ cannot occur. By Lemmas 4.4 and 4.7, $x_1, \ldots, x_i$ is a chordless path. We assume w.l.o.g. that x_1 is adjacent to a node in $P_{a_1c_1}$.

Case 1: x_i is of Type a.

Then x_i is of Type 7 w.r.t. Σ_1. If a_1 is the unique neighbor of x_1 in H, then either $H_1 \cup \{d_1, x_1, \ldots, x_i\}$ induces a Mickey Mouse (if $i \neq 2$) or $\{a_1, a_2, b_1, x_1, x_i\}$ induces a gem (if $i = 2$). Otherwise, Σ_1, x_i and the path $x_{i-1}, \ldots, x_i$ contradict Lemma 2.9.

Case 2: x_i is of Type b.

Node a_1 is the unique neighbor of x_1 in H, since otherwise, by substituting $x_1, \ldots, x_i$ into H for an appropriate subpath of $P_{a_1c_1}$, we obtain a connected diamond that satisfies the conditions of Theorem 4.9, contradicting our choice of H. If $i > 2$, then $H_1 \cup \{d_1, x_1, \ldots, x_i\}$ is a Mickey Mouse. Hence, $i = 2$. Let $S = (N(a_1) \cup N(a_2) \cup N(b_1)) \setminus \{x_1, a'_1, a'_2, b'_1, b'_2\}$ and let $P = p_1, \ldots, p_k$ be a direct connection from x_1 to $H \setminus S$ in $G \setminus S$. W.l.o.g. p_k has a neighbor in Σ_1. Then x_1, P or a subpath of it (in case x_i has a neighbor in P) is an attachment of x_i to Σ_1. By Lemma 2.8, p_k is of Type 2, 5 or 8 w.r.t. Σ_1. If p_k is of Type 2 w.r.t. Σ_1, then $P_{a_1c_1} \cup (P_{b_1d_1} \setminus d_1) \cup P \cup x_1$ is a $3PC(a_1, p_k)$. If p_k has a neighbor in $P_{b_1d_1} \setminus d_1$ or $P_{a_2c_2}$, then x_1, P contradicts Lemma 2.17 applied to Σ_1. Hence, the neighbors of p_k in Σ_1 are contained in $P_{a_1c_1} \cup d_1$. If p_k is of Type 5 w.r.t. Σ_1, then $T = P_{a_1c_1} \cup P_{b_1d_1} \cup P \cup x_1$ contains a $3PC(a_1, p_k)$. So let r be the unique neighbor of p_k in Σ_1. If $r \neq a'_1$ then T is a $3PC(a_1, r)$, and otherwise $T \cup x_i$ contains a proper wheel with center a_1.

Case 3: x_i is of Type c.

Then x_i is of Type 4 w.r.t. both Σ_1 and Σ_2. If x_1 has a neighbor in $P_{a_1c_1} \setminus c_1$, then Lemma 2.10 is contradicted. So c_1 is the unique neighbor of x_1 in H. But then either $P_{a_1c_1} \cup P_{b_1d_1} \cup \{x_1, \ldots, x_i\}$ is a Mickey Mouse (if $i > 2$) or $\{c_1, c_2, d_2, x_1, x_i\}$ induces a gem (if $i = 2$).

Case 4: x_i is of Type d.

Let Σ' be the $3PC(c_1d_1d_2, y)$ induced by $H_2 \cup P_{a_1c_1}$. Then x_i is of Type 7 w.r.t. Σ'. Let $S = (N(c_1) \cup N(d_1) \cup N(d_2)) \setminus \{x_i, c_1', d_1', d_2'\}$ and let $P = p_1, \ldots, p_k$ be a direct connection from x_i to $H \setminus S$ in $G \setminus S$. First suppose that all the neighbors of p_k in H are contained in $P_{a_2c_2}$. Then H' obtained from H by substituting x_i, P for an appropriate subpath of $P_{a_2c_2}$ satisfies the conditions of Theorem 4.9, and hence contradicts our choice of H. So p_k has a neighbor in Σ'. By Lemma 2.9, p_k is either of Type 6 w.r.t. Σ' with neighbors in P_{c_1y} path of Σ' or it is of Type 4 w.r.t. Σ' adjacent to y and the neighbors of y in P_{d_1y} and P_{d_2y} paths of Σ'. If the neighbors of p_k in H are contained in H_2, then $\Sigma' \cup P \cup x_i$ is a connected diamond that contradicts our choice of H. So p_k has a neighbor in $P_{a_1c_1}$, and hence it is of Type 6 w.r.t. Σ'. If p_k is adjacent to a_1 and b_1, then $\Sigma' \cup P \cup x_i$ is a connected diamond that satisfies the conditions of Theorem 4.9, and hence contradicts our choice of H. So the neighbors of p_k in H are contained in $P_{a_1c_1}$. Let r be the neighbor of p_k in $P_{a_1c_1}$ that is closest to a_1, and let P' be the ra_1-subpath of $P_{a_1c_1}$. If c_2 has a neighbor in P, then $P' \cup P \cup P_{b_1d_1} \cup c_2$ either induces a proper wheel (if c_2 has at least two neighbors in P) or a $3PC(d_2, \cdot)$. Otherwise, $H_2 \cup P_{a_2c_2} \cup P \cup P' \cup x_i$ satisfies the conditions of Theorem 4.9, and hence our choice of H is contradicted.

Case 5: b_1 is the unique neighbor of x_i in H.

Then $x_1, \ldots, x_i$ contradicts Lemma 2.17 applied to Σ_1. □

5.2 Decomposable $3PC(\Delta, \cdot)$

In this section we assume that G does not contain a connected diamond. So by Lemma 2.22, the only strongly adjacent nodes to a $3PC(\Delta, \cdot)$ are of Type 3 or 6.

Lemma 5.4 *Let u be a node of Type 3 w.r.t. $\Sigma = 3PC(a_1a_2a_3, a_4)$ and let $P = u_1, \ldots, u_n$ be an attachment of u to Σ such that u_n is of Type 8a w.r.t. Σ adjacent to a node in $P_{a_1a_4}$. Let Σ' be the $3PC(ua_2a_3, a_4)$ contained in $(\Sigma \cup P \cup u) \setminus a_1$. Then Q is a crosspath for Σ if and only if Q is a crosspath for Σ'.*

Proof: Let $Q = q_1, \ldots, q_m$ be a crosspath for Σ. First assume that Q is an a_5-crosspath. If Q is not an a_5-crosspath for Σ', then some node of Q is adjacent to or coincident with a node in $P \cup u$. Let q_i be the node of highest index adjacent to a node in $P \cup u$. Since the only strongly adjacent nodes to Σ are of Type 3 and 6, $i \neq m$. But then q_iq_m-subpath of Q contradicts Lemma 2.17 applied to Σ'.

Now assume w.l.o.g. that Q is an a_6-crosspath. Let r_1 and r_2 be the neighbors of q_m in Σ. Node u cannot be adjacent to Q, since otherwise $P_{a_1a_4} \cup P_{a_3a_4} \cup Q \cup u$ contains a $3PC(ua_1a_3, q_mr_1r_2)$ or an even wheel. Suppose that a node of P is adjacent to or coincident with a node of Q. Let q_i be the node of Q with lowest index adjacent to a node of P and let u_j be the node of P with highest index adjacent to q_i. If $i \neq m$, then the path $q_1, \ldots, q_i, u_j, \ldots, u_n$ contradicts Lemma 2.17 applied to Σ. So $i = m$. But then r_1 and r_2 are contained in $P_{a_1a_4}$, since otherwise q_m violates Lemma 2.7 in Σ'. Hence, Q is an a_6-crosspath w.r.t. Σ'. So we may assume that no node of P is adjacent to or coincident with a node of Q. If Q is not an a_6-crosspath for Σ', then q_m has a neighbor in $\Sigma \setminus \Sigma'$. But then Q and an appropriate subpath of $P_{a_1a_4}$ contradict Lemma 2.17 applied to Σ'.

The converse holds by symmetry since a_1 is of Type 3 w.r.t. Σ' attached to Σ' by the path $\Sigma \setminus \Sigma'$. □

Lemma 5.5 *Let $\Sigma = 3PC(a_1a_2a_3, a_4)$ and let $P = x_1, \ldots, x_n$ be an a_5-crosspath to $P_{a_2a_4}$. Let u be a node of Type 8b w.r.t. Σ with an attachment $Q = y_1, \ldots, y_m$ such that the neighbors of y_m in Σ are contained in $P_{a_1a_4}$. Let Σ' be the $3PC(a_1a_2a_3, a_4)$ obtained by substituting u, Q into Σ. Then Σ' has no crosspath.*

Proof: By Lemma 2.12, node y_m is of Type 6a in Σ. Let Σ'' be the $3PC(\Delta, \cdot)$ induced by $P_{a_2a_4} \cup P_{a_1a_4} \cup Q \cup u$. We first show that no node of $Q \cup u$ is adjacent to or coincident with a node of P. Node u is not adjacent to a node of P, since otherwise $P \cup P' \cup \{u, a_5\}$, where P' is an $x'a_4$-subpath of $P_{a_2a_4}$ where x' is the neighbor of x_n in $P_{a_2a_4}$ that is closer to a_4, induces a proper wheel with center u or a $3PC(\cdot, \cdot)$. Now suppose that a node of Q is adjacent to or coincident with a node of P. Let x_i be a node of P with highest index adjacent to a node of Q and let y_j be the node of Q with highest index adjacent to x_i. $i \neq 1$, since otherwise x_1 violates Lemma 2.7 in Σ''. But then the path $y_m, \ldots, y_j, x_i, \ldots, x_n$ contradicts Lemma 2.17 applied to Σ.

Hence, P is an a_5-crosspath w.r.t. Σ''. By Lemma 2.18, Σ'' has neither an u-crosspath nor an a_6-crosspath, and Σ has neither an a_6-crosspath nor an a_7-crosspath. Let y be the neighbor of y_m in $P_{a_1a_4}$ that is closer to a_4 and let P'' be the ya_5-subpath of $P_{a_1a_4}$. Suppose that $R = r_1, \ldots, r_k$ is a crosspath for Σ'.

First assume that R is a u-crosspath w.r.t. Σ'. No node of P'' is adjacent to or coincident with a node of R, since otherwise by Lemma 2.17, a subpath of R is a u-crosspath w.r.t. Σ''. Since R cannot be a u-crosspath w.r.t. Σ'', the neighbors of r_k in Σ' are contained in $P_{a_3a_4}$. But then R together with an appropriate subpath of $P_{a_3a_4}$ is a u-crosspath w.r.t. Σ''.

Now assume that R is an a_6-crosspath or an a_7-crosspath w.r.t. Σ'. No node of P'' is adjacent to or coincident with a node of R, since otherwise, by Lemma 2.17, a subpath of R is an a_6-crosspath or an a_7-crosspath w.r.t. Σ. Since R cannot be an a_6-crosspath or an a_7-crosspath w.r.t. Σ, and R cannot be an a_6-crosspath w.r.t. Σ'', R is an a_7-crosspath w.r.t. Σ' and the neighbors of r_k in Σ' are contained in Q. But then R together with an appropriate subpath of Q is an a_7-crosspath w.r.t. Σ. □

Lemma 5.6 *Let Σ be a $3PC(a_1a_2a_3, a_4)$ and let $P = x_1, \ldots, x_n$ be a chordless path with one endnode adjacent to a node in $P_{a_1a_4} \setminus \{a_4\}$, the other to a node in $P_{a_2a_4} \setminus \{a_4\}$ and no intermediate node adjacent to any node in $\Sigma \setminus \{a_4\}$. If node a_4 has a neighbor in P then a_4 has exactly one neighbor in P, the two endnodes of P are of Type 6a w.r.t. Σ, and Σ has no crosspath.*

Proof: Note that $n > 1$, since otherwise there exists a wheel with center x_1.

Claim 1: Node a_4 has at most one neighbor in P.

Proof of Claim 1: If node a_4 is adjacent to more than two nodes in P, then there exists a wheel with center a_4, a contradiction. So assume that a_4 has two neighbors in P, say x_p and x_q with $p < q$. Note that x_p and x_q must be adjacent, otherwise there exists a $3PC(x_p, x_q)$. If x_1 is of Type 8a w.r.t. Σ, adjacent to a node r in $P_{a_1a_4}$, then r is adjacent to a_4 since otherwise there exists a $3PC(r, a_4)$. But then P together with the $P_{a_1 r}$ subpath of $P_{a_1 a_4}$ and the $P_{a_2 t}$ subpath of $P_{a_2 a_4}$, where t is the neighbor of x_n in $P_{a_2 a_4}$ closest to a_2, and path P makes a the rim of a wheel with center a_4. Similarly if x_1 is of Type 6b there exists a wheel with center a_4. Thus x_1 is of Type 6a and by symmetry x_n is also of Type 6a. Let r and s be the neighbors of x_1 in $P_{a_1 a_4}$ with s closer to a_4 than r. Now either there exists a $3PC(x_1 rs, x_p x_q a_4)$, or $x_p = x_1$ in which case there exists a wheel with center x_p. This completes the proof of Claim 1.

By Claim 1, a_4 has a unique neighbor in P. Let this be node x_q.

Claim 2: Nodes x_1 and x_n are of Type 6a w.r.t. Σ

Proof of Claim 2: If $q \neq 1$ and $q \neq n$, then if x_1 and x_n are not of Type 6a, they are of Type 8a. Assume node x_1 is of Type 8a and let r be the neighbor of x_1 in $P_{a_1 a_4}$. Now either there exists a $3PC(x_q, r)$, or x_n is of Type 8a, adjacent to the neighbor of a_4 in $P_{a_2 a_4}$. But then there exists a wheel with center a_4. So we may assume w.l.o.g. that $q = 1$ and x_1 is of Type 6b. Let x_n be of Type 8a with neighbor s in $P_{a_2 a_4}$. If s is not adjacent to a_4 there exists a $3PC(s, a_4)$, induced by $P_{a_2 a_4} \cup P_{a_3 a_4} \cup P$, otherwise there exists a wheel with center a_4. If x_n is of Type 6a w.r.t. Σ with neighors r and s, s closer to a_4 than r, then there exists a $3PC(x_n sr, x_1 a_4 a_5)$, where a_5 is the neighbor of a_4 in path $P_{a_1 a_4}$. This completes the proof of Claim 2.

By Claim 2, x_1 and x_n are of Type 6a w.r.t. Σ. Let Σ' be the $3PC(\Delta, \cdot)$ obtained by substituting $x_q, \ldots, x_1$ for the appropriate subpath of $P_{a_1 a_4}$. Σ' has an x_q-crosspath. Node a_5 is of Type 8b w.r.t. Σ' and the path induced by $\Sigma \setminus \Sigma'$ consists of node a_5 and its attachment to Σ', that satisfies the conditions of Lemma 5.5. Note that in applying the lemma the roles of Σ and Σ' are interchanged, with a_5 being a node of Type 8b attached to Σ'. Thus Σ has no crosspath. □

Lemma 5.7 *Let Σ be a decomposable $3PC(a_1 a_2 b_1, c_1)$ that has an e_1-crosspath to $P_{a_2 c_1}$. Let u be a Type 3 node w.r.t. Σ and let $P = x_1, \ldots, x_n$ be its attachment to Σ such that x_n is of Type 8b w.r.t. Σ. Let Σ' be a $3PC(\Delta, \cdot)$ obtained from Σ by substituting u and P for $P_{b_1 c_1}$. Then Σ' does not have an x_n-crosspath.*

Proof: Let Q be an e_1-crosspath w.r.t. Σ. If no node of Q is adjacent to a node of $P \cup u$, then Q is an e_1-crosspath w.r.t. Σ' and hence the result follows from Lemma 2.18. If a node of Q is adjacent to a node of $P \cup u$, then by Lemma 2.17, a subpath of Q is an e_1-crosspath w.r.t. Σ' and hence the result follows from Lemma 2.18. □

The following theorem implies Theorem 2.24.

Theorem 5.8 *Let Σ be a decomposable $3PC(\Delta, \cdot)$ and $H = \Sigma \cup u_H$ its extension. The 2-join $H_1|H_2$ of H extends to a 2-join of G.*

Proof: Assume that the 2-join $H_1|H_2$ of H does not extend to a 2-join of G. By Theorem 4.5, there exists a blocking sequence $S = x_1, \ldots, x_n$. W.l.o.g. we assume that H and S are chosen so that the size of S is minimized. Let x_j be the node of S with lowest index that is adjacent to a node in H_2.

Case 1: Node x_j is of Type 3 w.r.t. Σ.

By Corollary 2.14, x_j is attached to Σ. By Lemma 2.8 and Lemma 2.22, every attachment of x_j to Σ ends in a Type 8 node.

Suppose that x_j has an attachment that ends in a Type 8a node w.r.t. Σ, adjacent to a node in H_1. Let $Q = y_1, \ldots, y_m$ be such an attachment, with y_1 adjacent to x_j and y_m w.l.o.g. adjacent to a node in $P_{a_1c_1} \setminus c_1$. Let Σ' be the $3PC(\Delta, \cdot)$ obtained by substituting x_j and Q into Σ. By Lemma 2.22, u_H is of the same type w.r.t. Σ' as it is w.r.t. Σ. If u_H is of Type 3 w.r.t. Σ (and Σ') and it has an attachment to Σ' that ends in a Type 8a node w.r.t. Σ' adjacent to a node in Q, then it also has an attachment to Σ that ends in a Type 8a node w.r.t. Σ adjacent to a node in $P_{a_1c_1} \setminus c_1$. Hence every attachment of u_H to Σ' ends in $P_{b_1c_1}$. Now, by Lemma 5.4, Σ' is decomposable, since any crosspath w.r.t. Σ' is also a crosspath w.r.t. Σ. Let $H' = \Sigma' \cup u_H$ and $H_1' = H' \setminus H_2$. H' has a 2-join with partition $H_1'|H_2$ with special sets $A' = \{a_2, x_j\}$, $B' = B$, $C' = C$ and $D' = D$. By Theorem 4.9 the set S contains a blocking sequence for the 2-join $H_1'|H_2$ of H'. But this contradicts our choice of H.

Hence no attachment of x_j ends in a Type 8a node w.r.t. Σ adjacent to a node in $H_1 \setminus c_1$. But then $H' = \Sigma \cup x_j$ is an extension of a decomposable $3PC(\Delta, \cdot)$. Let $H_2' = H' \setminus H_1$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A' = A$, $B' = \{b_1, x_j\}$, $C' = C$ and $D' = \{d_1\}$. By Theorem 4.9, the set S contains a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H.

Case 2: Node x_j is not of Type 3 w.r.t. Σ.

By Lemma 4.8 $x_1, \ldots, x_j$ is a chordless path. By Lemma 2.22 and the definition of a blocking sequence, x_1 is either of Type 8a w.r.t. Σ with neighbors in $H_1 \setminus c_1$ or of Type 6 with both neighbors in H_1. By Lemma 4.3 nodes $x_2, \ldots, x_{j-1}$ are either not adjacent to any node of H or are of Type 8b w.r.t. Σ.

First suppose that x_j is adjacent to a node of Σ. By Lemma 2.22 and the assumption that x_j is not of Type 3 w.r.t. Σ, all of the neighbors of x_j in Σ are contained in $P_{b_1c_1}$ and x_j is either of Type 8a or 6 w.r.t. Σ. Let $H_2' = P_{b_1c_1} \cup x_j$ and $H' = \Sigma \cup x_j$. Then $H_1|H_2'$ is a 2-join of H' with special sets $A' = A$, $B' = B$, $C' = C$ and D' containing node d_1 and possibly x_j, if x_j is of Type 6b w.r.t. Σ. By Theorem 4.9, the set S contains a blocking sequence for the 2-join $H_1|H_2'$ of H', contradicting our choice of H.

Hence x_j is not adjacent to a node of Σ, so it must be adjacent to u_H. Assume that u_H is of Type 8a or 6 w.r.t. Σ. Node c_1 must be adjacent to a node of $x_2, \ldots, x_j$, since otherwise by Lemma 2.17, $x_1, \ldots, x_j, u_H$ is a crosspath w.r.t. Σ, contradicting the assumption that Σ is decomposable. If c_1 has a neighbor in $x_2, \ldots, x_j$, then by Lemma 5.6, c_1 has exactly one neighbor in $x_2, \ldots, x_j$, nodes x_1 and u_H are of Type 6a w.r.t. Σ and Σ has no crosspath. But this contradicts the assumption that Σ is decomposable since the graph induced by $\Sigma \cup \{x_1, \ldots, x_j, u_H\}$ contains a $3PC(\Delta, \cdot)$ with a crosspath.

Hence u_H is of Type 3 w.r.t. Σ. First assume that no node of $x_2, \ldots, x_j$ is adjacent to c_1. Node x_1 must be adjacent to a_1 or a_2, since otherwise $x_1, \ldots, x_j$ is an attachment of u_H to Σ that ends in $H_1 \setminus c_1$ which contradicts the assumption that $\Sigma \cup u_H$ is an extension of a decomposble $3PC(\Delta, \cdot)$. So assume w.l.o.g. that x_1 is adjacent to a_1. Then the node set $H_1 \cup \{x_1, \ldots, x_j, u_H\}$ induces either a Mickey Mouse (if x_1 is of Type 8 w.r.t. Σ) or a proper wheel with center a_1 (if x_1 is of Type 6 w.r.t. Σ). So c_1 must be adjacent to a node of $x_2, \ldots, x_j$. First suppose that x_1 has a neighbor in $P_{a_1c_1}$. Let x_1' be the neighbor of x_1

in $P_{a_1c_1}$ that is closest to a_1 and let $P_{a_1x_1'}$ be the a_1x_1'-subpath of $P_{a_1c_1}$. Let H' be the hole induced by the node set $P_{a_1x_1'} \cup \{x_1, \ldots, x_j, u_H\}$. Node c_1 has at most two neighbors in H', since otherwise (H', c_1) is a wheel, contradicting Lemma 2.21. In particular, x_1 is not of Type 6b w.r.t. Σ. If c_1 has two nonadjacent neighbors in H', then $H' \cup c_1$ induces a $3PC(\cdot, \cdot)$. If c_1 has two adjacent neighbors in H', then the node set $P_{a_2c_1} \cup P_{a_1x_1'} \cup \{x_1, \ldots, x_j, u_H\}$ induces a $3PC(\Delta, \Delta)$. Hence c_1 has a unique neighbor x_i, $i > 1$, in $x_1, \ldots, x_j$. By an analogous argument the same conclusion holds if x_1 has a neighbor in $P_{a_2c_1}$. Let Σ' be a $3PC(a_1a_2u_H, c_1)$ obtained from Σ by substituting $x_i, \ldots, x_j, u_H$ for $P_{b_1c_1}$. Then $x_1, \ldots, x_{i-1}$ is an x_i-crosspath w.r.t. Σ', contradicting Lemma 5.7. □

5.3 Decomposable Connected Triangles

In this section, we assume that G does not contain a connected diamond.

Lemma 5.9 *Strongly adjacent nodes to a connected triangles $T(a_1a_2b_1, c_1c_2d_1, u, v)$ are of the following types:*

Type a: Adjacent to a_1, a_2 and b_1.

Type b: Adjacent to c_1, c_2 and d_1.

Type c: Adjacent to two adjacent nodes in T that belong to a segment of T.

Proof: Strongly adjacent nodes to T are strongly adjacent to at least on of $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$. By Lemma 2.22 the only strongly adjacent nodes to Σ_i are of Type 3 and Type 6. As a consequence, if w is strongly adjacent to T, all its neighbors must be in Σ_i for some $i = 1, 2, 3$ or 4. Suppose that w is not of Type a, b or c. Then $N(w) \cap T = \{u, v\}$. By Lemma 2.14, w is attached to Σ_1. Let $W = w_1, \ldots, w_n$ be an attachment of w to Σ_1. By Lemma 2.11 and Lemma 2.22, w_n is not strongly adjacent to Σ_1, with a neighbor in $P_{a_1u} \setminus \{u, v, v'\}$, where v' is the neighbor of v in Σ_1 distinct from u. A node of P_{c_1v} must be adjacent to a node of W, since otherwise there is a $3PC(a_1a_2b_1, wuv)$ contained in $T \cup W \cup w$. But then a subpath of P_{c_1v} contradicts Lemma 2.17 applied to a $3PC(a_1a_2b_1, u)$ obtained from Σ_1 by substituting w and its attachment W into Σ_1. □

Proof of Theorem 2.26: Assume otherwise. Let $w \notin T$ be adjacent to $b_1 = d_1$ but no other node of T and let Q be a chordless path from w to T in the graph obtained from G by removing the star $b_1 \cup N(b_1) \setminus w$. By Lemma 5.9, each intermediate node of Q can be adjacent to at most one node in the set $\{a_1, a_2, c_1, c_2\}$. But, if such an adjacency exists, the closest such node in Q creates a Mickey Mouse. On the other hand, when no such adjacency exists, there is a $3PC(b_1, u)$ or $3PC(b_1, v)$. □

The following theorem implies Theorem 2.28.

Theorem 5.10 *Let H be an extension of a decomposable connected triangles T, with 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G.*

Proof: Suppose not and let H be chosen so that the size of the blocking sequence $S = x_1, \ldots, x_n$ for the 2-join $H_1|H_2$ is minimized. By Remark 4.2 and Lemma 5.9, x_1 has a neighbor in H_1, x_n has a neighbor in H_2, x_1 and x_n are either not strongly adjacent to T or of Type c w.r.t. T (Lemma 5.9) or x_n is adjacent to node w and no other node of H. By Lemma 4.3 and Lemma 5.9, x_i, $1 < i < n$, is either of Type a or b w.r.t. T (Lemma 5.9), or does not have a neighbor in H, or the unique neighbor of x_i in H is b_1 or d_1. Let x_j be the node of lowest index adjacent to a node in H with $j > 1$. Note that x_j has a neighbor in H_2. By Lemma 4.8, $Q = x_1, \ldots, x_j$ induces a chordless path. If w is the unique neighbor of x_j in H, then let $P = x_1, \ldots, x_j, w$ and otherwise let $P = Q$.

Case 1: w is the unique neighbor of x_j in H, or x_j is adjacent to a node of T but is not strongly adjacent to T, or x_j is of Type c w.r.t. T (Lemma 5.9).

By Lemma 2.17, P is contained in a v-crosspath w.r.t. Σ_1 or a u-crosspath w.r.t. Σ_2. But this contradicts the assumption that T is decomposable.

Case 2: x_j is of Type a or b w.r.t. T (Lemma 5.9).

Note that $P = Q$. We assume w.l.o.g. that x_j is of Type a. If x_1 is adjacent to a_1 only in T, we must have $j = 2$, otherwise there is a Mickey Mouse. By Corollary 2.14, x_2 has an attachment $R = y_1, \ldots y_m$ to Σ_1. By Lemma 2.8 and Lemma 2.22, y_m is not strongly adjacent to Σ_1. Let y be the unique neighbor of y_m in Σ_1. If y is a node of $P_{b_1 u}$ or $P_{a_2 u}$, let Σ' be obtained by sustituting x_2 and R into Σ. Otherwise let Σ' be a $3PC(x_2 a_1 b_1, y)$ that is induced by the node set $P_{b_1 u} \cup P_{a_1 u} \cup R \cup x_2$. Then x_1 is of Type 7 w.r.t. Σ', which contradicts Lemma 2.22. Hence a_1 cannot be the unique neighbor of x_1 in T and similarly a_2 cannot be the unique neighbor of x_1 in T.

Let a_1' be the neighbor of a_1 on $P_{a_1,v}$ and let a_2' be the neighbor of a_2 on $P_{a_2 u}$. Node x_1 cannot be adjacent to a_1' only (or a_2' only) or $\{a_1, a_1'\}$ (or $\{a_2, a_2'\}$) in T since, in each case, there is a proper wheel. Now, by Lemma 2.8, the path P is an attachment of x_j to Σ_1 or Σ_2. The node x_1 is not strongly adjacent to T. If x_1 is adjacent to $P_{c_1 v} \setminus v$ or $P_{c_2 u} \setminus u$, say x_1 has a neighbor z in $P_{c_1 v} \setminus v$, then there is a $3PC(z, u)$. So x_1 is adjacent to some node in $P_{a_1 v}$ or $P_{a_2 u}$, say $P_{a_1 v}$. But now substituting x_j and P for a_1 and the appropriate subpath of $P_{a_1 v}$, we obtain connected triangles T' that are decomposable by Lemma 5.4, and by Theorem 4.9 S contains a shorter blocking sequence for $H' = T' \cup w$ contradicting the choice of H. □

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