Balanced $0, \pm 1$ Matrices
Part I: Decomposition

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Abstract

A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This paper extends the decomposition of balanced $0, 1$ matrices obtained by Conforti, Cornuéjols and Rao to the class of balanced $0, \pm 1$ matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced $0, \pm 1$ matrices.

Keywords: balanced matrix, decomposition, recognition algorithm, 2-join, 6-join, extended star cutset

Running head: Decomposition of balanced $0, \pm 1$ matrices

1 Introduction

A $0, 1$ matrix is *balanced* if for every square submatrix with two ones per row and column, the number of ones is a multiple of four. This notion was introduced by Berge [1] and extended to $0, \pm 1$ matrices by Truemper [16]. A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four.

This paper extends the decomposition of balanced $0, 1$ matrices obtained by Conforti, Cornuéjols and Rao [8] to the class of balanced $0, \pm 1$ matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced $0, \pm 1$ matrices. The algorithm is discussed in a sequel paper.

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The class of balanced $0, \pm 1$ matrices properly includes totally unimodular $0, \pm 1$ matrices. (A matrix is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$.) The fact that every totally unimodular matrix is balanced is implied, for example, by Camion's theorem [3] which states that a $0, \pm 1$ matrix is totally unimodular if and only if, in every square submatrix with an *even number* of nonzero entries per row and column, the sum of the entries is a multiple of four. Therefore our work is related to Seymour's decomposition and recognition of totally unimodular matrices [15].

In Section 3 we show that, to understand the structure of balanced $0, \pm 1$ matrices, it is sufficient to understand the structure of their zero-nonzero pattern. Such $0, 1$ matrices are said to be *balanceable*. Clearly balanced $0, 1$ matrices are balanceable but the converse is not true: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is balanceable but not balanced. Section 5 describes the cutsets used in our decomposition theorem and Section 6 states the theorem and outlines its proof. In Section 7, we relate our result to Seymour's [15] decomposition theorem for totally unimodular matrices. The proofs are given in Section 8 and Section 9. The necessary definitions and notation are introduced in Section 4.

Interestingly, a number of polyhedral results known for balanced $0, 1$ matrices and totally unimodular matrices can be generalized to balanced $0, \pm 1$ matrices. It follows that several problems in propositional logic can be solved in polynomial time by linear programming when the underlying clauses are "balanced". These results are reviewed in Section 2.

2 Bicoloring, Polyhedra and Propositional Logic

Berge [1] introduced the following notion. A $0, 1$ matrix is *bicolorable* if its columns can be partitioned into blue and red columns in such a way that every row with two or more $1's$ contains a 1 in a blue column and a 1 in a red column. This notion provides the following characterization of balanced $0, 1$ matrices.

Theorem 2.1 (Berge [1]) *A $0, 1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

Ghouila-Houri [14] introduced the notion of *equitable bicoloring* for a $0, \pm 1$ matrix A as follows. The columns of A are partitioned into blue columns and red columns in such a way that, for every row of A, the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

Theorem 2.2 (Ghouila-Houri [14]) *A $0, \pm 1$ matrix A is totally unimodular if and only if every submatrix of A has an equitable bicoloring.*

A $0, \pm 1$ matrix A is *bicolorable* if its columns can be partitioned into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. For a $0, 1$ matrix, this definition coincides with Berge's notion of bicoloring. Clearly, if a $0, \pm 1$ matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable.

Theorem 2.3 (Conforti, Cornuéjols [6]) *A* $0, \pm 1$ *matrix* A *is balanced if and only if every submatrix of* A *is bicolorable.*

Balanced $0, 1$ matrices are important in integer programming due to the fact that several polytopes, such as the set covering, packing and partitioning polytopes, only have integral extreme points when the constraint matrix is balanced. Such integrality results were first observed by Berge [2] and then expanded upon by Fulkerson, Hoffman and Oppenheim [12]. In the case of balanced $0, \pm 1$ matrices, similar integrality results were proved by Conforti and Cornuéjols [6] for the generalized set covering, packing and partitioning polytopes.

Given a $0, \pm 1$ matrix A, let $n(A)$ denote the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A.

Theorem 2.4 (Conforti, Cornuéjols [6]) *Let* M *be a* $0, \pm 1$ *matrix. Then the following statements are equivalent:*
(i) M *is balanced.*
(ii) For each submatrix A *of* M*, the generalized set covering polytope* $\{x : Ax \geq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ *is integral.*
(iii) For each submatrix A *of* M*, the generalized set packing polytope* $\{x : Ax \leq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ *is integral.*
(iv) For each submatrix A *of* M*, the generalized set partitioning polytope* $\{x : Ax = \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ *is integral.*

Several problems in propositional logic can be written as generalized set covering problems. For example, the satisfiability problem in conjunctive normal form (SAT) is to find whether the formula

$$\bigwedge_{i \in S} \left(\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j \right)$$

is true. This is the case if and only if the system of inequalities

$$\sum_{j \in P_i} x_j + \sum_{j \in N_i} (1 - x_j) \geq 1 \text{ for all } i \in S$$

has a $0, 1$ solution vector x. This is a generalized set covering problem

$$\begin{aligned} & Ax \geq \mathbf{1} - n(A) \\ & x \in \{0, 1\}^n. \end{aligned}$$

Given a set of clauses $\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j$ with weights w_i, MAXSAT consists of finding a truth assignment which satisfies a maximum weight set of clauses. MAXSAT can be formulated as the integer program

$$\begin{aligned} Min \quad & \textstyle\sum_{i=1}^{m} w_i s_i \\ & Ax + s \geq 1 - n(A) \\ & x \in \{0, 1\}^n, s \in \{0, 1\}^m. \end{aligned}$$

Similarly, the inference problem in propositional logic can be formulated as

$$min \ \{cx : Ax \geq \mathbf{1} - n(A),\ x \in \{0, 1\}^n\}.$$

The above three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [4],[17]. MAXSAT remains NP-hard for Horn clauses with at most two literals [13]. A consequence of Theorem 2.4 is the following.

Corollary 2.5 *SAT, MAXSAT and logical inference can be solved in polynomial time by linear programming when the corresponding* $0, \pm 1$ *matrix* A *is balanced.*

In fact SAT and logical inference can be solved by repeated application of unit resolution when the underlying $0, \pm 1$ matrix A is balanced [5]. These results are surveyed in [7].

3 Balanceable $0, 1$ Matrices

In this section, we consider the following question: given a $0, 1$ matrix, is it possible to turn some of the 1's into -1's in order to obtain a balanced $0, \pm 1$ matrix? A $0, 1$ matrix for which such a signing exists is called a *balanceable* matrix.

Given a $0, 1$ matrix A, the *bipartite graph representation of* A is the bipartite graph $G = (V^r, V^c; E)$ having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} of A equals 1. The sets V^r and V^c are the *sides* of the bipartition. We say that G is *balanced* if A is a balanced matrix.

A *signed graph* is a graph G, together with an assignment of weights $+1, -1$ to the edges of G. To a $0, \pm 1$ matrix corresponds its signed bipartite graph representation. A signed bipartite graph G is *balanced* if it is the signed bipartite graph representation of a balanced $0, \pm 1$ matrix. Thus a signed bipartite graph G is balanced if and only if, in every hole H of G, the *weight* of the hole, i.e. the sum of the weights of the edges in H, is a multiple of four. (A *hole* in a graph is a chordless cycle).

A bipartite graph G is *balanceable* if there exists a signing of its edges so that the resulting signed graph is balanced.

Remark 3.1 *Since cuts and cycles of a connected graph have even intersection, it follows that, if a connected signed bipartite graph* G *is balanced, then the signed bipartite graph* G'*, obtained by switching signs on the edges of a cut, is also balanced.*

For every edge uv of a spanning tree, there is a cut containing uv and no other edge of the tree (such cuts are known as *fundamental cuts*). Thus, if G is a connected balanceable bipartite graph, the edges of a spanning tree can be signed arbitrarily and then the remaining edges can still be signed so that G is a balanced signed bipartite graph. This was already observed by Camion [3] in the context of $0, 1$ matrices that can be signed to be totally unimodular. So Remark 3.1 implies that a bipartite graph G is balanceable if and only if the following signing algorithm produces a balanced signed bipartite graph:

Signing Algorithm

Choose a spanning forest of G*, sign its edges arbitrarily and recursively choose an edge* uv*, which closes a hole* H *of* G *with the previously chosen edges, and sign* uv *so that the sum of the weights of the edges in* H *is a multiple of four.*

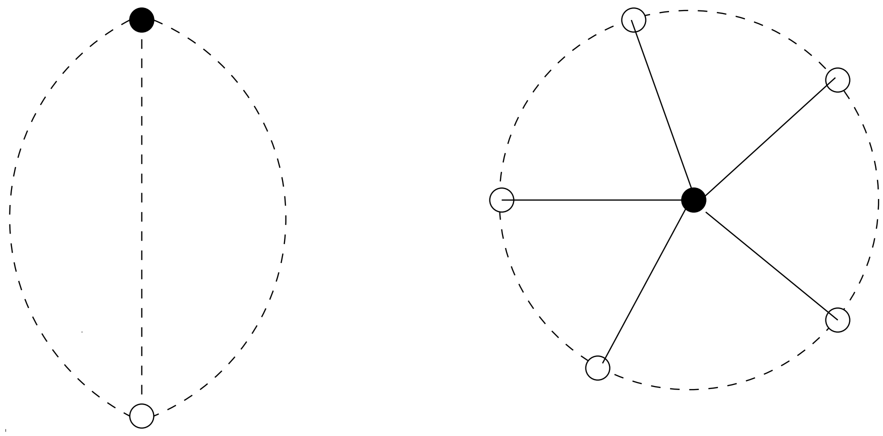

Figure 1: 3-path configuration and odd wheel

Note that, in the signing algorithm, the edge uv can be chosen to close the smallest length hole with the previously chosen edges. Such a hole H is also a hole in G.

It follows from this signing algorithm that, up to the signing of a spanning forest, a balanceable bipartite graph has only one signing that makes it balanced. Consequently, the problem of recognizing whether a bipartite graph is balanceable is equivalent to the problem of recognizing whether a signed bipartite graph is balanced.

Let G be a bipartite graph. Let u, v be two nonadjacent nodes in opposite sides of the bipartition. A *3-path configuration connecting u and v*, denoted by $3PC(u, v)$, is defined by three chordless paths P_1, P_2, P_3 with endnodes u and v, such that the node set $V(P_i) \cup V(P_j)$ induces a hole for $i \neq j$ and $i, j \in \{1, 2, 3\}$. In particular, none of the three paths is an edge. A 3-path configuration is shown in Figure 1. (In all figures black nodes and white nodes are nodes on opposite sides of the bipartition. A solid line denotes an edge, a dashed one a path that is not an edge). Since paths P_1, P_2, P_3 of a 3-path configuration are of length one or three modulo four, the sum of the weights of the edges in each path is also one or three modulo four. It follows that two of the three paths induce a hole of weight two modulo four. So a bipartite graph which contains a 3-path configuration as an induced subgraph is not balanceable.

A *wheel*, denoted by (H, x), is defined by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, x_2, \ldots, x_n$. If n is even, the wheel is an *even wheel*, otherwise it is an *odd wheel* (for example see Figure 1). An edge xx_i is a *spoke*. A subpath of H connecting x_i and x_j is called a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. Consider a wheel which is signed to be balanced. By Remark 3.1, all spokes of the wheel can be assumed to have weight 1. This implies that the sum of the weights of the edges in each sector is two modulo four. Hence the wheel must be an even wheel else the hole H has weight two modulo four.

So, balanceable bipartite graphs contain neither odd wheels nor 3-path configurations as induced subgraphs. This fact is used extensively in our proofs in this paper. The following theorem of Truemper [16] states that the converse is also true.

Theorem 3.2 (Truemper [16]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel nor a 3-path configuration as an induced subgraph.*

4 Additional Definitions and Notation

Let G be a bipartite graph where the two sides of the bipartition are V^r and V^c. G *contains* a graph Σ if Σ is an induced subgraph of G. $N(v)$ refers to the set of nodes adjacent to node v. A node $v \notin V(\Sigma)$ is *strongly adjacent* to Σ if $|N(v) \cap V(\Sigma)| \geq 2$. A node v is a *twin* of a node $x \in V(\Sigma)$ with respect to Σ if $N(v) \cap V(\Sigma) = N(x) \cap V(\Sigma)$.

A *path* P is a sequence of distinct nodes $x_1, x_2, \ldots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. Let x_i and x_l be two nodes of P, where $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the $x_i x_l$-subpath of P and is denoted by $P_{x_i x_l}$. We write $P = x_1, \ldots, x_{i-1}, P_{x_i x_l}, x_{l+1}, \ldots, x_n$ or $P = x_1, \ldots, x_i, P_{x_i x_l}, x_l, \ldots, x_n$. A cycle C is a sequence of nodes $x_1, x_2, \ldots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, x_2, \ldots, x_n$ form a path and $x_1 x_n$ is an edge. The node set of a path or a cycle Q is denoted by $V(Q)$.

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, x_2, \ldots, x_n$ *connects* A *and* B if one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between* A *and* B if, in the subgraph induced by the node set $V(P) \cup A \cup B$, no path connecting A and B is shorter than P. A direct connection P between A and B *avoids* C if $V(P) \cap C = \phi$. The direct connection P is said to be *from* A *to* B if x_1 is adjacent to some node in A and x_n to some node in B.

5 Cutsets

In this section we introduce the operations needed for our decomposition result. A set S of nodes (edges) of a connected graph G is a *node cutset* (an *edge cutset* respectively) if the subgraph $G \setminus S$, obtained from G by removing the nodes (edges) in S, is disconnected.

Extended Star Cutsets

A *biclique* is a complete bipartite graph K_{AB} where the two sides of the bipartition A and B are both nonempty.

In a connected bipartite graph G, an *extended star* $(x; T; A; R)$ is defined by disjoint subsets T, A, R of $V(G)$ and a node $x \in T$ such that

(i) $A \cup R \subseteq N(x)$,

(ii) the node set $T \cup A$ induces a biclique (with node set T on one side of the bipartition and node set A on the other),

(iii) if $|T| \geq 2$, then $|A| \geq 2$.

This concept was introduced in [8]. An *extended star cutset* is an extended star $(x; T; A; R)$ where $T \cup A \cup R$ is a node cutset. When $R = \emptyset$ the extended star is a biclique, and the cutset is called a *biclique cutset.*

Joins

Let G be a connected bipartite graph containing a biclique $K_{A_1A_2}$ with the property that its edge set $E(K_{A_1A_2})$ is a cutset of G and no connected component of $G' = G \setminus E(K_{A_1A_2})$ contains both a node of A_1 and a node of A_2. For $i = 1, 2$, let G'_i be the union of the components of G' containing a node of A_i. The edge set $E(K_{A_1A_2})$ is a *1-join* if the graphs G'_1 and G'_2 each contains at least two nodes. This concept was introduced by Cunningham and Edmonds [11].

Let G be a connected bipartite graph with more than four nodes, containing bicliques $K_{A_1A_2}$ and $K_{B_1B_2}$, where A_1, A_2, B_1, B_2 are disjoint nonempty node sets. The edge set $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is a *2-join* if it satisfies the following properties:

(i) The graph $G' = G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ is disconnected.

(ii) Every connected component of G' has a nonempty intersection with exactly two of the sets A_1, A_2, B_1, B_2 and these two sets are either A_1 and B_1 or A_2 and B_2. For $i = 1, 2$, let G'_i be the subgraph of G' containing all its connected components that have nonempty intersection with A_i and B_i.

(iii) If $|A_1| = |B_1| = 1$, then G'_1 is not a chordless path or $A_2 \cup B_2$ induces a biclique. If $|A_2| = |B_2| = 1$, then G'_2 is not a chordless path or $A_1 \cup B_1$ induces a biclique.

This concept was introduced by Cornuéjols and Cunningham [10] and was extensively used in [8]. In the present paper, 2-joins are needed in the statement of the main theorem, which builds on the work of [8], but do not occur in the proofs.

In a connected bipartite graph G, let A_i, $i = 1, \ldots, 6$ be disjoint, nonempty node sets such that, for each i, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. (Note that for convenience of notation the modulo 6 function is assumed to return values between 1 and 6, instead of the usual 0 to 5). The edge set $E(A)$ is a *6-join* if

(i) The graph $G' = G \setminus E(A)$ is disconnected.

(ii) The nodes of G can be partitioned into V_T and V_B so that $A_1 \cup A_3 \cup A_5 \subseteq V_T$, $A_2 \cup V_4 \cup V_6 \subseteq V_B$ and the only adjacencies between V_T and V_B are the edges of $E(A)$.

(iii) $|V_T| \geq 4$ and $|V_B| \geq 4$.

When the graph G comprises more than one connected component, we say that G has a 1-join, a 2-join, a 6-join or an extended star cutset if at least one of its connected components does.

6 The Main Theorem

A bipartite graph is *strongly balanceable* if it is balanceable and contains no cycle with exactly one chord. Strongly balanceable bipartite graphs can be recognized in polynomial time [9]. R_{10} is the balanceable bipartite graph defined by the cycle $x_1, \ldots, x_{10}, x_1$ of length 10 with

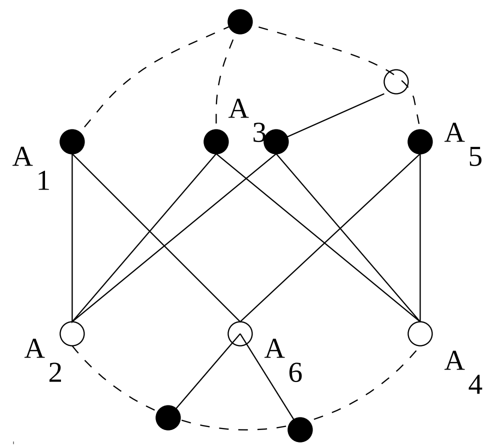

Figure 2: A 6-join

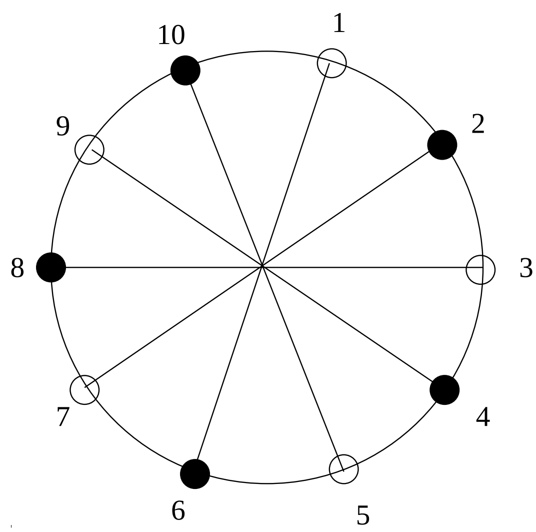

Figure 3: R_{10}

chords $x_i x_{i+5}$, $1 \leq i \leq 5$ (see Figure 3). For example, a proper signing of R_{10} is to assign weight $+1$ to the edges of the cycle $x_1, \ldots, x_{10}, x_1$ and -1 to the chords.

We can now state the decomposition theorem for balanceable bipartite graphs:

Theorem 6.1 *A balanceable bipartite graph that is not strongly balanceable is either R_{10} or contains a 2-join, a 6-join or an extended star cutset.*

The key idea in the proof of Theorem 6.1 is that if a balanceable bipartite graph G is not strongly balanceable, then one of the three following cases occurs: (i) the graph G contains R_{10} as an induced subgraph, or (ii) it contains a certain induced subgraph which forces a 6-join or an extended star cutset of G, or (iii) an earlier result of Conforti, Cornuéjols and Rao [8] applies.

Connected 6-Holes

A *triad* is a bipartite graph consisting of three internally node-disjoint paths $t, \ldots, u$; $t, \ldots, v$ and $t, \ldots, w$, where t, u, v, w are distinct nodes and belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes u, v and w are called the *attachments* and t is called the *meet* of the triad.

A *fan* consists of a chordless path $P = x, \ldots, y$ together with a node z not in P adjacent to a positive even number of nodes in P, where x, y and z belong to the same side of the bipartition and are called the *attachments* of the fan. Node z is the *center* of the fan and the edges connecting z to P are the *spokes*.

A *connected 6-hole* Σ is a bipartite graph induced by two disjoint node sets $T(\Sigma)$ and $B(\Sigma)$ such that each induces either a triad or a fan, the attachments of $B(\Sigma)$ and $T(\Sigma)$ induce a 6-hole and there are no other adjacencies between the nodes of $T(\Sigma)$ and $B(\Sigma)$. $T(\Sigma)$ and $B(\Sigma)$ are the *sides* of Σ, $T(\Sigma)$ is the *top* and $B(\Sigma)$ the *bottom*.

In this paper we will prove the following two theorems.

Theorem 6.2 *A balanceable bipartite graph that contains R_{10} as a proper induced subgraph has a biclique cutset.*

Theorem 6.3 *A balanceable bipartite graph that contains a connected 6-hole as an induced subgraph has an extended star cutset or a 6-join.*

Now Theorem 6.1 follows from Theorems 6.2, 6.3 and the following result.

Theorem 6.4 [8] *A balanceable bipartite graph not containing R_{10} or a connected 6-hole as induced subgraphs either is strongly balanceable or contains a 2-join or an extended star cutset.*

A signed bipartite graph is *strongly balanced* if it is balanced and contains no cycle with exactly one chord. A corollary of Theorem 6.1 and of the signing algorithm is the following result.

Theorem 6.5 *A signed bipartite graph that is balanced but not strongly balanced is either R_{10} with proper signing or it contains a 2-join, a 6-join or an extended star cutset.*

Conjecture 6.6 *If a $0, \pm 1$ matrix is balanced but not totally unimodular, then the underlying signed bipartite graph contains an extended star cutset.*

The restriction of this conjecture to 0,1 matrices is true: a proof can be found in [8].

7 Connection with Seymour's Decomposition of Totally Unimodular Matrices

Seymour [15] discovered a decomposition theorem for $0,1$ matrices that can be signed to be totally unimodular. The decompositions involved in his theorem are 1-separations, 2-separations and 3-separations. A matrix B has a *k-separation* if its rows and columns can be partitioned so that, after permutation of rows and columns,

$$B = \begin{pmatrix} A^1 & D^2 \\ D^1 & A^2 \end{pmatrix}$$

where $r(D^1) + r(D^2) = k - 1$ and the number of rows plus number of columns of A^i is at least k, for $i = 1, 2$ (here $r(C)$ denotes the GF(2)-rank of the $0,1$ matrix C).

For a 1-separation, $r(D^1) + r(D^2) = 0$. Thus both D^1 and D^2 are identically zero. The bipartite graph corresponding to the matrix B is disconnected.

For the 2-separation, $r(D^1) + r(D^2) = 1$, thus w.l.o.g. D^2 has rank zero and is identically zero. Since $r(D^1) = 1$, after permutation of rows and columns, $D^1 = \begin{pmatrix} \mathbf{0} & E \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where E is a matrix all of whose entries are 1. The 2-separation in the bipartite graph representation of B corresponds to a 1-join.

For the 3-separation, $r(D^1) + r(D^2) = 2$. If both D^1 and D^2 have rank 1 then, after permutation of rows and columns,

$$D^1 = \begin{pmatrix} \mathbf{0} & E^1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad D^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ E^2 & \mathbf{0} \end{pmatrix}$$

where E^1 and E^2 are matrices whose entries are all 1. This 3-separation in the bipartite graph representation of B corresponds to a 2-join.

When $r(D^1) = 2$ or $r(D^2) = 2$, it can be shown that the resulting 3-separation corresponds to a 2-join, a 6-join or to one of two other decompositions which each contain an extended star cutset.

In order to prove his decomposition theorem, Seymour used matroid theory. A matroid is *regular* if it is binary and its partial representations can be signed to be totally unimodular (see [18] for relevant definitions in matroid theory). The elementary families in Seymour's decomposition theorem consist of graphic matroids, cographic matroids and a 10-element matroid called $\mathcal{R}_{10}$. $\mathcal{R}_{10}$ has exactly two partial representations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The bipartite graph representations are shown in Figure 4.

Theorem 7.1 (Seymour [15]) *A regular matroid is either graphic, cographic, the 10-element matroid $\mathcal{R}_{10}$, or it contains a 1-, 2- or 3-separation.*

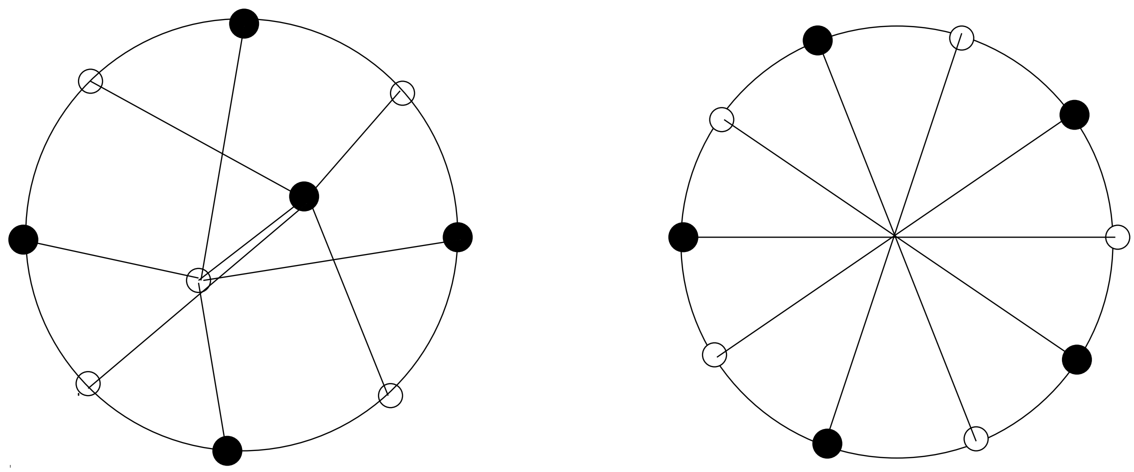

Figure 4: Representations of $\mathcal{R}_{10}$

In order to prove Theorem 7.1, Seymour first showed that a regular matroid which is not graphic or cographic either contains a 1- or 2-separation or contains an $\mathcal{R}_{10}$ or an $\mathcal{R}_{12}$ minor, where $\mathcal{R}_{12}$ is a 12-element matroid having the following matrix as one of its partial representations.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that the bipartite graph representation of this matrix is a connected 6-hole where both sides are fans. So, this first part in Seymour's proof has some similarity with Theorem 6.4 stated above for balanceable bipartite graphs.

Then Seymour showed that, if a regular matroid contains an $\mathcal{R}_{10}$ minor, either it is $\mathcal{R}_{10}$ itself or it contains a 1-separation or a 2-separation. This is similar to Theorem 6.2.

Seymour completed his proof by showing that, for a regular matroid which contains an $\mathcal{R}_{12}$ minor, the 3-separation of $\mathcal{R}_{12}$ induces a 3-separation for the matroid. We show that for a balanceable bipartite graph, which contains a connected 6-hole as an induced subgraph, either the 6-join of the connected 6-hole induces a 6-join of the whole graph or there is an extended star cutset (Theorem 6.3).

Our proof differs significantly from Seymour's for the following reason: a regular matroid may have a large number of partial representations which lead to nonisomorphic bipartite graphs. This is the case for $\mathcal{R}_{12}$. All these partial representations are related through pivoting. In the case of $0, 1$ balanceable matrices there is no underlying matroid, so pivoting cannot help reduce the number of cases. Since our proof is broken down differently from Seymour's, we do not consider all these cases explicitly either.

8 Splitter Theorem for R_{10}

An *extended* R_{10} is a bipartite graph induced by ten nonempty pairwise disjoint node sets $T_1, \ldots, T_{10}$ such that for every $1 \le i \le 10$, the node sets $T_i \cup T_{i-1}$, $T_i \cup T_{i+1}$ and $T_i \cup T_{i+5}$ all induce bicliques and these are the only edges in the graph. Throughout this section, all the indices are taken modulo 10.

We consider a balanceable bipartite graph G which contains a node induced subgraph R isomorphic to R_{10}. We denote its node set by $\{1, \ldots, 10\}$ and for each $i = 1, \ldots, 10$, node i is adjacent to nodes $i-1, i+1$ and $i+5$.

In this section we give a proof of Theorem 6.2. The first step in the proof of the theorem is to study the structure of the strongly adjacent nodes to R.

Theorem 8.1 *Let R be an R_{10} of G. If w is a strongly adjacent node to R, then w is a twin of a node in $V(R)$ with respect to R.*

Proof: First, assume that w has exactly two neighbors in R. If the neighbors of w in R are nodes 1 and 3, the hole $w, 1, 6, 7, 8, 3, w$ induces an odd wheel with center 2. If the neighbors of w in R are nodes 1 and 5, the hole $w, 1, 2, 7, 8, 9, 4, 5, w$ is an odd wheel with center 10. The other cases where w has two neighbors in R are isomorphic.

We now assume that node w is adjacent to at least three nodes in R. If node w is adjacent to nodes $i, i+2, i+4$, then there exists an odd wheel $i, i+1, i+2, i+3, i+4, i+5, i$ with center w. So w is adjacent to exactly three nodes $i, i+2, i+6$, showing that w is a twin of $i+1$. □

Definition 8.2 *Let R be an R_{10} of G. For $1 \le i \le 10$, let $T_i(R)$ be the set of nodes comprising node i in R and all the twins of node i with respect to R. Let R^* be the graph induced by the node set $\cup_{i=1}^{10} T_i(R)$.*

Lemma 8.3 *R^* is an extended R_{10}.*

Proof: Let $u \in T_i(R)$ and $v \in T_j(R)$, where $1 \le i, j \le 10$. Let R' be the R_{10} obtained from R by substituting node u for node i. Now by Theorem 8.1, node v is twin of node j in R'. Hence nodes u and v are adjacent if and only if nodes i and j are adjacent. □

Theorem 8.4 *R^* satisfies the following two properties:*
(i) If node w is strongly adjacent to R^* then for some $1 \le i \le 10$, $N(w) \cap V(R^*) \subseteq T_i(R)$.
(ii) If R' is an R_{10} induced by the node set $\{x_1, \ldots, x_{10}\}$ where $x_i \in T_i(R)$ for $1 \le i \le 10$, then $T_i(R') = T_i(R)$.

Proof: To prove (i), assume that w is adjacent to $w_i \in T_i(R)$ and $w_j \in T_j(R)$, $i \ne j$. Let $R_{w_i w_j}$ be an R_{10} obtained from R by replacing node i with w_i and node j with w_j. Node w is now strongly adjacent to $R_{w_i w_j}$, so by Theorem 8.1, node w is a twin of a node in $R_{w_i w_j}$. Hence w is adjacent to a node k of R. Let R_{w_i} be an R_{10} obtained from R by replacing node i by w_i. Since w is adjacent to k and w_i, it is strongly adjacent to R_{w_i}, hence, by Theorem 8.1, w is adjacent to a node $l \ne k$ of R. Now w is a strongly adjacent node of R and, by Theorem 8.1, must be a twin of a node of R. Hence $w \in V(R^*)$, which contradicts our choice of w.

To prove (ii), note that Lemma 8.3 implies $T_i(R) \subseteq T_i(R')$, so it is enough to show that $T_i(R') \subseteq T_i(R)$. Let $u \in T_i(R')$ and suppose that $u \notin T_i(R)$. Then node u is strongly adjacent to R^* and by (i) we have a contradiction. □

Remark 8.5 *Considering Theorem 8.4, we can simplify the notation by replacing $T_i(R)$ by T_i.*

Definition 8.6 *For $1 \leq i \leq 10$, let K_i be the complete bipartite graph induced by the node set $T_{i-1} \cup T_i \cup T_{i+1} \cup T_{i+5}$.*

We now study the structure of paths between the nodes of R^*.

Lemma 8.7 *If $P = x_1, \dots, x_n$ is a direct connection from T_i to $V(R^*) \setminus T_i$ in $G \setminus E(K_i)$, then the neighbors of x_n in R^* belong to a unique set T_j, where $j = i-1, i+1$ or $i+5$.*

Proof: Assume w.l.o.g. that x_1 is adjacent to node i. By Theorem 8.4 (i), $n > 1$ and node x_n has neighbors in exactly one T_j. Assume that for some $j \notin \{i-1, i+1, i+5\}$, x_n is adjacent to a node $v_j \in T_j$.

If $j = i+2$ then the hole $i, x_1, P, x_n, v_{i+2}, i+7, i+6, i+5, i$ induces an odd wheel with center $i+1$. If $j = i+3$ then the paths $P_1 = i, x_1, P, x_n, v_{i+3}; P_2 = i, i+1, i+2, v_{i+3}$ and $P_3 = i, i-1, i+4, v_{i+3}$ induce a $3PC(i, v_{i+3})$. If $j = i+4$ then the hole $i, x_1, P, x_n, v_{i+4}, i+3, i+8, i+7, i+6, i+1, i$ induces an odd wheel with center $i+2$. This completes the proof since the remaining cases are isomorphic to the above three. □

Lemma 8.8 *There cannot exist a path $P = x_1, \dots, x_n$ with nodes belonging to $V(G) \setminus V(R^*)$ such that x_1 is adjacent to a node $v_i \in T_i$ and x_n is adjacent to a node $v_j \in T_j$, where $i \neq j$ and v_i and v_j are not adjacent.*

Proof: Let P be a shortest path contradicting the lemma. Hence P does not contain an intermediate node adjacent to a node in $T_i \cup T_j$. By Theorem 8.1, $n > 1$. If no node x_l of P, $2 \leq l \leq n-1$, is adjacent to a node in $V(R^*)$ then P is a direct connection from T_i to $V(R^*) \setminus T_i$ in $G \setminus E(K_i)$ contradicting Lemma 8.7.

Let $w \in T_k$, $k \neq i, j$, be adjacent to a node of P. By minimality of P, w is adjacent to v_i and v_j and no node of $V(P) \setminus \{x_1, x_n\}$ is adjacent to a node of $V(R^*) \setminus T_k$. By symmetry, there are two cases to consider: k, j are either $i+1, i+2$ or $i+1, i+6$. In the first case, let $H_1 = v_i, P, v_{i+2}, i+3, i+4, i-1, v_i$ and $H_2 = v_i, P, v_{i+2}, i+7, i+6, i+5, v_i$. Now either H_1 or H_2 induces an odd wheel with center w depending on the number of neighbors of w in P. In the second case, the hole $v_i, P, v_{i+6}, i+7, i+2, i+3, i+4, i-1, v_i$ induces an odd wheel with center $i+5$. □

Proof of Theorem 6.2: Let G be a balanceable bipartite graph. Let R be an R_{10} of G. By Lemma 8.3, R^* is an extended R_{10}. Assume that $V(G) \neq V(R^*)$. Let w be a node in $V(G) \setminus V(R^*)$ adjacent to a node in T_l. If the biclique K_l is not a cutset of G, separating w from $V(R^*)$, then a path contradicting Lemma 8.8 exists. Hence $V(G) = V(R^*)$. If G is not R_{10}, then at least one of the node sets $T_i(R)$ has cardinality greater than one. W.l.o.g. let u and v be two nodes in $T_1(R)$. Now $\{u\} \cup N(u)$ is a biclique cutset separating v from the rest of the graph. □

9 Connected 6-Hole

Let Σ be a connected 6-hole induced by $T(\Sigma)$ and $B(\Sigma)$ in a balanceable bipartite graph G. In this section, we prove that either G contains an extended star cutset or it has a 6-join which separates the top and the bottom of Σ (Theorem 6.3).

We denote by $H = h_1, h_2, h_3, h_4, h_5, h_6, h_1$ the 6-hole of Σ and we assume that $h_1, h_3, h_5 \in T(\Sigma)$ and $h_2, h_4, h_6 \in B(\Sigma)$. We also assume $h_1, h_3, h_5 \in V^c$ and $h_2, h_4, h_6 \in V^r$. Throughout the remainder of the paper indices referring to the hole will be taken modulo 6. If $T(\Sigma)$ is a triad, then the three paths defining it are denoted by P_1, P_3 and P_5 with endnodes h_1, h_3 and h_5 respectively and the meet is denoted by t.

The idea of the proof is to extend the 6-join of Σ into a 6-join of G. Namely, we aim to find node sets $H_1, H_2, \ldots, H_6$ such that $h_i \in H_i$, for $1 \leq i \leq 6$, and $E(\cup_{i=1}^{6} H_i)$ is a 6-join for G separating $T(\Sigma)$ from $B(\Sigma)$. If this is not possible, we detect an extended star cutset in G.

Remark 9.1 *Let h_i and h_j be two distinct attachments of a side X of Σ. There is a unique chordless path in X, connecting h_i and h_j. This path is denoted by P_{ij}. Also any pair of nodes in $V(\Sigma)$ are contained in a hole of Σ.*

Definition 9.2 *A* tripod *with attachments x, y, z is either a fan where we allow the center to have any positive number (even or odd) of neighbors in the path P, or a triad where the meet is not adjacent to any of the attachments but is not restricted to be in the same side of the bipartition as the attachements.*

Lemma 9.3 *Let G be a bipartite graph and let x, y, z be distinct nodes in the same side of the bipartition such that both G and $G \setminus \{x, y, z\}$ are connected. Then G contains a tripod with attachments x, y and z.*

Proof: Let G' be a minimal subgraph of G such that x, y, z are in G' and both G', $G' \setminus \{x, y, z\}$ are connected. We show that G' is a tripod with attachments x, y, z.

Let $P_{xy} = x, y_1, \ldots, y_m, y$ be a shortest xy-path in $G' \setminus \{z\}$, P_{xz} and P_{yz} similarly defined. Assume w.l.o.g. that P_{xy} is not shorter than any of the other two. If P_{xy} contains a neighbor of z then $V(G') = V(P_{xy}) \cup \{z\}$ and G' is a tripod. Otherwise let $P_z = x_1, \ldots, x_n$, be a direct connection in G' from z to $V(P_{xy}) \setminus \{x, y\}$. By the minimality of G', $V(G') = V(P_{xy}) \cup V(P_z)$ and x_n has a unique neighbor, say x_{n+1}, in P_{xy}. If x has a neighbor in P_z, by the minimality of G' and the fact that $V(G') = V(P_{xy}) \cup V(P_z)$, y has no neighbor in P_z and x_{n+1} is adjacent to x. Now this contradicts our choice of P_{xy}.

By symmetry, neither x nor y have neighbors in P_z and if x_{n+1} is adjacent to x or y our choice of P_{xy} is contradicted, so G' is a tripod. □

Lemma 9.4 *In a balanceable bipartite graph G, let T and B be node disjoint tripods with attachments h_1, h_3, h_5 and h_2, h_4, h_6. If $h_1, h_2, h_3, h_4, h_5, h_6, h_1$ is a 6-hole of G and no other adjacency exists between T and B, then T is a fan or a triad and so is B. Therefore T and B are the top and bottom of a connected 6-hole.*

Proof: Let Σ be the graph induced by $V(T) \cup V(B)$. Let P_{13} be a shortest h_1h_3-path in $T \setminus \{h_5\}$. If P_{13} contains neighbors of h_5 and T is not a fan, then Σ contains an odd wheel. So by symmetry we can assume that T contains three chordless paths $t, \ldots, h_1$, $t, \ldots, h_3$ and $t, \ldots, h_5$ and t not adjacent to any of the nodes h_1, h_3 and h_5. If t and h_1 are on opposite sides of the bipartition, Σ contains a $3PC(t, h_1)$. Therefore T is a triad. Similarly, B is a fan or a triad. □

9.1 Strongly Adjacent Nodes and Direct Connections

Theorem 9.5 *Let Σ be a connected 6-hole in a balanceable bipartite graph G. Let $P = x_1, \ldots, x_n$ (we allow $n = 1$) be a direct connection between $T(\Sigma)$ and $B(\Sigma)$ in $G \setminus E(H)$ such that either x_1 has a neighbor in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ or x_n has a neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ or both. Then either x_1 has exactly two neighbors in $\{h_1, h_3, h_5\}$ and no other neighbor in $T(\Sigma)$ or x_n has exactly two neighbors in $\{h_2, h_4, h_6\}$ and no other neighbor in $B(\Sigma)$.*

To prove the above theorem, we use the following result about the structure of strongly adjacent nodes to an even wheel. Two sectors of a wheel (W, v) are *adjacent* if they have a common endnode. A *bicoloring* of (W, v) is an assignment of two colors to the intermediate nodes of its sectors so that the nodes in the same sector have the same color and nodes of adjacent sectors have distinct colors. The neighbors of v are left unpainted. Note that a wheel is bicolorable if and only if it is even.

Lemma 9.6 *Let (W, v), $v \in V^r$, be a bicolored wheel in a balanceable bipartite graph, and let $u \in V^c \backslash N(v)$ be a node with neighbors in at least two distinct sectors of the wheel (W, v). Then u satisfies one of the following properties:*

Type a *Node u has exactly two neighbors in W and these neighbors belong to two distinct sectors having the same color.*

Type b *There exists one sector, say S_j with endnodes v_i and v_k, such that u has a positive even number of neighbors in S_j and has exactly two neighbors in $V(W) \backslash V(S_j)$, adjacent to v_i and v_k respectively.*

Proof: Assume first that u has neighbors in at least three different sectors, say S_i, S_j, S_k. If none of these sectors is adjacent to both of the other two, then there exist three unpainted nodes v_i, v_j, v_k, such that $v_i \in V(S_i) \setminus (V(S_j) \cup V(S_k))$, $v_j \in V(S_j) \setminus (V(S_i) \cup V(S_k))$, $v_k \in V(S_k) \setminus (V(S_i) \cup V(S_j))$. This implies the existence of a $3PC(u, v)$, where each of the nodes v_i, v_j, v_k belongs to a distinct path of the 3-path configuration. So u has neighbors in exactly three sectors and one of them is adjacent to the other two, say S_j is adjacent to both S_i and S_k. Let v_i be the unpainted node in $V(S_i) \cap V(S_j)$ and v_k the unpainted node in $V(S_j) \cap V(S_k)$. Then, there is a $3PC(u, v)$ unless node u has a unique neighbor u_i in S_i which is adjacent to v_i and a unique neighbor u_k in S_k which is adjacent to v_k. When this is the case, node u has an even number of neighbors in S_j (else (H, u) is an odd wheel) and u is of Type b.

Assume now that u has neighbors in exactly two sectors of the wheel, say S_j and S_k. If these two sectors are adjacent, let v_i be their common endnode and v_j, v_k the other endnodes of S_j and S_k respectively. Let H' be the hole obtained from H by replacing $S_j \cup S_k$ by the shortest path in $S_j \cup S_k \cup \{u\} \setminus \{v_i\}$. The wheel (H', v) is an odd wheel. So the sectors S_j and S_k are not adjacent. If u has three neighbors or more on H, say two or more in S_j and at least one in S_k, then denote by v_j and v_{j-1} the endnodes of S_j and by v_k one of the endnodes of S_k. There exists a $3PC(u, v)$ where each of the nodes v_j, v_{j-1}, and v_k belongs to a different path. Therefore u has only two neighbors in H, say $u_j \in V(S_j)$ and $u_k \in V(S_k)$. Let C_1 and C_2 be the holes formed by the node u and the two $u_j u_k$-subpaths of H, respectively. In order for neither (C_1, v) nor (C_2, v) to be an odd wheel, the sectors S_j and S_k must be of the same color and u is of Type a. □

Proof of Theorem 9.5:

Recall that $h_1, h_3, h_5 \in V^c$ and $h_2, h_4, h_6 \in V^r$. We first show that $x_1 \in V^r$ or $x_n \in V^c$ or both. Assume the contrary, i.e. $x_1 \in V^c$ and $x_n \in V^r$. Now all neighbors of x_1 in $V(\Sigma)$ are in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and all neighbors of x_n are in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$. Assume first that $B(\Sigma)$ is a triad, let $b \in V^r$ be the meet of $B(\Sigma)$ and let P_2 be the path in $B(\Sigma) \setminus \{h_4, h_6\}$ with endnodes b and h_2. P_4, P_6 are similarly defined. If x_n has neighbors in at least two of the paths, say P_2 and P_4, then h_2, h_4 and x_1 are intermediate nodes in the three paths of a $3PC(h_3, x_n)$. So we can assume w.l.o.g. that all the neighbors of x_n are in P_2. Now h_4, h_6 and x_n are intermediate nodes in the three paths of a $3PC(h_5, b)$. So, by symmetry, both $T(\Sigma)$ and $B(\Sigma)$ are fans. Assume h_6 is the center of $B(\Sigma)$, so all the neighbors of x_n are in P_{24}. If x_n has more than one neighbor in P_{24}, there is a $3PC(h_3, x_n)$. So by symmetry x_1 has a unique neighbor in $T(\Sigma)$, say x_0 and x_n has a unique neighbor in $B(\Sigma)$, say x_{n+1}, but now we have a $3PC(x_0, x_{n+1})$. Thus we have that $x_1 \in V^r$ or $x_n \in V^c$.

Since either x_1 has a neighbor in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ or x_n has a neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$, we can assume w.l.o.g. that $x_n \in V^c$ and that x_1 has a neighbor in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$. We show that x_n has exactly two neighbors in $\{h_2, h_4, h_6\}$ and no other neighbor in $B(\Sigma)$.

Case 1: $B(\Sigma)$ is a triad.

Let $b \in V^r$ be the meet of $B(\Sigma)$, P_2 be the path in $B(\Sigma)$ with endnodes b and h_2. P_4, P_6 are similarly defined. Let n_i, $i = 2, 4, 6$, be the number of neighbors of x_n in $P_i \setminus \{b\}$.

Assume first that x_n and b are adjacent. Then $n_2 + n_6$ is positive, else h_2, h_6 and x_n are intermediate nodes in the three paths of a $3PC(h_1, b)$. Now $n_2 + n_6$ is odd, else there is an odd wheel with center x_n. By symmetry, $n_2 + n_4$ and $n_4 + n_6$ are also odd, but this is impossible.

Assume now that x_n and b are nonadjacent. If n_2, n_4 and n_6 are all positive, then we have a $3PC(x_n, b)$. So assume w.l.o.g. $n_4 = 0$. If $n_6 = 0$, then h_4, h_6 and x_n are intermediate nodes in the three paths of a $3PC(h_5, b)$. So by symmetry n_2 and n_6 are both positive. Let b_2, b_6 be respectively the neighbors of x_n, closest to b in P_2 and P_6.

If $b_2 \neq h_2$, then h_3, b_2 and b_6 are intermediate nodes in the three paths of a $3PC(x_n, b)$. So $b_2 = h_2$ and by symmetry, $b_6 = h_6$ and the theorem holds in this case.

Case 2: $B(\Sigma)$ is a fan.

Assume h_6 is the center of the fan, let ℓ be the number of neighbors of x_n in P_{24} and let $b_1, \ldots, b_k$ be the neighbors of h_6, encountered in this order when traversing P_{24} from h_2 to h_4, where k is positive and even.

If $\ell = 0$, then h_6 is the only neighbor of x_n in $B(\Sigma)$ and b_1, b_k and x_n are intermediate nodes in the three paths of a $3PC(h_3, h_6)$.

If $\ell = 1$, let y_1 be the unique neighbor of x_n in P_{24}. If $y_1 \neq h_2$ and $y_1 \neq h_4$, then h_2, h_4 and x_n are intermediate nodes in the three paths of a $3PC(h_3, y_1)$. So we assume w.l.o.g. that $y_1 = h_2$. Let Q be a shortest path between h_5 and x_n in $P \cup T(\Sigma) \setminus \{h_1, h_3\}$ and let $C = x_n, h_2, P_{24}, h_4, h_5, Q, x_n$. Then x_n is adjacent to h_6, else (C, h_6) is an odd wheel. So x_n is adjacent to h_2, h_6 and no other node in $B(\Sigma)$, so the theorem holds in this case.

Assume now $\ell \geq 2$. Then ℓ is even, else (C, x_n) is an odd wheel, where $C = h_2, P_{24}, h_4, h_3, h_2$. Let y_1, y_ℓ be the neighbors of x_n, closest to h_2, h_4 in P_{24}.

Assume first that x_n is adjacent to h_6. Then y_1 belongs to $(P_{24})_{h_2 b_1}$. For, if not, let Q be a shortest path between h_3 and x_n, in $P \cup T(\Sigma) \setminus \{h_1, h_5\}$ and let R be a shortest path between h_1 and x_n, in $P \cup T(\Sigma) \setminus \{h_3, h_5\}$. Let $C_1 = y_1, x_n, R, h_1, h_2, (P_{24})_{h_2 y_1}, y_1, x_n$. Then h_6 has an even number of neighbors in $(P_{24})_{h_2 y_1}$, else (C_1, h_6) is an odd wheel. Let $C_2 = y_1, x_n, Q, h_3, h_2, (P_{24})_{h_2 y_1}, y_1, x_n$. Now (C_2, h_6) is an odd wheel. So y_1 belongs to $(P_{24})_{h_2 b_1}$ and by symmetry, y_ℓ belongs to $(P_{24})_{h_4 b_k}$. If $y_1 \neq h_2$, then the following three paths induce a $3PC(x_n, h_2)$:

$x_n, y_1, (P_{24})_{y_1 h_2}, h_2$; $x_n, y_\ell, (P_{24})_{y_\ell h_4}, h_4, h_3, h_2$; x_n, h_6, h_1, h_2.

So $y_1 = h_2$ and by symmetry, $y_\ell = h_4$. Let $H = h_1, h_2, h_3, h_4, h_5, h_6, h_1$, now (H, x_n) is an odd wheel.

Assume finally that $\ell \geq 2$ and x_n is not adjacent to h_6. Let $C = h_2, P_{24}, h_4, h_5, P_{51}, h_1, h_2$. Then (C, h_6) is a wheel, $h_6 \in V^r$ and $x_n \in V^c$ is strongly adjacent to C and is not adjacent to h_6. First suppose that the neighbors of x_n in C are all contained in the same sector of (C, h_6), say sector S. If S does not contain h_2 nor h_4, then there is a $3PC(x_n, h_6)$ in which two of the paths use the endnodes of S and the third path is contained in $T(\Sigma) \cup P$. Now w.l.o.g. assume that S contains h_2. Let Q be a shortest path between y_ℓ and h_5, contained in $P \cup T(\Sigma) \setminus \{h_1, h_3\}$ and let $C_1 = y_\ell, Q, h_5, h_4, (P_{24})_{h_4 y_\ell}, y_\ell$ then (C_1, h_6) is an odd wheel.

So no sector contains all the neighbors of x_n and Lemma 9.6 can be applied. If x_n is of Type a[9.6] with neighbors y_1 and y_2 in C, then since $\ell \geq 2$, $y_1, y_2 \in B(\Sigma)$. Now y_1, y_2 must coincide with h_2, h_4, else there is a $3PC(x_n, h_6)$ and the theorem holds in this case.

So x_n is of Type b[9.6]. If all the neighbors of x_n in C belong to $B(\Sigma)$, there is a $3PC(x_n, h_6)$. If all but one of the neighbors of x_n in C belong to $B(\Sigma)$, then (C_2, x_n) is an odd wheel, where $C_2 = h_3, h_2, P_{24}, h_4, h_3$. □

Lemma 9.7 *Let Σ be a connected 6-hole in a balanceable bipartite graph G. A strongly adjacent node w to Σ is of one of the following types:*

Type a *Either $T(\Sigma)$ or $B(\Sigma)$ contains all the neighbors of w, and w has a neighbor in $V(\Sigma) \setminus H$.*

Type b *Node w is adjacent to exactly two nodes of Σ and these two nodes belong to the 6-hole of Σ. Such a node w is called a* fork.

Type c *Node w has neighbors in both $T(\Sigma)$ and $B(\Sigma)$, and either w has exactly two neighbors in $\{h_1, h_3, h_5\}$ and no other neighbor in $T(\Sigma)$, or w has exactly two neighbors in $\{h_2, h_4, h_6\}$ and no other neighbor in $B(\Sigma)$.*

Proof: If all the neighbors of w are either in $T(\Sigma)$ or in $B(\Sigma)$ and w has a neighbor in $V(\Sigma) \setminus H$, then w is of Type a.

If all the neighbors of w are either in $T(\Sigma)$ or in $B(\Sigma)$, say in $T(\Sigma)$ and w has no neighbor in $V(\Sigma) \setminus H$, then w has exactly two neighbors in $\{h_1, h_3, h_5\}$, else (H, w) is an odd wheel, and hence w is of Type b.

Finally, if w has neighbors in both $T(\Sigma)$ and $B(\Sigma)$, then $P = w$ is a direct connection between $T(\Sigma)$ and $B(\Sigma)$ in $\Sigma \setminus E(H)$ such that w has a neighbor in $T(\Sigma) \cup B(\Sigma) \setminus \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and by Theorem 9.5, w is of Type c. □

Lemma 9.8 *Let Σ be a connected 6-hole in a balanceable bipartite graph G. Every direct connection $P = x_1, \dots, x_n$ from $T(\Sigma)$ to $B(\Sigma)$ in $G \setminus E(H)$ is of one of the following types:*

a) $n = 1$ and x_1 is a strongly adjacent node of Type c [9.7].

b) One endnode of P is a fork, adjacent to h_{i-1} and h_{i+1}, and the other endnode of P is adjacent to a node of $V(\Sigma) \setminus V(H)$.

c) Nodes x_1 and x_n are not strongly adjacent to Σ and their unique neighbors in Σ are two adjacent nodes of H.

d) One endnode of P is a fork, say x_1 is adjacent to h_1 and h_3, and x_n has a unique neighbor in Σ which is h_2.

e) Node x_1 is a fork, say adjacent to h_1 and h_3, and x_n is also a fork, adjacent to h_2 and either h_4 or h_6.

Proof: If x_1 or x_n has a neighbor in $(T(\Sigma) \cup B(\Sigma)) \setminus \{h_1, h_2, h_3, h_4, h_5, h_6\}$, by Theorem 9.5 we have a) or b). So $n > 1$, x_1 has no neighbor in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and x_n has no neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ and by Lemma 9.7, x_1 and x_n are either not strongly adjacent to Σ or they are forks.

Assume both x_1 and x_n have a unique neighbor in H, where x_1 is adjacent to say h_1. If x_n is adjacent to h_4 we have a $3PC(h_1, h_4)$, otherwise we have c).

Assume now x_1 is a fork, adjacent to say h_1 and h_3 and x_n has a unique neighbor in H. If x_n is adjacent to h_2 we have d) and if x_n is adjacent to say h_6 the following three paths give a $3PC(h_3, h_6)$:

h_3, x_1, P, x_n, h_6; h_3, P_{35}, h_5, h_6; h_3, h_2, P_{26}, h_6.

Finally assume x_1 is a fork, adjacent to say h_1 and h_3 and x_n is also a fork. If x_n is adjacent to h_2 we have e). Otherwise let $C = h_1, h_2, h_3, h_4, x_n, h_6, h_1$. If $n = 2$, (C, x_1) is an odd wheel, otherwise $C \cup P$ contains a $3PC(x_1, x_n)$. □

Lemma 9.9 *Let $P = x_1, \dots, x_n$ be a direct connection between $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and $B(\Sigma)$ avoiding $\{h_1, h_3, h_5\}$ in $G \setminus E(H)$, with x_1 adjacent to a node in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and x_n adjacent to a node in $B(\Sigma)$. Then x_n has exactly two neighbors in $\{h_2, h_4, h_6\}$ and no other neighbor in $B(\Sigma)$.*

Proof: Assume not and choose P and Σ as a counterexample to the lemma with P shortest. Now at least one intermediate node of P is adjacent to a node in $\{h_1, h_3, h_5\}$, else the lemma holds as a consequence of Theorem 9.5.

We show that x_n has no neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$. Assume not and let x_k be the node of P with highest index, adjacent to a node in $\{h_1, h_3, h_5\}$ (possibly $k = n$). Then

$P_{x_kx_n}$ is a direct connection between $T(\Sigma)$ and $B(\Sigma)$ in $\Sigma \setminus E(H)$ where x_n has a neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$. So by Theorem 9.5, x_k has exactly two neighbors in $\{h_1, h_3, h_5\}$, say h_1 and h_3, and no other neighbor in $T(\Sigma)$. By Lemma 9.3, $B(\Sigma) \cup P_{x_kx_n}$ contains a tripod $B(\Sigma')$ with attachments x_k, h_4 and h_6. Let Σ' be a connected 6-hole with top $T(\Sigma') = T(\Sigma)$ and bottom $B(\Sigma')$. (Σ' exists by Lemma 9.4). Now $P_{x_1x_{k-1}}$ is a direct connection from $T(\Sigma') \setminus \{h_1, h_3, h_5\}$ to $B(\Sigma')$ avoiding $\{h_1, h_3, h_5\}$ and, since x_k is the unique neighbor of x_{k-1} in $B(\Sigma')$, this contradicts our choice of P and Σ.

So we assume w.l.o.g. that node x_n is adjacent to h_2 and no other node of $B(\Sigma)$.

We now show that at most two of the nodes in $\{h_1, h_3, h_5\}$ have neighbors in P. If all three nodes h_1, h_3, h_5 have neighbors in P, then by Lemma 9.3, there exists a tripod $T(\Sigma')$ contained in $(P \setminus \{x_n\}) \cup \{h_1, h_3, h_5\}$ with attachments h_1, h_3 and h_5. Let Σ' be a connected 6-hole with top $T(\Sigma')$ and bottom $B(\Sigma') = B(\Sigma)$ (Again, Σ' exists by Lemma 9.4). Now a subpath of P is a direct connection between $T(\Sigma') \setminus \{h_1, h_3, h_5\}$ and $B(\Sigma')$ avoiding $\{h_1, h_3, h_5\}$ and this contradicts our choice of P and Σ.

We show that h_5 has no neighbors in P. Assume not and let x_l be the node of highest index adjacent to h_5. By the previous argument, either h_1 or h_3 has no neighbors in P. W.l.o.g. assume node h_3 has no neighbors in P. Now there exists a $3PC(h_5, h_2)$ with paths $h_5, x_l, P_{x_lx_n}, x_n, h_2$; h_5, P_{53}, h_3, h_2 and h_5, h_6, P_{62}, h_2.

So we assume w.l.o.g. that h_1 is adjacent to an intermediate node of P while h_5 is not. Let Q be a shortest path in $\{x_1\} \cup T(\Sigma) \setminus \{h_1, h_3\}$ between x_1 and h_5. Now one of the two holes $x_n, P, x_1, Q, h_5, h_6, P_{62}, x_n$ or $x_n, P, x_1, Q, h_5, h_4, P_{42}, x_n$ induces an odd wheel with center h_1. □

9.2 Extreme Connected 6-Holes

Definition 9.10 *An* extreme connected 6-hole $E\Sigma$ *is a subgraph of G containing six nonempty node sets $H_1, \dots, H_6$ such that, if H is the graph induced by $H_1 \cup \dots \cup H_6$, then $E(H)$ is a 6-join of $E\Sigma$ separating subgraphs T and B, where $V(T) \cup V(B) = V(E\Sigma)$, $H_1 \cup H_3 \cup H_5 \subset V(T)$, $H_2 \cup H_4 \cup H_6 \subset V(B)$ and the three following properties hold:*

(1) Let $T'_1, \dots, T'_m$ be the connected components of the graph T' induced by $V(T) \setminus (H_1 \cup H_3 \cup H_5)$. Then $m \geq 1$, each T'_j has at least one neighbor in each of the sets H_1, H_3, H_5 and each node in $H_1 \cup H_3 \cup H_5$ has at least one neighbor in T'.

The graph B' induced by $V(B) \setminus (H_2 \cup H_4 \cup H_6)$ is nonempty and connected. Each node in $H_2 \cup H_4 \cup H_6$ has at least one neighbor in B'.

(2) For $i = 1, 3, 5$ and $j = 1, \dots, m$, let H_i^j be the set of nodes in H_i with a neighbor in T'_j and let T_j be the graph induced by the node set $V(T'_j) \cup H_1^j \cup H_3^j \cup H_5^j$. Let the H-intersection graph of $E\Sigma$ be defined as follows: its node set is $\{t_1, \dots, t_m\}$ and t_j is adjacent to t_k if at least two of the following three sets are nonempty:

$$H_1^j \cap H_1^k, \quad H_3^j \cap H_3^k, \quad H_5^j \cap H_5^k.$$

Then the H-intersection graph of $E\Sigma$ is connected.

(3) $V(E\Sigma)$ is maximal, subject to (1) and (2).

Lemma 9.11 *The graphs T_j and B satisfy the following properties:*

(1) For every index j and triple of nodes $h_1 \in H_1^j$, $h_3 \in H_3^j$, $h_5 \in H_5^j$, T_j contains a fan or a triad with attachments h_1, h_3, h_5 and all other nodes in T'_j.

For every triple of nodes $h_2 \in H_2$, $h_4 \in H_4$, $h_6 \in H_6$, B contains a fan or a triad with attachments h_2, h_4, h_6 and all other nodes in B'.

(2) Let S be a fan or a triad in T_j satisfying (1) and s be any node of $T_j \setminus S$. Then either s is adjacent to a node in $S \setminus \{h_1, h_3, h_5\}$ or T'_j contains a direct connection between s and $S \setminus \{h_1, h_3, h_5\}$.

Let R be a fan or a triad in B satisfying (1) and r be any node of $B \setminus R$. Then either r is adjacent to a node in $R \setminus \{h_2, h_4, h_6\}$ or B' contains a direct connection between r and $R \setminus \{h_2, h_4, h_6\}$.

Proof: By definition, T'_j is a connected graph and since h_1, h_3, h_5 all have neighbors in T'_j, then the graph induced by $V(T'_j) \cup \{h_1, h_3, h_5\}$ is also connected. So by Lemma 9.3, T_j contains a tripod S with attachments h_1, h_3, h_5 and all other nodes in $V(T'_j)$. The same argument shows that for every three nodes $h_2 \in H_2$, $h_4 \in H_4$, $h_6 \in H_6$, B contains a tripod R with attachments h_2, h_4, h_6 and all other nodes in B'. Now by Lemma 9.4 applied to the graph induced by $V(S) \cup V(R)$, we have that S and R are indeed fans or triads and (1) follows.

Since $S \setminus \{h_1, h_3, h_5\} \subseteq T'_j$ and by definition $V(T'_j) \cup \{s\}$ induces a connected graph, then the first part of (2) follows. The proof of the second part is identical. □

Theorem 9.12 *Let $E\Sigma$ be an extreme connected 6-hole in a balanceable bipartite graph G and let U be a connected component of $G \setminus V(E\Sigma)$, with neighbors in T and in B. Then U has no neighbor in $T' \cup B'$.*

Proof: Assume not. Since U has neighbors of both T and B, then either U contains a direct connection between T' and B avoiding $H_1 \cup H_3 \cup H_5$, or U contains a direct connection between T and B' avoiding $H_2 \cup H_4 \cup H_6$ (or both). Among all these direct connections, let $Q = y_1, \ldots, y_\ell$ be a shortest one. (Possibly $\ell = 1$).

Case 1: Node y_1 has a neighbor in T', y_ℓ has a neighbor in B and no intermediate node of Q is adjacent to a node in $H_2 \cup H_4 \cup H_6$.

We assume that y_1 has a neighbor in T'_j.

Claim 1: Node y_ℓ has no neighbor in B'.

Proof of Claim 1: Assume y_ℓ has a neighbor in B'. Let $T(\Sigma)$ be any fan or triad in T_j with attachments $h_1 \in H_1^j$, $h_3 \in H_3^j$, $h_5 \in H_5^j$ and $B(\Sigma)$ be any fan or triad in B with attachments $h_2 \in H_2$, $h_4 \in H_4$, $h_6 \in H_6$. Let Σ be the connected 6-hole in $E\Sigma$ with $T(\Sigma)$ as top and $B(\Sigma)$ as bottom. (By Lemma 9.11(1) such a Σ exists). By Lemma 9.11(2), there exists a direct connection $P = x_1, \ldots, x_n$ from $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ to $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ avoiding $\{h_1, h_2, h_3, h_4, h_5, h_6\}$, such that $P = P_{T'_j}, Q, P_{B'}$ where $P_{T'_j} \subset T'_j$ and $P_{B'} \subset B'$. Possibly $P_{T'_j}$

or $P_{B'}$ or both are empty. If no intermediate node of P is adjacent to a node in $\{h_2, h_4, h_6\}$, then P is a direct connection from $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and $B(\Sigma)$, avoiding $\{h_1, h_3, h_5\}$ and, since x_n has a neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$, P contradicts Lemma 9.9. So at least one intermediate node of P has a neighbor in $\{h_2, h_4, h_6\}$ and the same argument shows that at least one intermediate node of P has a neighbor in $\{h_1, h_3, h_5\}$. Let x_r be the intermediate node of P with highest index with a neighbor in $\{h_1, h_3, h_5\}$, and let x_s be the intermediate node of P with lowest index with a neighbor in $\{h_2, h_4, h_6\}$. By construction, the intermediate nodes of P that have neighbors in $\{h_1, h_3, h_5\}$ belong to $P_{T'_j}$ or Q and the intermediate nodes of P that have neighbors in $\{h_2, h_4, h_6\}$ are either y_ℓ or belong to $P_{B'}$. Clearly y_ℓ cannot have neighbors in both $\{h_1, h_3, h_5\}$ and $\{h_2, h_4, h_6\}$. This shows that $r < s$. Now $P_{x_1 x_s}$ is a direct connection from $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ to $B(\Sigma)$, avoiding $\{h_1, h_3, h_5\}$ and by Lemma 9.9, x_s has exactly two neighbors in $B(\Sigma)$, say h_2 and h_6. By Lemma 9.3, $T_j \cup P_{x_1 x_s}$ contains a tripod $T(\Sigma')$ with attachments x_s, h_3, h_5 and all other nodes in $T'_j \cup P_{x_1 x_{s-1}}$. Let Σ' be the connected 6-hole with top $T(\Sigma')$ and bottom $B(\Sigma') = B(\Sigma)$. Now $P_{x_{s+1} x_n}$ is a direct connection from $T(\Sigma')$ to $B(\Sigma') \setminus \{h_2, h_4, h_6\}$ avoiding $\{h_2, h_4, h_6\}$ and x_s is the unique neighbor of x_{s+1} in $T(\Sigma')$, a contradiction to Lemma 9.9. This completes the proof of Claim 1.

Claim 2: Node y_ℓ is adjacent to all nodes in exactly two of the sets H_2, H_4, H_6 and to no other node of B.

Proof of Claim 2: By Claim 1, $N(y_\ell) \cap V(B) \subseteq H_2 \cup H_4 \cup H_6$. Assume y_ℓ is adjacent to $h_2 \in H_2$. Let $B(\Sigma)$ be any fan or triad in B with attachments h_2, $h_4 \in H_4$, $h_6 \in H_6$, let $T(\Sigma)$ be any fan or triad in T_j having attachments $h_1 \in H_1^j$, $h_3 \in H_3^j$, $h_5 \in H_5^j$ and let Σ be the connected 6-hole with top $T(\Sigma)$ and bottom $B(\Sigma)$. Such a choice of Σ is possible by Lemma 9.11(1). Now by Lemma 9.11(2), there exists a direct connection R from $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ to $B(\Sigma)$ avoiding $\{h_1, h_3, h_5\}$, such that $R = R_{T'_j}, Q$, where $V(R_{T'_j}) \subset V(T'_j)$ and possibly $V(R_{T'_j})$ is empty. So by Lemma 9.9 applied to Σ and R, y_ℓ has exactly two neighbors in $B(\Sigma)$, say h_2 and h_6.

Let Σ' be any connected 6-hole with top $T(\Sigma') = T(\Sigma)$ and bottom $B(\Sigma')$ with exactly two common attachments with $B(\Sigma)$. (Again by Lemma 9.11(1), Σ' exists.) Now R is a direct connection from $T(\Sigma') \setminus \{h_1, h_3, h_5\}$ to $B(\Sigma')$ avoiding $\{h_1, h_3, h_5\}$. So by Lemma 9.9, y_ℓ is adjacent to the new attachment h' of $B(\Sigma')$ if and only if $h' \in H_2 \cup H_6$. By Lemma 9.11(1), every node h' in $H_2 \cup H_4 \cup H_6 \setminus \{h_2, h_4, h_6\}$ is the attachment of such $B(\Sigma')$. So, by Lemma 9.9, y_ℓ is adjacent to all nodes in $H_2 \cup H_6$ and to no other node of B and this completes the proof of Claim 2.

By Claim 2 we may assume w.l.o.g. that y_ℓ is adjacent to all nodes in $H_2 \cup H_6$ and to no other node of B. Let $E\Sigma^*$ be the subgraph of G, induced by $V(E\Sigma) \cup V(Q)$, where $H_1^* = H_1 \cup \{y_\ell\}$, $H_i^* = H_i$ for all the other indices, and let H^* the graph induced by $H_1^* \cup \ldots \cup H_6^*$. By Claim 2, $E(H^*)$ is a 6-join of $E\Sigma^*$, separating $T^* = T \cup Q$ from $B^* = B$.

If $\ell = 1$, then let $T_k'^* = T'_k$ for $k = 1, \ldots, m$, let $H_1^{k*} = H_1^k \cup \{y_1\}$ if and only if y_1 has a neighbor in T'_k and $H_i^{k*} = H_i^k$ in all other cases. By construction, $T_1'^*, \ldots, T_m'^*$ are the connected components of the graph induced by $V(T') \setminus (H_1^* \cup H_3^* \cup H_5^*)$ and H_i^{k*} contains the nodes in H_i^* with a neighbor in $T_k'^*$. The H^*-intersection graph of $E\Sigma^*$ is connected since $H_i^k \subseteq H_i^{k*}$ for all k and $i = 1, 3, 5$. So $E\Sigma^*$ satisfies Properties (1) and (2) of Definition 9.10, contradicting the assumption that $E\Sigma$ is extreme.

If $\ell > 1$, assume w.l.o.g. that y_1 has no neighbor in $T'_1, \ldots, T'_{p-1}$ and has at least one neighbor in each of $T'_p, \ldots, T'_m$. Let $T'^*_1 = T'_1, \ldots, T'^*_{p-1} = T'_{p-1}$ and let T'^*_p be the connected component induced by $V(T'_p) \cup \ldots \cup V(T'_m) \cup \{y_1, \ldots, y_{\ell-1}\}$. For $k = 1, \ldots, p-1$, let $H^{k*}_i = H^k_i$. Finally, let H^{p*}_1 contain $\{y_\ell\} \cup \cup^m_{k=p} H^k_1$ together with all nodes of H_1 with a neighbor in $Q_{y_1 y_{\ell-1}}$, and for $i = 3, 5$, let H^{p*}_i contain $\cup^m_{k=p} H^k_i$ together with all nodes in H_i with a neighbor in $Q_{y_1 y_{\ell-1}}$. By construction, $T'^*_1, \ldots, T'^*_p$ are the connected components of the graph induced by $V(T^*) \setminus (H^*_1 \cup H^*_3 \cup H^*_5)$ and H^{k*}_i contains the nodes in H^*_i with a neighbor in T'^*_k.

The H^*-intersection graph of $E\Sigma^*$ is connected since for $i = 1, 3, 5$, $H^k_i \subseteq H^{k*}_i$ for all $k = 1, \ldots, p-1$, and $H^k_i \subseteq H^{p*}_i$ for all $k = p, \ldots, m$. From this it follows that $E\Sigma^*$ satisfies Properties (1) and (2) of Definition 9.10, a contradiction to the assumption that $E\Sigma$ is extreme.

Case 2: Node y_1 has a neighbor in T, y_ℓ has a neighbor in B' and no intermediate node of Q is adjacent to a node in $H_1 \cup H_3 \cup H_5$.

The same proof given for Claim 1 shows that y_1 has no neighbor in T', so $N(y_1) \cap V(T) \subseteq H_1 \cup H_3 \cup H_5$.

Claim 3: Node y_1 is adjacent to all the nodes in exactly two of the sets H_1, H_3, H_5 and to no other node of T.

Proof of Claim 3: By Case 1 we may assume that y_1 is not adjacent to a node of T'. W.l.o.g. assume y_1 has a neighbor in $h_1 \in H^j_1$, let $T(\Sigma)$ be any fan or triad in T_j having attachments h_1, $h_3 \in H^j_3$, $h_5 \in H^j_5$, $B(\Sigma)$ be any fan or triad in B with attachments $h_2 \in H_2$, $h_4 \in H_4$, $h_6 \in H_6$ and let Σ be a connected 6-hole with top $T(\Sigma)$ and bottom $B(\Sigma)$. Now by Lemma 9.11(2), there exists a direct connection R from $T(\Sigma)$ and $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ avoiding $\{h_2, h_4, h_6\}$, such that $R = Q, R_{B'}$, where $V(R_{B'}) \subset V(B')$ and possibly $V(R_{B'})$ is empty. So by Lemma 9.9 applied to Σ and R, y_1 has exactly two neighbors in $T(\Sigma)$, say h_1 and h_3. Now the same argument used in the proof of Claim 2 shows that y_1 is adjacent to all the nodes in $H^j_1 \cup H^j_3$ and no other node of T_j.

Choose now T_k such that at least two of the following three sets are nonempty: $H^j_1 \cap H^k_1$, $H^j_3 \cap H^k_3$, $H^j_5 \cap H^k_5$. (This choice is possible by Property (2) of Definition 9.10). Let $T(\Sigma')$ be any fan or triad in T_k having attachments $h'_1 \in H^k_1$, $h'_3 \in H^k_3$, $h'_5 \in H^k_5$, where at least two of these attachments are in T_j and let Σ' be a connected 6-hole with top $T(\Sigma')$ and bottom $B(\Sigma') = B(\Sigma)$. (Lemma 9.11(1) shows that Σ' exists). Now since at least two of the attachments of $T(\Sigma')$ are in T_j and y_1 is adjacent to all the nodes in $H^j_1 \cup H^j_3$ and no other node of T_j, by Lemma 9.9 applied to Σ' and R, we have that h'_1 and h'_3 are the unique neighbors of y_1 in $T(\Sigma')$. This shows that y_1 is adjacent to all the nodes in $H^k_1 \cup H^k_3$ and no other node of T_k. Now by Property (2) of Definition 9.10, we obtain that y_1 is adjacent to all the nodes in $H_1 \cup H_3$ and no other node of T and the proof of Claim 3 is complete.

Let $E\Sigma^*$ be the subgraph of G, induced by $V(E\Sigma) \cup V(Q)$, where $H^*_2 = H_2 \cup \{y_1\}$, $H^*_i = H_i$ for all the other indices and H^* is the subgraph induced by $H^*_1 \cup \ldots \cup H^*_6$. By Claim 3, $E(H^*)$ is a 6-join of $E\Sigma^*$, separating $T^* = T$ from $B^* = B \cup Q$. Now $V(B^*) \setminus (H^*_2 \cup H^*_4 \cup H^*_6)$ induces a connected graph, since B' is connected and y_ℓ has a neighbor in B'. So $E\Sigma^*$ satisfies Properties (1) and (2) of Definition 9.10, a contradiction to the fact that $E\Sigma$ is extreme. □

9.3 Extended Star Cutsets and 6-Joins

Lemma 9.13 *Let Σ be a connected 6-hole in a balanceable bipartite graph G and let $i \in \{1,2,3\}$. Let P be a direct connection from h_i to h_{i+3} such that the nodes of P are in $G \setminus V(\Sigma)$, have no neighbors in $V(\Sigma) \setminus \{h_1, \ldots, h_6\}$ and no proper subpath of P is a direct connection from h_j to h_{j+3}, for $j \in \{1,2,3\}$. Then exactly two of the nodes h_{i-2}, h_{i-1}, h_{i+1}, h_{i+2}, have a neighbor in P and these two nodes are either $\{h_{i+1}, h_{i+2}\}$ or $\{h_{i-1}, h_{i-2}\}$.*

Proof: Let $P = x_1, \ldots, x_n$ with x_1 adjacent to h_i and x_n to h_{i+3}. Assume w.l.o.g. that $i = 3$.

We first show that if P contains no neighbor of h_5, then P contains neighbors of h_2 and h_1.

If P contains no neighbors of h_2, there exists a $3PC(h_3, h_6)$ where the three paths are P; h_3, P_{35}, h_6 and h_3, h_2, P_{26}, h_6. So P must contain a neighbor of h_2. If P contains no neighbors of h_1, then one of the two holes $h_3, P, h_6, h_1, P_{13}, h_3$ or $h_3, P, h_6, h_5, P_{53}, h_3$ makes an odd wheel with center h_2.

We now show that if P contains no neighbor of h_5, then P contains no neighbor of h_4.

Since P contains no neighbor of h_5, then P contains neighbors of h_2 and h_1. If P has a neighbor of h_4, then there exists a direct connection P' from h_1 to h_4 in $G \setminus V(\Sigma)$ using nodes in P. By minimality of P, $P = P'$. Thus x_1 is adjacent to h_1 and x_n to h_4, and nodes h_1 and h_4 have no other neighbors in P. Let x_j be the neighbor of h_2 with the highest index. Then $x_j, \ldots, x_n, h_4, h_5, P_{51}, h_1, h_2, x_j$ makes an odd wheel with center h_6.

So, if node h_5 has no neighbors in $V(P)$, the lemma holds. Now by symmetry, if any one of the nodes $\{h_1, h_2, h_4, h_5\}$ has no neighbors in P, we are done. If all four nodes have neighbors in P, then P contains a direct connection P' from h_1 to h_4 in $G \setminus V(\Sigma)$ and a direct connection P'' from h_5 to h_2 in $G \setminus V(\Sigma)$. By minimality of P, $P = P' = P''$. But then x_1 is adjacent to h_1, h_3 and h_5. Consequently (H, x_1) is an odd wheel. □

Theorem 9.14 *Let $E\Sigma$ be an extreme connected 6-hole in a balanceable bipartite graph G and let U be a connected component of $G \setminus V(E\Sigma)$, with neighbors in both T and in B. If for some i, both H_i and H_{i+3} contain neighbors of U, then there exists an extended star cutset, separating at least one node of U from $E\Sigma$.*

Proof: Let U be a connected component of $G \setminus V(E\Sigma)$ with neighbors in H_i and H_{i+3} for some $i = 1, 2$ or 3. By Theorem 9.12, all the neighbors of U in $E\Sigma$ belong to H. So U contains a direct connection from H_i and H_{i+3} with no neighbor in $V(E\Sigma) \setminus H$. Among all these direct connections and possible choices of i, let $P = x_1, \ldots, x_n$ be a shortest one and assume w.l.o.g. that x_1 is adjacent to a node $h_3 \in H_3^j$ and x_n to a node $h_6 \in H_6$.

Claim 1: Either every node in $H_1^j \cup H_2$ has a neighbor in P and no node in $H_4 \cup H_5$ has a neighbor in P, or every node in $H_4 \cup H_5^j$ has a neighbor in P and no node in $H_1 \cup H_2$ has a neighbor in P.

Proof of Claim 1: For every pair of nodes $h_1 \in H_1^j$ and $h_5 \in H_5^j$, let $T(\Sigma)$ be a fan or a triad in T_j with attachments h_1, h_3, h_5. For every pair of nodes $h_2 \in H_2$ and $h_4 \in H_4$, let $B(\Sigma)$ be a fan or a triad in B with attachments h_2, h_4, h_6 and let Σ be the connected 6-hole with $T(\Sigma)$ as top and $B(\Sigma)$ as bottom. By Lemma 9.13, we can assume that both h_1 and h_2 have neighbors in P, while h_4 and h_5 do not have neighbors in P. By Lemma 9.11(1), it follows readily that every node in $H_1^j \cup H_2$ has a neighbor in P and no node in $H_5^j \cup H_4$ has a neighbor in P.

It remains to show that no node in $H_5 \setminus H_5^j$ is adjacent to a node of P. Assume not and let $h_5' \in H_5^k \setminus H_5^j$ be such a node. Let Σ' be a connected 6-hole having top $T(\Sigma') \subset T_k$ with attachments h_5' and arbitrarily chosen nodes $h_1' \in H_1^k$ and $h_3' \in H_3^k$ and bottom $B(\Sigma') = B(\Sigma)$. By the previous argument, h_2 has a neighbor in P, while h_4 has no neighbor in P. So P contains a direct connection P' between h_5' and h_2 and by the minimality of P, $P' = P$. So x_1 is the unique neighbor of h_5' in P and x_n is the unique neighbor of h_2 in P. Now, by Lemma 9.13 applied to Σ', P and $i = 2$, node h_1' has a neighbor in P, since h_4 has no neighbor in P. Let x_j be a neighbor of h_1' with lowest index. Since x_n is the unique neighbor of h_2 or h_6 in P, then $P_{x_1x_j}$ contains no neighbor in $\{h_2, h_4, h_6\}$. Let H^* be the 6-hole $h_1', h_2, h_3, h_4, h_5', h_6$, and consider the graph G^* induced by $V(H^*) \cup V(P_{x_1x_j})$. Then x_1 is adjacent to h_5', h_3 and x_j to h_1'. So if $j = 1$, G^* is an odd wheel with center x_1 and, if $j > 1$, G^* contains a $3PC(x_1, h_1')$ and this completes the proof of Claim 1.

By Claim 1, we can assume w.l.o.g. that every node in $H_1^j \cup H_2$ has a neighbor in P and no node in $H_4 \cup H_5$ has a neighbor in P. Let h_2^* be a node in H_2. Let S be the extended star $(h_2^*; H_2; H_1 \cup H_3; N(h_2^*) \setminus V(E\Sigma))$. We show that S is an extended star cutset separating x_1 from $E\Sigma$.

Assume not. Then the connected component U contains a direct connection $Q = y_1, \dots, y_q$ from x_1 to $V(E\Sigma) \setminus V(S)$, avoiding $V(S)$. Since y_q belongs to U, by Theorem 9.12, $N(y_q) \cap V(E\Sigma) \subset V(H)$. Let $Q' = x_1, y_1, \dots, y_q$.

Case 1: y_q is adjacent to a node $h_6' \in H_6$.

Let Σ be a connected 6-hole containing h_2^*, h_3, h_6' and arbitrary other attachments $h_1 \in H_1^j$, $h_4 \in H_4$, $h_5 \in H_5^j$. Let Q'' be a minimal subpath of Q' which is a direct connection from h_3 to h_6' or from h_1 to h_4. Since Q'' has no neighbors in $V(\Sigma) \setminus \{h_1, h_3, h_4, h_6'\}$, it is a minimal direct connection satisfying the assumptions of Lemma 9.13 relative to Σ. But then, by Lemma 9.13, h_2^* or h_5 has a neighbor in Q'', contradicting the choice of Q or the assumption that x_1 is not adjacent to h_5.

Case 2: y_q is adjacent to a node $h_4 \in H_4$.

We can assume that y_q is not adjacent to any node in H_6. Therefore no node in $H_5 \cup H_6$ has a neighbor in Q'. Let $R = r_1 (= y_q), \dots, r_t$ be a direct connection from h_4 to H_1^j with $V(R) \subseteq V(P) \cup V(Q)$ (such R exists since P has at least one neighbor in H_1^j). We can assume w.l.o.g. that $r_1, \dots, r_s \in V(Q) \setminus V(P)$ for some $s \leq t$ and $r_j \in V(P)$ for $j > s$. Let $h_1 \in H_1^j$ be a neighbor of r_t and let Σ be a connected 6-hole with attachments h_1, h_2^*, h_3, h_4, h_6 and an arbitrary node $h_5 \in H_5^j$. Node h_5 has no neighbor in R. Furthermore, h_6 has no neighbor in R since x_n is the only node of $V(P) \cup V(Q)$ adjacent to h_6 and R cannot contain x_n. Therefore R is a minimal set of nodes satisfying the assumptions of Lemma 9.13 relative to Σ. This implies that both h_2^* and h_3 have neighbors in R. Let $C_1 = h_4, r_1, R, r_t, h_1, h_6, P_{64}, h_4$ (P_{64} is defined in Remark 9.1). Then R has an odd number of neighbors of h_3, else (C_1, h_3) is an odd wheel. No node of Q' is adjacent to h_1 since, otherwise, some subpath of Q' would be a direct connection from h_1 to h_4 violating Lemma 9.13 in Σ (since neither h_2^* nor h_6 has a neighbor in Q'). Therefore, by construction of R, if r_ℓ denotes the node of lowest index adjacent to h_2^*, then all the neighbors of h_3 in R are in $R_{r_1r_\ell}$. Let $C_2 = h_4, r_1, R_{r_1r_\ell}, r_\ell, h_2^*, h_1, P_{15}, h_5, h_4$.

Since h_3 has an even number of neighbors in P_{15} and an odd number of neighbors in $R_{r_1 r_\ell}$, then (C_2, h_3) is an odd wheel.

Case 3: y_q is adjacent to a node $h_5 \in H_5^k$.

Then Q has no neighbors of $H_4 \cup H_6$. We first show that no node in H_2 is adjacent to a node of Q. Let y_s be the node of highest index adjacent to a node $t_2 \in H_2$. Let Σ be a connected 6-hole containing t_2, h_5, h_6 and arbitrary other attachements $h_1 \in H_1^k$, $h_3' \in H_3^k$, $h_4 \in H_4$. Now $Q_{y_s y_q} \subseteq U$ is a subpath of Q, and is a direct connection from t_2 to h_5 satisfying the assumptions of Lemma 9.13 relative to Σ. So $Q_{y_s y_q}$ must contain a neighbor of h_4 or h_6, which is a contradiction. So Q has no neighbor in $H_2 \cup H_4 \cup H_6$.

Let x_ℓ be the node of P with lowest index adjacent to a node in H_2, say t_2. Let Σ be a connected 6-hole with attachements t_2, h_5, h_6 and arbitrary $h_1 \in H_1^k$, $h_3' \in H_3^k$, $h_4 \in H_4$. Now $P_{x_1 x_\ell} \cup Q$ contains a direct connection P' from t_2 to h_5 in $G \setminus V(\Sigma)$ with no neighbors in $V(\Sigma) \setminus \{h_1, t_2, h_3', h_4, h_5, h_6\}$. Either P' is a direct connection from t_2 to h_5 satisfying the assumptions of Lemma 9.13 relative to Σ, or a subpath P'' of P' is a direct connection from h_3' to h_6 satisfying these assumptions (these are the only two possibilities since h_4 has no neighbor in P'). In both cases, Lemma 9.13 implies that P' contains a neighbor of h_1 and that $x_\ell = x_n$ (since h_4 has no neighbor in P' and h_6 is only adjacent to x_n in P' or P''). But now the nodes of $P_{x_1 x_{n-1}} \cup Q$ have no neighbors in H_2. So the nodes of $P_{x_1 x_{n-1}} \cup Q$ have no neighbors in B. Since the graph induced by $V(P_{x_1 x_{n-1}}) \cup V(Q) \cup \{h_1, h_3, h_5\}$ is connected, by Lemma 9.3, there exists a tripod Y with attachments h_1, h_3 and h_5, contained in $V(P_{x_1 x_{n-1}}) \cup V(Q) \cup \{h_1, h_3, h_5\}$.

Let $E\Sigma^*$ be the subgraph of G, induced by $V(E\Sigma) \cup V(Y)$. Let $H_i^* = H_i$ and let H^* be the graph induced by $H_1^* \cup \ldots \cup H_6^*$. Then $E(H^*)$ is a 6-join of $E\Sigma^*$, separating $T^* = T \cup Y$ from $B^* = B$. Now the connected components of $V(T^*) \setminus \{H_1^* \cup H_3^* \cup H_5^*\}$ are the same as the ones for $E\Sigma^*$ except for a new one, namely $Y^* = Y \setminus \{h_1, h_3, h_5\}$. Let H_i^Y denote the set of neighbors of Y^* in H_i, for $i = 1, 3, 5$. Since $h_1 \in H_1^Y \cap H_1^k$ and $h_5 \in H_5^Y \cap H_5^k$, the H^*-intersection graph of $E\Sigma^*$ is connected. It follows that $E\Sigma^*$ satisfies Properties (1) and (2) of Definition 9.10, a contradiction to the fact that $E\Sigma$ is extreme. □

Now we can prove Theorem 6.4.

Theorem 6.4 *A balanceable bipartite graph G that contains a connected 6-hole as an induced subgraph, has an extended star cutset or a 6-join.*

Proof: Since a connected 6-hole satisfies (1) and (2) of Definition 9.10, the assumption that G contains a connected 6-hole implies that G contains an extreme connected 6-hole $E\Sigma$. Let $U_1, \ldots, U_k$ be the connected components of $G \setminus V(E\Sigma)$ having at least one neighbor in T and at least one neighbor in B. Note that $E(H)$ is a 6-join of G, separating T and B if and only if no such component exists. By Theorem 9.12, no connected component U_j has a neighbor in $T' \cup B'$, so H contains all the neighbors of U_j. If all the neighbors of U_j belong to $H_{i-1} \cup H_i \cup H_{i+1}$ for some i, then $K_{H_i, H_{i-1} \cup H_{i+1}}$ is a biclique cutset, separating U_j and $E\Sigma$. Otherwise U_j has neighbors in H_i and in H_{i+3}, for some i. Now, by Theorem 9.14, there exists an extended star cutset, separating at least one node of U_j from $E\Sigma$. □

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