A Structure Theorem for Graphs with No Cycle with a Unique Chord and Its Consequences

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Abstract: We give a structural description of the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord as an induced subgraph. Our main theorem states that any connected graph in $\mathcal{C}$ is either in some simple basic class or has a decomposition. Basic classes are chordless cycles, cliques, bipartite graphs with one side containing only nodes of degree 2 and induced subgraphs of the famous Heawood or Petersen

graph. Decompositions are node cutsets consisting of one or two nodes and edge cutsets called 1-joins. Our decomposition theorem actually gives a complete structure theorem for $\mathcal{C}$, i.e. every graph in $\mathcal{C}$ can be built from basic graphs that can be explicitly constructed, and gluing them together by prescribed composition operations, and all graphs built this way are in $\mathcal{C}$. This has several consequences: an $\mathcal{O}(nm)$-time algorithm to decide whether a graph is in $\mathcal{C}$, an $\mathcal{O}(n+m)$-time algorithm that finds a maximum clique of any graph in $\mathcal{C}$, and an $\mathcal{O}(nm)$-time coloring algorithm for graphs in $\mathcal{C}$. We prove that every graph in $\mathcal{C}$ is either 3-colorable or has a coloring with ω colors where ω is the size of a largest clique. The problem of finding a maximum stable set for a graph in $\mathcal{C}$ is known to be NP-hard. © 2009 Wiley Periodicals, Inc. J Graph Theory 63: 31–67, 2010

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1. MOTIVATION

In this paper all graphs are simple. We give a structural characterization of graphs that do not contain a cycle with a unique chord as an induced subgraph. For the sake of conciseness we call as $\mathcal{C}$ this class of graphs. Our main result, Theorem 2.1, states that every connected graph in $\mathcal{C}$ is either in some simple basic class or has a particular decomposition. Basic classes are chordless cycles, cliques, bipartite graphs with one side containing only nodes of degree 2, and graphs that are isomorphic to an induced subgraph of the famous Heawood or Petersen graph. Our decompositions are node cutsets consisting of one or two nodes or an edge cutset called a 1-join. The definitions and the precise statement are given in Section 2. The proof is given in Section 3. Both Petersen and Heawood graphs were discovered at the end of the XIXth century in the research on the four-color conjecture, see [20, 13]. It is interesting to us to have them both as sporadic basic graphs. Note that our theorem works in two directions: a graph is in $\mathcal{C}$ if and only if it can be constructed by gluing basic graphs along our decompositions (this is proved in Section 4). Such structure theorems are stronger than the usual decomposition theorems and there are not so many of them (see [3] for a survey). This is our first motivation.

Our structural characterization allows us to prove properties of classical invariants. We prove in Section 6 that every graph G in $\mathcal{C}$ satisfies either $\chi(G) = \omega(G)$ or $\chi(G) \leq 3$ (where $\chi(G)$ denotes the chromatic number and $\omega(G)$ denotes the size of a maximum clique). This is a strengthening of the classical Vizing bound $\chi(G) \leq \omega(G) + 1$. So this class of graphs belongs to the family of χ-bounded graphs, introduced by Gyárfás [12] as a natural extension of perfect graphs: a family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function f* if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$. A natural question to ask is: what choices of forbidden induced subgraphs guarantee that a family of graphs is χ-bounded? Much research has been done in this area, for a survey

see [22]. We note that most of that research has been done on classes of graphs obtained by forbidding a finite number of graphs. Since there are graphs with arbitrarily large chromatic number and girth [11], in order for a family of graphs defined by forbidding a finite number of graphs (as induced subgraphs) to be χ-bounded, at least one of these forbidden graphs needs to be acyclic. Vizing's Theorem [25] states that for a simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ (where $\Delta(G)$ denotes the maximum vertex degree of G, and $\chi'(G)$ denotes the chromatic index of G, i.e. the minimum number of colors needed to color the edges of G so that no two adjacent edges receive the same color). This implies that the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$. This special upper bound for the chromatic number is called the *Vizing bound*. We obtain the Vizing bound for the chromatic number by forbidding a family of graphs, none of which is acyclic. Our result is algorithmic: we provide an $\mathcal{O}(nm)$ algorithm that computes an optimal coloring of every graph in $\mathcal{C}$. Furthermore, it is easy to see that there exists an $\mathcal{O}(n+m)$ algorithm that computes a maximum clique for every graph in $\mathcal{C}$, and it follows from a construction of Poljak [21] that finding a maximum stable set of a graph in $\mathcal{C}$ is NP-hard (see Section 7). All this is our second motivation.

A third motivation is the detection of induced subgraphs. A *subdivisible graph* (*s-graph* for short) is a triple $B = (V, D, F)$ such that $(V, D \cup F)$ is a graph and $D \cap F = \emptyset$. The edges in D are said to be *real edges of B* while the edges in F are said to be *subdivisible edges of B*. A *realization* of B is a graph obtained from B by subdividing edges of F into paths of arbitrary length (at least 1). The problem Π_B is the decision problem whose input is a graph G and whose question is "Does G contain a realization of B as an induced subgraph?". In the discussion below, by "detection problem," we mean "problem Π_B for some fixed s-graph B." This is restrictive since a lot of detection problems of great interest (such as the detection of odd holes, where a *hole* is an induced cycle of length at least 4) are not of that kind.

Let $H_{1|1}$ be the s-graph on nodes a, b, c, d with real edges ab, ac, ad and subdivisible edges bd, cd. We also define for $k, l \geq 1$ the s-graph $H_{k|l}$ obtained from $H_{1|1}$ by subdividing the edge ab into a path of length k and the edge ac into a path of length l. See Figure 1 where real edges are represented as straight lines and subdivisible edges as dashed lines. The question in problem $\Pi_{H_{1|1}}$ can be rephrased as "Does G contain a cycle with a unique chord?" or "Is G not in $\mathcal{C}$?". The existence of a polynomial time algorithm was an open question. A consequence of our structural description of $\mathcal{C}$ is an $\mathcal{O}(nm)$-time algorithm for $\Pi_{H_{1|1}}$ (see Section 5). This is a solution to the recognition problem for the class $\mathcal{C}$ and it is interesting for reasons explained below.

Several problem Π_B's can be solved in polynomial time by non-trivial algorithms (such as detecting pyramids in [2] and thetas in [4]) and others that may look similar at first glance are NP-complete (see [1, 18, 16] for a survey). A general criterion on an s-graph that decides whether the related decision problem is NP-complete or polynomial would be of interest. Our solution of $\Pi_{H_{1|1}}$ gives some insight in the quest for such a criterion.

A very powerful tool for solving detection problems is the algorithm three-in-a-tree of Chudnovsky and Seymour (see [4]). This algorithm decides in time $\mathcal{O}(n^4)$ whether three given nodes of a given graph G are in an induced tree of G. In [4, 16] it is observed that every detection problem Π_B for which a polynomial time algorithm is
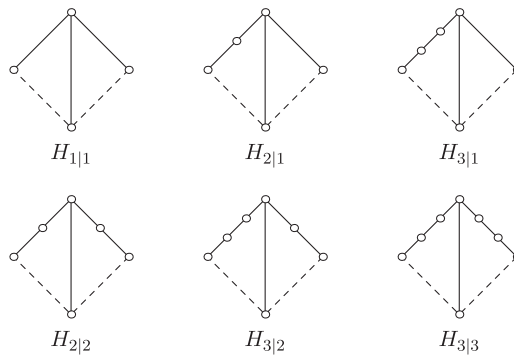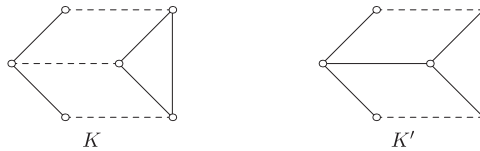
FIGURE 1. Some s-graphs.

FIGURE 2. Some s-graphs.

known can be solved easily by a brute force enumeration or by using three-in-a-tree. But as far as we can see, three-in-a-tree cannot be used to solve $\Pi_{H_{1|1}}$, so our solution of $\Pi_{H_{1|1}}$ yields the first example of a detection problem that does not fall under the scope of three-in-a-tree. Is there a good reason for that? We claim that a polynomial time algorithm for $\Pi_{H_{1|1}}$ exists thanks to what we call *degeneracy*. Let us explain this. Every statement that we give from here on to the end of the section is under the assumption that $P \neq$ NP.

Degeneracy has to deal with the following question: does putting bounds on the lengths of the paths in realizations of an s-graph affect the complexity of the related detection problem? For upper bounds, the answer can be found in previous research. First, putting upper bounds may turn the complexity from NP-complete to polynomial. This follows from a simple observation: let B be any s-graph. A realization of B, where the lengths of the paths arising from the subdivisions of subdivisible edges are bounded by an integer N, has a number of nodes bounded by a fixed integer N' (that depends only on N and the size of B). So, such a realization can be detected in time $\mathcal{O}(n^{N'})$ by a brute force enumeration. But surprisingly, putting upper bounds in another way may also turn the complexity from polynomial to NP-complete: in [2], a polynomial time algorithm for Π_K is given, while in [19] it is proved that $\Pi_{K'}$ is NP-complete, where K, K' are the s-graphs represented in Figure 2. Note that Π_K is usually called the pyramid (or 3PC($\Delta, \cdot$)) detection problem.

Can putting lower bounds turn the complexity from polynomial to NP-complete? Our recognition algorithm for $\mathcal{C}$ shows that the answer is yes since in Section 8 we also prove that the problem $\Pi_{H_{3|3}}$ is NP-complete. A realization of $H_{3|3}$ is simply a realization of $H_{1|1}$ where every subdivisible edge is subdivided into a path of length

at least 3. We believe that a satisfactory structural description of the class $\mathcal{C}'$ of graphs that do not contain a realization of $H_{3|3}$ is hopeless because $\Pi_{H_{3|3}}$ is NP-complete. So why is there a decomposition theorem for $\mathcal{C}$? Simply because degenerate small graphs like the *diamond* (that is, the cycle on four nodes with exactly one chord) are forbidden in $\mathcal{C}$, not in $\mathcal{C}'$, and this helps a lot in our proof of Theorem 2.1 (the decomposition theorem for $\mathcal{C}$). This is what we call the *degeneracy* of the class $\mathcal{C}$. It is clear that degeneracy can help in solving detection problems, and our results give a first example of this phenomenon.

So the last question is: can putting lower bounds turn the complexity from NP-complete to polynomial? We do not know the answer. Also, we were not able to solve the following questions: what is the complexity of the problems $\Pi_{H_{2|1}}$, $\Pi_{H_{3|1}}$, $\Pi_{H_{2|2}}$, and $\Pi_{H_{3|2}}$? The related classes of graphs are not degenerate enough to allow us to decompose, and they are too degenerate to allow us to find an NP-completeness proof.

A fourth motivation is that our class $\mathcal{C}$ is related to well-studied classes. It is a generalization of strongly balanceable graphs, see [7] for a survey. A bipartite graph is *balanceable* if there exists a signing of its edges with $+1$ and -1 so that the weight of every hole is a multiple of 4. A bipartite graph is *strongly balanceable* if it is balanceable and it does not contain a cycle with a unique chord. There is an excluded induced subgraph characterization of balanceable bipartite graphs due to Truemper [24]. A *wheel* in a graph consists of a hole H and a node v that has at least three neighbors in H, and the wheel is *odd* if v has an odd number of neighbors in H. In a bipartite graph G, a 3-*odd-path configuration* consists of two non-adjacent nodes u and v that are on opposite sides of the bipartition of G, together with three internally node-disjoint uv-paths, such that there are no other edges in G among the nodes of the three paths. A bipartite graph is balanceable if and only if it does not contain an odd wheel nor a 3-odd-path configuration [24]. So a bipartite graph is strongly balanceable if and only if it does not contain a 3-odd-path configuration nor a cycle with a unique chord.

A bipartite graph is *restricted balanceable* if there exists a signing of its edges with $+1$ and -1 so that the weight of every cycle is a multiple of 4. Conforti and Rao [8] show that a strongly balanceable graph is either restricted balanceable or has a 1-join, which enables them to recognize the class of strongly balanceable graphs (they decompose along 1-joins, and then directly recognize restricted balanceable graphs). A bipartite graph is 2-*bipartite* if all the nodes in one side of the bipartition have degree at most 2. Yannakakis [26] shows that a restricted balanceable graph is either 2-bipartite or has a 1-cutset or a 2-join consisting of two edges (this is an edge cutset that consists of two edges that have no common endnode), and hence obtains a linear time recognition algorithm for restricted balanceable graphs.

We note that the basic graphs from our decomposition theorem that do not have any of our cutsets, and are balanceable, are in fact 2-bipartite.

Class $\mathcal{C}$ is contained in another well-studied class of graphs, the *cap-free* graphs (where a cap is a graph that consists of a hole and a node that has exactly two neighbors on this hole, and these two neighbors are adjacent) [6]. In [6] cap-free graphs are decomposed with 1-amalgams (a generalization of a 1-join) into triangulated graphs and biconnected triangle-free graphs together with at most one additional node that is adjacent to all other nodes of the graph. This decomposition theorem is then used to

recognize strongly even-signable and strongly odd-signable graphs in polynomial time, where a graph is *strongly even-signable* if its edges can be signed with 0 and 1 so that every cycle of length ≥ 4 with at most one chord has even weight and every triangle has odd weight, and a graph is *strongly odd-signable* if its edges can be signed with 0 and 1 so that cycles of length 4 with one chord are of even weight and all other cycles with at most one chord are of odd weight.

2. THE MAIN THEOREM

We say that a graph G *contains* a graph H if H is isomorphic to an induced subgraph of G. A graph G is *H-free* if it does not contain H. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. A *cycle* C in a graph G is a sequence of nodes $v_1 v_2 \ldots v_n v_1$ that are distinct except for the first and the last node, such that for $i = 1, \ldots, n-1$, $v_i v_{i+1}$ is an edge and $v_n v_1$ is an edge (these are the edges of C). An edge of G with both endnodes in C is called a *chord* of C if it is not an edge of C. One can similarly define a path and a chord of a path. In this paper we will only use what is in the literature known as chordless paths, so for the convenience, in this paper (as in [5]) we define a path as follows: a *path* P in a graph G is a sequence of distinct nodes $v_1 \ldots v_n$ such that for $i = 1, \ldots, n-1$, $v_i v_{i+1}$ is an edge and these are the only edges of G that have both endnodes in $\{v_1, \ldots, v_n\}$. Such a path P is also called a *$v_1 v_n$-path*. A *hole* is a chordless cycle of length at least 4. A *triangle* is a cycle of length 3. A *square* is a hole of length 4. A cycle in a graph is *Hamiltonian* is every node of the graph is in the cycle. Let us define our basic classes:

The *Petersen graph* is the graph on nodes $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that $\{a_1, \ldots, a_5\}$ and $\{b_1, \ldots, b_5\}$ both induce a C_5 with nodes in their natural order, and such that the only edges between the a_i's and the b_i's are $a_1 b_1, a_2 b_4, a_3 b_2, a_4 b_5, a_5 b_3$. See Figure 3.

The *Heawood graph* is the graph on $\{a_1, \ldots, a_{14}\}$ so that $\{a_1, \ldots, a_{14}\}$ is a Hamiltonian cycle with nodes in their natural order, and such that the only other edges are $a_1 a_{10}, a_2 a_7, a_3 a_{12}, a_4 a_9, a_5 a_{14}, a_6 a_{11}, a_8 a_{13}$. See Figure 4.

It can be checked that both Petersen and Heawood graphs are in $\mathcal{C}$. Note that since the Petersen graph and the Heawood graph are both vertex-transitive, and are not themselves a cycle with a unique chord, to check that they are in $\mathcal{C}$, it suffices to delete one node, and then check that there is no cycle with a unique chord. Also the Petersen graph has girth 5 so a cycle with a unique chord in it must contain at least eight nodes. The Heawood graph has girth 6 so a cycle with a unique chord in it must contain at least ten nodes. For the Petersen graph, deleting a node yields a Hamiltonian graph, and it is easy to check that it does not contain a cycle with a unique chord. For the Heawood graph, it is useful to notice that deleting one node yields the Petersen graph with edges $a_1 b_1, b_3 b_4, a_3 a_4$ subdivided.

Let us define our last basic class. A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where X is the set of all degree 2 nodes of G and Y is the set of all nodes of G with degree at least 3. A strongly 2-bipartite graph is clearly in $\mathcal{C}$ because any chord of a cycle is an edge linking two nodes of degree at least 3, so every cycle in a strongly 2-bipartite graph is chordless.
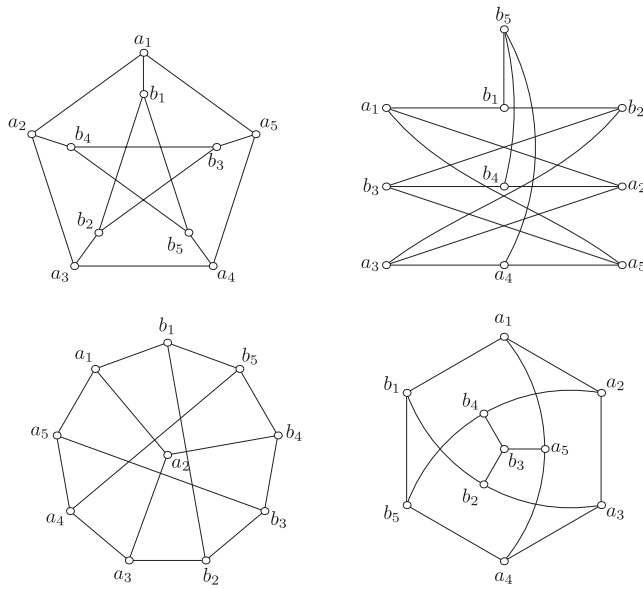
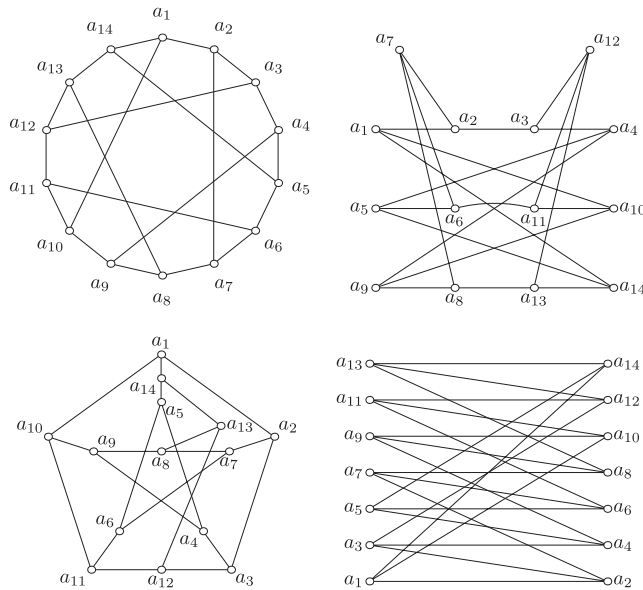FIGURE 3. Four ways to draw the Petersen graph.

FIGURE 4. Four ways to draw the Heawood graph.

We now define cutsets used in our decomposition theorem:

- A 1-*cutset* of a connected graph G is a node v such that $V(G)$ can be partitioned into non-empty sets X, Y, and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.
- A *proper* 2-*cutset* of a connected graph G is a pair of non-adjacent nodes a, b, both of degree at least 3, such that $V(G)$ can be partitioned into non-empty sets X, Y, and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there are no edges between X and Y; and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.
- A 1-*join* of a graph G is a partition of $V(G)$ into sets X and Y such that there exist sets A, B satisfying:

  ○ $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
  ○ $|X| \geq 2$ and $|Y| \geq 2$;
  ○ there are all possible edges between A and B;
  ○ there are no other edges between X and Y.

  We say that (X, Y, A, B) is a *split* of this 1-join. The sets A, B are *special sets* with respect to this 1-join.
  1-Joins were first introduced by Cunningham [9]. In our paper we will use a special type of a 1-join called a *proper* 1-*join*: a 1-join such that A and B are stable sets of G of size at least two. Note that a square admits a proper 1-join.

Our main result is the following decomposition theorem:

Theorem 2.1. *Let G be a connected graph that does not contain a cycle with a unique chord. Then either G is strongly 2-bipartite, or G is a hole of length at least 7, or G is a clique, or G is an induced subgraph of the Petersen or the Heawood graph, or G has a 1-cutset, a proper 2-cutset, or a proper 1-join.*

The following intermediate results are proved in the next section. Theorem 2.1 follows from Theorems 2.3 and 2.4 (more precisely, it follows from 2.4 for square-free graphs, and from 2.3 for graphs that contain a square).

Theorem 2.2. *Let G be a connected graph that does not contain a cycle with a unique chord. If G contains a triangle then either G is a clique, or one node of the maximal clique that contains this triangle is a 1-cutset of G.*

Theorem 2.3. *Let G be a connected graph that does not contain a cycle with a unique chord. Suppose that G contains either a square, the Petersen graph, or the Heawood graph. Then either G is the Petersen graph or G is the Heawood graph or G has a 1-cutset or a proper 1-join.*

Theorem 2.4. *Let G be a connected square-free graph that does not contain a cycle with a unique chord. Then either G is strongly 2-bipartite, or G is a hole of length at least 7, or G is a clique, or G is an induced subgraph of the Petersen or the Heawood graph, or G has a 1-cutset or a proper 2-cutset.*

3. PROOF OF THEOREMS 2.2, 2.3, AND 2.4

We first need two lemmas:

Lemma 3.1.   *Let G be a graph in $\mathcal{C}$, H a hole of G, and v a node of $G \setminus H$. Then v has at most two neighbors in H, and these two neighbors are not adjacent.*

Proof.   If v has at least three neighbors in H, then H contains a subpath P with exactly three neighbors of v and $V(P) \cup \{v\}$ induces a cycle of G with a unique chord, a contradiction. If v has two neighbors in H, they must be non-adjacent for otherwise $H \cup \{v\}$ is a cycle with a unique chord.   ∎

In a connected graph G two nodes a and b form a *2-cutset* if $G \setminus \{a, b\}$ is disconnected.

Lemma 3.2.   *Let G be a connected graph that has no 1-cutset. If $\{a, b\}$ is a 2-cutset of G and ab is an edge, then $G \notin \mathcal{C}$.*

Proof.   Suppose $\{a, b\}$ is a 2-cutset of G, and ab is an edge. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus \{a, b\}$. Since G is connected and has no 1-cutset, for every $i \in \{1, \ldots, k\}$, both a and b have a neighbor in C_i. Let G' be the graph obtained from G by removing the edge ab. So for every $i \in \{1, \ldots, k\}$, there is an ab-path P_i in G' whose interior nodes are contained in C_i. Then $G[V(P_1) \cup V(P_2)]$ is a cycle with a unique chord, and hence $G \notin \mathcal{C}$.   ∎

If H is any induced subgraph of G and D is a subset of nodes of $G \setminus H$, the *attachment of D over H* is the set of all nodes of H that have at least one neighbor in D. When clear from context we do not distinguish between a graph and its node set, so we also refer to the attachment of $G[D]$ over H.

A.   Proof of Theorem 2.2

Suppose G contains a triangle, and let C be a maximal clique of G that contains this triangle. In fact, C is unique or otherwise G contains a diamond. If $G \neq C$ and if no node of C is a 1-cutset of G then let D be a connected induced subgraph of $G \setminus C$, whose attachment over C contains at least two nodes, and that is minimal with respect to this property. So, D is a path with one end adjacent to $a \in C$, the other end adjacent to $b \in C \setminus \{a\}$, and $D \cup \{a, b\}$ induces a chordless cycle. If D has length 0, then its unique node (say u) must have a non-neighbor $c \in C$ since C is maximal. Hence, $\{u, a, b, c\}$ induces a diamond, a contradiction. If D has length at least 1 then let $c \neq a, b$ be any node of C. Then the hole induced by $D \cup \{a, b\}$ and node c contradict Lemma 3.1. This proves Theorem 2.2.

B.   Proof of Theorem 2.3

Claim 1.   *We may assume that G is triangle-free.*

Proof.   Clear by Theorem 2.2 (note that G cannot be a clique).   ∎

Claim 2.   *We may assume that G is square-free.*

Proof.   Assume G contains a square. Then G contains disjoint sets of nodes A and B such that $G[A]$ and $G[B]$ are both stable graphs, $|A|, |B| \geq 2$, and every node of A is adjacent to every node of B. Let us suppose that $A \cup B$ is chosen to be maximal with respect to this property. If $V(G) = A \cup B$ then (A, B) is a proper 1-join of G, so we may assume that there are nodes in $G \setminus (A \cup B)$.

(1) *Every component of* $G \setminus (A \cup B)$ *has neighbors only in* A *or only in* B.

Else let us take a connected induced subgraph D of $G \setminus (A \cup B)$, whose attachment over $A \cup B$ contains nodes of both A and B, and that is minimal with respect to this property. So $D = u \ldots v$ is a path, no interior node of which has a neighbor in $A \cup B$, and there exists $a \in A$, $b \in B$ such that $ua, vb \in E(G)$. By Claim 1, $u \neq v$, u has no neighbor in B, and v has no neighbor in A. By maximality of $A \cup B$, u has a non-neighbor $a' \in A$ and v has a non-neighbor $b' \in B$. Now, $D \cup \{a, b, a', b'\}$ is a cycle with a unique chord (namely ab), a contradiction. This proves (1).

From (1), it follows that G has a proper 1-join with special sets A, B.   ∎

Now, we just have to prove the following two claims:

Claim 3.   *If G contains the Petersen graph then the theorem holds.*

Proof.   Let $\Pi = \{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ be a set of ten nodes of G so that $G[\Pi]$ has adjacencies as in the definition of the Petersen graph. We may assume that there are some other nodes in G for otherwise the theorem holds.

(1) *A node of $G \setminus \Pi$ has at most one neighbor in Π.*

Otherwise G contains a triangle or a square, contrary to Claims 1 and 2. This proves (1).

Here below, we use symmetries in the Petersen graph to shorten the list of cases. First, the Petersen graph is edge-transitive, so up to an automorphism all edges are equivalent. But also, it is "distance-two-transitive," meaning that every induced P_3 is equivalent to every other induced P_3. To see this, it suffices to check that every induced P_3 is included in an induced C_5 and that removing any P_3 always yields the same graph.

(2) *The attachment of any component of $G \setminus \Pi$ over Π contains at most one node.*

Else let D be a connected induced subgraph of $G \setminus \Pi$ whose attachment over Π contains at least two nodes, and that is minimal with respect to this property. By minimality and up to symmetry, D is a path with one end adjacent to a_1 (and to no other node of Π by (1)) and one end adjacent to $x \in \{a_2, a_3\}$ (and to no other node of Π). Moreover, no interior node of D has a neighbor in Π. If $x = a_2$ then $D \cup \{a_1, \ldots, a_5\}$ is a cycle with a unique chord, a contradiction. If $x = a_3$ then $D \cup \{a_1, a_5, b_3, b_4, b_5, a_4, a_3\}$ is a cycle with a unique chord, a contradiction again. This proves (2).

From (2) it follows that G has a 1-cutset.   ∎

Claim 4.   *If G contains the Heawood graph then the theorem holds.*

Proof.   Let $\Pi = \{a_1, \ldots, a_{14}\}$ be a set of 14 nodes of G so that $G[\Pi]$ has adjacencies as in the definition of the Heawood graph. We may assume that there are some other

nodes in G for otherwise the theorem holds.

(1) *A node of $G \setminus \Pi$ has at most two neighbors in Π.*

Suppose that some node v in $G \setminus \Pi$ has at least two neighbors in Π. Since the Heawood graph is vertex-transitive we may assume $va_1 \in E(G)$. By Claims 1 and 2, v cannot be adjacent to a node at distance one or two from v, namely to any of $a_2, a_{14}, a_{10}, a_3, a_{13}, a_9, a_{11}, a_5, a_7$. So, the only other possible neighbors are a_4, a_6, a_8, a_{12}. But these four nodes are pairwise at distance two in Π, so by Claim 2 v can be adjacent to at most one of them. This proves (1).

(2) *The attachment of any component of $G \setminus \Pi$ over Π contains at most one node.*

Else let D be a connected induced subgraph of $G \setminus \Pi$ whose attachment over Π contains at least two nodes, and is minimal with respect to this property. By minimality and up to symmetry, D is a path, possibly of length 0, with one end adjacent to a_1, one end adjacent to a_i where $i \neq 1$, and no interior node of D has neighbors in Π. Note that, by assumption, if D is of length at least 1 then no end of D can have more than one neighbor in Π, because such an end would contradict the minimality of D. So, a_1, a_i are the only nodes of Π that have neighbors in D (when D is of length 0 this holds by (1)).

If $i = 2$ then $D \cup \{a_1, a_2, a_7, a_8, a_9, a_{10}\}$ is a cycle with a unique chord. If $i \in \{3, 4, 5\}$ then $D \cup \{a_i, a_{i+1}, \ldots, a_8, a_{13}, a_{14}, a_1\}$ is a cycle with a unique chord. If $i = 6$ then $D \cup \{a_6, a_5, a_4, a_9, a_8, a_{13}, a_{14}, a_1\}$ is a cycle with a unique chord. If $i \in \{7, 8\}$ then $D \cup \{a_1, a_2, \ldots, a_i\}$ is a cycle with a unique chord. If $i = 9$ then $D \cup \{a_9, \ldots, a_{14}\}$ is a cycle with a unique chord. If $i \in \{10, 11\}$ then $D \cup \{a_1, a_2, a_3, a_4, a_9, a_{10}, a_i\}$ is a cycle with a unique chord. If $i \in \{12, 13\}$ then $D \cup \{a_1, a_2, a_3, a_4, a_5, a_6, a_{11}, a_{12}, a_i\}$ is a cycle with a unique chord. If $i = 14$ then $D \cup \{a_1, a_2, a_7, a_8, a_{13}, a_{14}\}$ is a cycle with a unique chord. In every case, there is a contradiction. This proves (2).

From (2) it follows that G has a 1-cutset.   ∎

This proves Theorem 2.3.

C.   Proof of Theorem 2.4

We consider a graph G containing no cycle with a unique chord and no square. So:

Claim 1.   *G is square-free.*

Our proof now goes through 13 claims, most of them of the same kind: if some basic graph H is an induced subgraph of G, then either $G = H$ and so G itself is basic, or some nodes of $G \setminus H$ must be attached to H in a way that entails a proper 2-cutset. At the end of this process there are so many induced subgraphs forbidden in G such that we can prove that G is strongly 2-bipartite.

Claim 2.   *We may assume that G is triangle-free.*

Proof.   Clear by Theorem 2.2.   ∎

Claim 3.   *We may assume that G does not contain the Petersen graph.*

Proof. Clear by Theorem 2.3. Note that G cannot admit a proper 1-join since it is square-free. ∎

Claim 4. *We may assume that G does not contain the Heawood graph.*

Proof. Clear by Theorem 2.3. ∎

Claim 5. *We may assume that G does not contain the following configuration*: *three node-disjoint paths $X = x \ldots x'$, $Y = y \ldots y'$, and $Z = z \ldots z'$ of length at least 2 and with no edges between them. There are four more nodes a, b, c, d. The only edges except those from the paths are $ax, ay, az, bx', by, bz', cx', cy', cz, dx, dy', dz'$.*

Proof. Let $\Pi = X \cup Y \cup Z \cup \{a, b, c, d\}$. Nodes $a, b, c, d, x, x', y, y', z, z'$ are called here the *branch nodes* of Π. It is convenient to notice that $G[\Pi]$ can be obtained by subdividing the edges of any induced matching of size three of the Petersen graph. Note also that either $G[\Pi]$ is the Heawood graph with one node deleted (when X, Y, Z all have length 2), or $G[\Pi]$ has a proper 2-cutset (when one of the paths is of length at least 3, the proper 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \setminus \Pi$.

(1) *A node of $G \setminus \Pi$ has neighbors in at most one of the following sets*: $X, Y, Z, \{a\}, \{b\}, \{c\}, \{d\}$.

Let u be a node of $G \setminus \Pi$. Note that a, b, c, d are pairwise at distance two in Π, so by Claims 1 and 2, u can be adjacent to at most one of them.

Suppose first that $ux \in E(G)$. Then u can be adjacent to none of a, d, y, y', z, z' by Claims 1 and 2. If u is adjacent to some other branch node of $G[\Pi]$ distinct from x', then we may assume that u is adjacent to one of b, c (say b up to symmetry), but then u is not adjacent to c so $uxdy'czaybu$ is a cycle with a unique chord, a contradiction. Hence we may assume that u has a neighbor v in the interior of Y or Z (say Z up to symmetry) for otherwise the claim holds. By Lemma 3.1 and since $xdz'Zzcx'Xx$ is a hole, we note that u has exactly two neighbors in $X \cup Z$, namely x and v. If $vz' \notin E(G)$ then $xuvZzaybz'dx$ is a cycle with a unique chord, a contradiction. So $vz' \in E(G)$ and $xuvZzcx'bz'dx$ is a cycle with a unique chord (namely vz'), a contradiction again. Hence we may assume that $ux \notin E(G)$ and symmetrically $ux', uy, uy', uz, uz' \notin E(G)$.

Suppose now that $ua \in E(G)$. Then u cannot be adjacent to any other branch node of Π by the discussion above. So we may assume that u has a neighbor in the interior of X, Y, Z (say Z up to symmetry) for otherwise our claim holds. Now we define v to be the neighbor of u along Z closest to z and observe that $uvZzcx'byau$ is a cycle with a unique chord, a contradiction. Therefore we may assume that u is not adjacent to any branch node of Π.

We may suppose now that u has neighbors in the interior of at least two of the paths among X, Y, Z (say X, Z w.l.o.g.) for otherwise our claim holds. Let v (resp. w) be a neighbor of u in X (resp. Z). Since $X \cup Z \cup \{d, c\}$ induces a hole, by Lemma 3.1, $N(u) \cap (X \cup Z) = \{v, w\}$. If $vx \notin E(G)$ then $vuwZz'dxaybx'Xv$ is a cycle with a unique chord. So $vx \in E(G)$ and, symmetrically, $vx', wz, wz' \in E(G)$. If u has no neighbor in Y then $vuwz'dy'Yybx'v$ is a cycle with a unique chord, so u must have at least one neighbor w' in Y. By the same discussion that we have done above on X, Z, we can

prove that $w'y, w'y' \in E(G)$. Now we observe that $G[\Pi \cup \{u\}]$ is the Heawood graph, contradicting Claim 4. This proves (1).

(2) *The attachment of any component of $G \setminus \Pi$ is included in one of the sets $X, Y, Z, \{a\}, \{b\}, \{c\}, \{d\}$.*

Else let D be a connected induced subgraph of $G \setminus \Pi$ whose attachment overlaps two of the sets, and is minimal with respect to this property. By (1), D is a path of length at least 1, with ends u, v and u (resp. v) has neighbors in exactly one set S_u (resp. S_v) of $X, Y, Z, \{a\}, \{b\}, \{c\}, \{d\}$. Moreover, no interior node of D has a neighbor in Π. If $S_u = X$ and $S_v = \{a\}$ then let w be the neighbor of u closest to x along X. We observe that $ayYy'dxXwuDva$ is a cycle with a unique chord, a contradiction. Every case where there is an edge between S_u and S_v is symmetric, so we may assume that there is no edge between S_u and S_v. If $S_u = X$ and $S_v = Z$ then let w (resp. w') be the neighbor of u (resp. of v) closest to x' along X (resp. to z' along Z). If $w = x$ and $w' = z$ then $xdy'Yyazv Dux$ is a cycle with a unique chord, a contradiction. So up to symmetry we may assume $w' \neq z$. Hence $wXx'cy'Yybz'Zw'vDuw$ is a cycle with a unique chord, a contradiction. So up to symmetry we may assume that $S_u = \{a\}$ and $S_v = \{c\}$. But then $auDvcx'bz'Zza$ is a cycle with a unique chord, a contradiction. This proves (2).

By (2), either some component of $G \setminus \Pi$ attaches to a node of Π and there is a 1-cutset, or some component attaches to one of X, Y, Z (say X up to symmetry), and $\{x, x'\}$ is a proper 2-cutset. ∎

Claim 6. *We may assume that G does not contain the following configuration: three node-disjoint paths $X = x \ldots x'$, $Y = y \ldots y'$, and $Z = z \ldots z'$, of length at least 2, and such that the only edges between them are xy, yz, zx', $x'y'$, $y'z'$, and $z'x$.*

Proof. Note that $G[x, y, z, x', y', z']$ is a hole on six nodes. Also either $G[X \cup Y \cup Z]$ is the Petersen graph with one node deleted (when X, Y, Z have length 2), or $G[X \cup Y \cup Z]$ has a proper 2-cutset (when one of the paths is of length at least 3, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \setminus (X \cup Y \cup Z)$.

(1) *A node of $G \setminus (X \cup Y \cup Z)$ has neighbors in at most one of the sets X, Y, Z.*

Let u be a node of $G \setminus (X \cup Y \cup Z)$. Note that u has neighbors in at most one of the following sets: $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$, for otherwise G contains a triangle or a square, contradicting Claims 1 and 2.

If u has at least two neighbors among x, x', y, y', z, z' then we may assume by the paragraph above that $ux, ux' \in E(G)$. Since $X \cup Y$ and $X \cup Z$ both induce holes, every node in $X \cup Y \cup Z$ is in a hole going through x, x'. So, by Lemma 3.1 u has no other neighbors in $X \cup Y \cup Z$. Hence, from here on, we assume that u has at most one neighbor among x, x', y, y', z, z'.

If $ux \in E(G)$ then we may assume that u has neighbors in one of Y, Z, say Z up to symmetry. Let $v \in Z$ be a neighbor of u. Then by Lemma 3.1, since $X \cup Z$ induces a hole, v and x are the only neighbors of u in $X \cup Z$. So, $xuvZz'y'x'Xx$ is a cycle with a unique chord, a contradiction. Hence we may assume that u has no neighbors among x, x', y, y', z, z'.

If u has neighbors in the interior of at most one of X, Y, Z our claim holds, so let us suppose that u has neighbors in the interior of X and the interior of Y. Since $X \cup Y$ induces a hole, by Lemma 3.1, u has a unique neighbor $v \in X$ and a unique neighbor $w \in Y$. If u has no neighbor in Z then $xXvuwYyzZz'x$ is a cycle with a unique chord, a contradiction. So u has a neighbor $w' \in Z$ that is unique by Lemma 3.1. If $vx, wy \notin E(G)$ then $x'zyxz'y'YwuvXx'$ is a cycle with a unique chord, a contradiction. So, up to symmetry we may assume $vx \in E(G)$. If $wy' \notin E(G)$ then $x'y'z'xyYwuvXx'$ is a cycle with a unique chord, a contradiction, so $wy' \in E(G)$. By the same argument, we can prove that $w'z \in E(G)$. If $vx', wy, w'z \in E(G)$ then we observe that $G[X \cup Y \cup Z \cup \{u\}]$ is the Petersen graph, contradicting Claim 3. If $vx', wy, w'z \notin E(G)$ then we observe that the three paths $vXx', wYy, w'Zz'$ and nodes u, z, y', x have the same configuration as those in Claim 5, a contradiction. So, we may assume that $vx' \in E(G)$ and $wy \notin E(G)$. But then, $wuvxyzx'y'w$ is a cycle with a unique chord, a contradiction. This proves (1).

(2) *The attachment of any component of $G \setminus (X \cup Y \cup Z)$ is included in one of the sets X, Y, Z.*

Else let D be a connected induced subgraph of $G \setminus (X \cup Y \cup Z)$, whose attachment overlaps two of the sets, and is minimal with this property. By (1), D is a path of length at least 1, with ends u, v, and no interior node of D has a neighbor in $X \cup Y \cup Z$. We may assume that u has neighbors only in X and v only in Y. Let u' (resp. v') be the neighbor of u (resp. of v) closest to x along X (resp. to y along Y). If $u' \neq x'$ and $v' \neq y'$ then $zyYv'vDuu'Xxz'Zz$ is a cycle with a unique chord, a contradiction, so we may assume $v' = y'$. Let u'' be the neighbor of u closest to x' along X. If $u'' \neq x$ then $uu''Xx'zZz'y'vDu$ is a cycle with a unique chord, a contradiction. So, $u'' = x$ and $uxz'Zzx'y'vDu$ is a cycle with a unique chord, a contradiction. This proves (2).

By (2) one of $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$ is a proper 2-cutset of G. ∎

Claim 7. *We may assume that G does not contain the following configuration: four node-disjoint paths $X = a_1 \dots a_5$, $Y = a_2 \dots a_6$, $Z = a_3 \dots a_7$, and $T = a_4 \dots a_8$, of length at least 2, and such that the only edges between them are $a_1 a_2$, $a_2 a_3$, $a_3 a_4$, $a_4 a_5$, $a_5 a_6$, $a_6 a_7$, $a_7 a_8$, and $a_8 a_1$.*

Proof. Either $G[X \cup Y \cup Z \cup T]$ is obtained from the Heawood graph by deleting two adjacent nodes (when X, Y, Z, T have length 2), or $G[X \cup Y \cup Z \cup T]$ has a proper 2-cutset (when one of the paths is of length at least 3, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \setminus (X \cup Y \cup Z \cup T)$.

(1) *A node of $G \setminus (X \cup Y \cup Z \cup T)$ has at most two neighbors in $X \cup Y \cup Z \cup T$.*

For suppose that a node u of $G \setminus (X \cup Y \cup Z \cup T)$ has at least three neighbors in $X \cup Y \cup Z \cup T$. Since every pair of path from X, Y, Z, T can be embedded in a hole (for instance, $X \cup Y$ or $X \cup Z \cup \{a_2, a_6\}$ are holes, and the other cases are symmetric), by Lemma 3.1, the neighbors of u lie on three or four paths and every path contains at most one neighbor of u.

Suppose u is adjacent to one of the a_i's, say a_1. Then by Lemma 3.1 u has at most one neighbor in $Y \cup T$ since $Y \cup T \cup \{a_1, a_5\}$ is a hole. So up to symmetry we assume that u has a neighbor v in Y, no neighbor in T, and so u must have a neighbor w in Z. By Lemma 3.1 applied to the hole $X \cup Z \cup \{a_2, a_6\}$ and node u, v must be in the

interior of Y. If $w \neq a_3$ then $wZa_7a_8Ta_4a_5a_6Yvuw$ is a cycle with a unique chord. So $w = a_3$ and G contains a square, a contradiction to Claim 1. Hence we may assume that u has no neighbors among the a_i's.

Up to symmetry we assume that u has neighbors $x \in X$, $y \in Y$, $z \in Z$. These neighbors are unique and are in the interior of their respective paths. So $uxXa_5a_6Ya_2a_3Zzu$ is a cycle with a unique chord (namely uy), a contradiction. This proves (1).

(2) *The attachment of any component of $G \setminus (X \cup Y \cup Z \cup T)$ is included in one of the sets X, Y, Z, T.*

Else let D be a connected induced subgraph of $G \setminus (X \cup Y \cup Z \cup T)$, whose attachment overlaps two of the sets, and is minimal with respect this property. By the choice of D, the following hold. D is a path, possibly of length 0, with ends u, v, and we may assume up to symmetry that u has neighbors in X and that v has neighbors in Y or in Z. No interior node of D has neighbors in $X \cup Y \cup Z \cup T$. If $u \neq v$ then u has neighbors only in X and v only in Y or in Z. If $u = v$ then by (1) u has neighbors only in $X \cup Y$ or only in $X \cup Z$.

If v has neighbors in Y then let x be the neighbor of u closest to a_5 along X and y be the neighbor of v closest to a_6 along Y. If $x = a_1$ and $y = a_2$ then $D \cup \{a_1, \ldots, a_8\}$ is a cycle with a unique chord, so up to symmetry we may assume $x \neq a_1$. But then, $uxXa_5a_4Ta_8a_7a_6YyvDu$ is a cycle with a unique chord, a contradiction.

So v has neighbors in Z. We claim that v has a unique neighbor z in Z, that is in the interior of Z, and that Z has length 2. Else, up to the symmetry between a_3 and a_7, we may assume that the neighbor z of v closest to a_3 along Z is not a_7 and is not adjacent to a_7. Let x be the neighbor of u closest to a_5 along X. If $x = a_1$ then $a_1Xa_5a_6Ya_2a_3ZzvDua_1$ is a cycle with a unique chord. So $x \neq a_1$. If v is not adjacent to a_7, then $a_5a_6a_7a_8Ta_4a_3ZzvDuxXa_5$ is a cycle with a unique chord, a contradiction. So v is adjacent to a_7. In particular, $u \neq v$. By (1), a_7 and z are the only neighbors of v in Z. But then $a_5a_6a_7ZzvDuxXa_5$ is a cycle with a unique chord, a contradiction. So, our claim is proved. Similarly, it can be proved that u has a unique neighbor x in X, that this neighbor is in the interior of X, and that X has length 2. We observe that the three paths $xuDvz$, Y, T and nodes a_1, a_3, a_7, a_5 have the same configuration as those in Claim 5, a contradiction. This proves (2).

By (2), one of $\{a_1, a_5\}$, $\{a_2, a_6\}$, $\{a_3, a_7\}$, $\{a_4, a_8\}$ is a proper 2-cutset. ∎

Claim 8.   *We may assume that G does not contain the following configuration*: *five paths $P_{13} = a_1 \ldots a_3$, $P_{15} = a_1 \ldots a_5$, $P_{48} = a_4 \ldots a_8$, $P_{37} = a_3 \ldots a_7$, and $P_{57} = a_5 \ldots a_7$, node disjoint except for their ends, of length at least 2, and such that $G[P_{13} \cup P_{15} \cup P_{37} \cup P_{57}]$ is a hole and the only edges between this hole and P_{48} are a_3a_4, a_4a_5, a_7a_8, and a_8a_1.*

Proof.   We put $\Pi = P_{13} \cup P_{15} \cup P_{48} \cup P_{37} \cup P_{57}$. Either $G[\Pi]$ is the Heawood graph with three nodes inducing a P_3 deleted (when the five paths have length 2), or $G[\Pi]$ has a proper 2-cutset (when one of the paths is of length at least 3, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \setminus \Pi$.

(1) *A node of $G \setminus \Pi$ has at most two neighbors in Π.*

Let u be a node of $G \setminus \Pi$ and suppose that u has more than two neighbors in Π. By Lemma 3.1 and since $P_{13} \cup P_{15} \cup P_{37} \cup P_{57}$ is a hole, u has at most two neighbors

among these paths. So, u must have one neighbor in P_{48}, and this neighbor is unique since the union of P_{48} with any of the other paths yields a hole. For the same reason, u has a unique neighbor in exactly two paths among $P_{13}, P_{15}, P_{37}, P_{57}$. So there are two cases up to symmetry: either u has neighbors in two paths among $P_{13}, P_{15}, P_{37}, P_{57}$ that have a common end, or u has neighbors in two paths among $P_{13}, P_{15}, P_{37}, P_{57}$ that have no common ends.

In the first case, we may assume that u has neighbors $x \in P_{37}$, $y \in P_{48}$, and $z \in P_{13}$. Note that $x \neq a_3$ and $z \neq a_3$, for otherwise P_{13} or P_{37} would contain two neighbors of u. Suppose $y \neq a_4$. If $z \neq a_1$ then $xuyP_{48}a_8a_1P_{15}a_5P_{57}a_7P_{37}x$ is a cycle with a unique chord, a contradiction. If $z = a_1$ then by Claim 1, $x \neq a_7$ and hence $xuyP_{48}a_8a_1P_{13}a_3P_{37}x$ is a cycle with a unique chord, a contradiction. So $y = a_4$. But then, since G does not contain a square, $xa_3 \notin E(G)$ and hence $uxP_{37}a_7a_8a_1P_{13}a_3a_4u$ is a cycle with a unique chord, a contradiction.

In the second case, we may assume that u has neighbors x in P_{13}, y in P_{48}, and z in P_{57}. If $x = a_1$ then the previous case applies. Hence we may assume $x \neq a_1$ and symmetrically $z \neq a_5$. So, $uxP_{13}a_3a_4P_{48}a_8a_7P_{57}zu$ is a cycle with a unique chord (namely uy). This proves (1).

(2) *The attachment of any component of $G \setminus \Pi$ is included in one of the sets $P_{13}, P_{15}, P_{37}, P_{57}, P_{48}$.*

Else let D be a connected induced subgraph of $G \setminus \Pi$, whose attachment is not contained in one of the sets, and is minimal with respect to this property. By the choice of D the following hold. D is a path, possibly of length 0, with ends u, v, where u has neighbors in one of the sets $P_{13}, P_{15}, P_{37}, P_{57}, P_{48}$ that we denote by X_u, and v has neighbors in another one, say X_v. No interior node of D has neighbors in Π. If $u \neq v$ then u has neighbors only in X_u and v only in X_v. If $u = v$ then by (1) u has neighbors only in $X_u \cup X_v$.

If $X_u = P_{48}$ then up to symmetry we may assume $X_v = P_{57}$. Let x be the neighbor of u closest to a_8 along P_{48}. If $x \neq a_4$ then let y be the neighbor of v closest to a_7 along P_{57}. Then $uDvyP_{57}a_7P_{37}a_3P_{13}a_1a_8P_{48}xu$ is a cycle with a unique chord. If $x = a_4$ then let y be the neighbor of v closest to a_5 along P_{57}. Then $uDvyP_{57}a_5P_{15}a_1P_{13}a_3a_4u$ is a cycle with a unique chord. So, $X_u \neq P_{48}$ and symmetrically $X_v \neq P_{48}$.

If X_u, X_v are paths with a common end then we may assume $X_u = P_{37}$ and $X_v = P_{57}$. Let x be the neighbor of u closest to a_3 along P_{37} and y the neighbor of v closest to a_5 along P_{57}. We note that $x, y \neq a_7$ for otherwise the attachment of D is a single path P_{37} or P_{57} contrary to the definition of D. So, $xuDvyP_{57}a_5a_4P_{48}a_8a_1P_{13}a_3P_{37}x$ is a cycle with a unique chord.

If X_u, X_v are paths with no common end then we may assume $X_u = P_{13}$ and $X_v = P_{57}$. We claim that u has a unique neighbor in P_{13}, that is in the interior of P_{13}, and that P_{13} has length 2. Else, up to the symmetry between a_1 and a_3, we may assume that the neighbor x of u closest to a_3 along P_{13} is not a_1 and is not adjacent to a_1. Let y be the neighbor of v closest to a_5 along P_{57}. If $y = a_7$ then the previous case applies, i.e. the neighbors of u and v in Π are contained in the paths with a common end. So $y \neq a_7$. If u is not adjacent to a_1, then $xuDvyP_{57}a_5P_{15}a_1a_8P_{48}a_4a_3P_{13}x$ is a cycle with a unique chord. So u is adjacent to a_1. By (1), a_1 and x are the only neighbors of u in Π. But then $xuDvyP_{57}a_5P_{15}a_1P_{13}x$ is a cycle with a unique chord.

Our claim is proved, and similarly we can prove that v has a unique neighbor in P_{57}, that this neighbor is in the interior of P_{57}, and that P_{57} has length 2. Now we observe that the paths $xuDvy$, P_{15}, P_{37}, P_{48} have the same configuration as those in Claim 7, a contradiction. This proves (2).

By (2), one of $\{a_1, a_5\}$, $\{a_1, a_3\}$, $\{a_4, a_8\}$, $\{a_5, a_7\}$ $\{a_3, a_7\}$ is a proper 2-cutset.   ∎

Claim 9.   *We may assume that G does not contain a cycle with exactly two chords.*

Proof.   For let C be a cycle in G with exactly two chords ab, cd. We may assume up to the symmetry between c and d that a, c, b, d appear in this order along C for otherwise there is a cycle with a unique chord. We denote by P_{ac} the unique path in C from a to c that does not go through b, d. We define similarly P_{cb}, P_{bd}, P_{da}. We assume that C is a cycle with exactly two chords in G that has the fewest number of nodes.

If P_{cb} has length 1 then $P_{ac} \cup P_{bd}$ is a cycle with a unique chord unless P_{ad} also has length 1. But then $G[a, b, c, d]$ is a square or contains a triangle, a contradiction to Claims 1 and 2. So P_{cb} has length at least 2 and, symmetrically, P_{ac}, P_{bd}, P_{da} have all length at least 2.

Note that either C is the Petersen graph with two adjacent nodes deleted (when C is on eight nodes), or C has a proper 2-cutset (when C is on at least nine nodes). Hence we may assume that there are nodes in $G \setminus C$.

(1) *A node of $G \setminus C$ has at most two neighbors in C, and if it has two neighbors in C then these two neighbors are not included in one of the sets $P_{ac}, P_{cb}, P_{bd}, P_{da}$.*

Let u be a node of $G \setminus C$ that has at least three neighbors in C. Note that, by Lemma 3.1, u has at most one neighbor in each of P_{ac}, P_{cb}, P_{bd}, P_{da} because the union of any two of them forms a hole. So, up to symmetry we may assume that u has neighbors $x \in P_{ad}$, $y \in P_{ac}$, $z \in P_{bd}$ (and possibly one more in P_{cb}). If $y = c$ then $x \neq d$ and $xd \notin E(G)$ for otherwise G contains a square or a triangle, contradicting Claims 1 and 2, and hence $ucd P_{bd} ba P_{ad} xu$ has a unique chord (namely uz). So $c \neq y$ and, symmetrically, $b \neq z$. Hence $yuz P_{bd} d P_{ad} a P_{ac} y$ is a cycle with a unique chord (namely ux), a contradiction. So u has at most two neighbors in C.

Let x and y be two neighbors of u in C, and suppose that they both belong to the same path, say P_{bd}. W.l.o.g. x is closer to b on P_{bd}. By Claims 1 and 2, the xy-subpath P of P_{bd} is of length greater than 2. Let C' be the cycle induced by $(C \setminus P) \cup \{x, y\}$. Then C' is a cycle with exactly two chords that has fewer nodes than C, contradicting our choice of C. This proves (1).

(2) *The attachment of any component of $G \setminus C$ is included in one of the sets $P_{ac}, P_{cb}, P_{bd}, P_{da}$.*

Else let D be a connected induced subgraph of $G \setminus C$, whose attachment is not contained in one of the sets and is minimal with respect to this property. By the choice of D the following hold. D is a path possibly of length 0, with ends u, v, where u has neighbors in one of the sets $P_{ac}, P_{cb}, P_{bd}, P_{da}$ that we denote by X_u, and v has neighbors in another one, say X_v. No interior node of D has neighbors in C. If $u \neq v$ then u has neighbors only in X_u and v only in X_v. If $u = v$, then by (1) u has neighbors only in $X_u \cup X_v$. Let x be a neighbor of u in X_u, and y a neighbor of v in X_v. By (1), $(N(u) \cup N(v)) \cap C = \{x, y\}$.

If X_u and X_v share a common end then up to symmetry we assume $X_u = P_{ac}$, $X_v = P_{ad}$. Neither x nor y coincides with a for otherwise the attachment of D over C is in P_{ac} or P_{ad}, contrary to the definition of D. So, $uDvyP_{ad}dP_{bd}bP_{bc}cP_{ac}xu$ is a cycle with a unique chord.

So X_u and X_v do not share a common end, hence up to symmetry we assume $X_u = P_{ac}$, $X_v = P_{bd}$. By the previous paragraph, we may assume $x \notin \{a, c\}$ and $y \notin \{b, d\}$. If $xa, yb \notin E(G)$ then $x P_{ac} c P_{cb} ba P_{ad} d P_{bd} yv Dux$ is a cycle with a unique chord. So, up to symmetry we assume $xa \in E(G)$. If $yd \notin E(G)$ then $xuDvyP_{bd}baP_{ad}dcP_{ac}x$ is a cycle with a unique chord, a contradiction. So, $yd \in E(G)$. Since $xabP_{bc}cdyvDux$ cannot be a cycle with a unique chord, xc, yb are either both in $E(G)$ or both not in $E(G)$. In the first case, the three paths $xuDvy$, P_{ad}, P_{bc} have the same configuration as those in Claim 6. In the second case, the five paths $xuDvy$, P_{ad}, P_{bc}, $x P_{ac}c$, $y P_{bd}b$ have the same configuration as those in Claim 8. This proves (2).

By (2) one of $\{a, c\}$, $\{c, b\}$, $\{b, d\}$, $\{d, a\}$ is a proper 2-cutset. ∎

Claim 10. *We may assume that G does not contain a cycle with exactly three chords.*

Proof. Let C be a cycle in G with exactly three chords ab, cd, ef say. Up to symmetry we may assume that a, c, e, b, d, f appear in this order along the cycle and are pairwise distinct for otherwise C contains a cycle with a unique chord. We denote by P_{ac} the unique path from a to c in C that does not go through e, b, d, f. We define similarly $P_{ce}, P_{eb}, P_{bd}, P_{df}, P_{fa}$. If $G[\{a, b, c, d, e, f\}]$ contains only three edges then $P_{af} \cup P_{fd} \cup P_{ce} \cup P_{eb}$ is a cycle with a unique chord (namely fe), a contradiction. Hence, up to symmetry we may assume $ac \in E(G)$. Now $P_{af} \cup P_{db} \cup P_{ce}$ is a cycle with one, two, or three chords: ac and possibly fd and eb. By Claim 9, this cycle must have three chords, so $eb, fd \in E(G)$. Note that $bd \notin E(G)$ since G contains no square by Claim 1, and similarly $af, ce \notin E(G)$. Now we observe that the paths P_{af}, P_{ce}, P_{bd} have the same configuration as those in Claim 6, a contradiction. ∎

Claim 11. *We may assume that G does not contain a cycle with at least one chord.*

Proof. Let C be a cycle in G with at least one chord ab. We choose C minimal with this property. Cycle C must have another chord cd, and we may assume that a, d, b, c are pairwise distinct and in this order along C for otherwise C contains a cycle with at least one chord that contradicts the minimality of C. By Claim 9, C must have another chord ef, and again we may assume that a, e, d, b, f, c are pairwise distinct and in this order along C because of the minimality of C. By Claim 10, C must have again another chord gh, and again we may assume that a, g, e, d, b, h, f, c are pairwise distinct and in this order along C because of the minimality of C. Now, the path from a to f along C that goes through c and the path from e to b along C that goes through d form a cycle smaller than C with at least one chord (namely cd), a contradiction. ∎

A *non-induced path* P in a graph G is a sequence of distinct nodes $v_1 \ldots v_n$ such that for $i = 1, \ldots, n - 1$, $v_i v_{i+1}$ is an edge (these are the *edges of the path*). There might be other edges: the *chords of the path*.

Claim 12. *We may assume that G does not contain the following configuration: five non-induced paths $P = a \ldots c$, $Q = a \ldots c$, $R = b \ldots d$, $S = b \ldots d$, and $X = c \ldots d$, node*

disjoint except for their ends, of length at least 1, *except for X that can be of length* 0 *or more, together with edge ab.*

Proof.   Let $\Pi = P \cup Q \cup R \cup S \cup X$. We suppose that Π is chosen subject to the minimality of X. Note that the only edges of $G[\Pi]$ are the edges of the non-induced paths P, Q, R, S, X, and ab. Indeed, every pair of nodes $\{x, y\} \subseteq \Pi$ can be embedded into a cycle containing only edges of the non-induced paths and ab, so if xy is an edge of $G[\Pi]$ that contradicts our statement then there is a cycle C of $G[\Pi]$ such that xy is a chord of C, and this contradicts Claim 11. In particular, the five non-induced paths have no chords, so they are in fact paths. Note that P, Q, R, S are all of length at least 2 for if P (say) is of length 1, then the unique edge of P is a chord of a cycle of $G[\Pi]$, and this contradicts Claim 11.

We now show that $\{a, c\}$ is a proper 2-cutset of G. Assume not. Then there is a path $D = u \ldots v$ in $G \setminus \Pi$ such that u has a neighbor x in $(P \cup Q) \setminus \{a, c\}$ (say in $Q \setminus \{a, c\}$) and v has a neighbor y in $(X \cup R \cup S) \setminus \{c\}$.

Suppose that $y \in X \setminus \{c\}$. Then the five non-induced paths $aQxuDvy$, $aPcXy$, R, S, and yXd form a configuration that contradicts the minimality of X.

So $y \notin X \setminus \{c\}$, and hence w.l.o.g. $y \in R \setminus \{d\}$. So $uxQaPcXdSbRyvDu$ is a cycle with at least one chord (namely ab), contradicting Claim 11.   ∎

Claim 13.   *We may assume that G does not contain the following induced subgraph (that we call I): six nodes a, b, c, d, e, f with the following edges: ab, ac, ad, be, bf.*

Proof.   We may assume that G has no 1-cutset. Hence, by Lemma 3.2, $\{a, b\}$ is not a cutset of G. So there exists a path in $G \setminus \{a, b\}$ with an end having neighbors in $\{c, d\}$ and an end having neighbors in $\{e, f\}$. We choose a minimal such path $D = u \ldots v$. Up to symmetries and since G contains no square and no triangle by Claims 1 and 2, we may assume that u has a unique neighbor in $\{c, d\}$, and that v has a unique neighbor in $\{e, f\}$. W.l.o.g. $du, vf \in E(G)$. Note that from the minimality of D, c, e have no neighbor in D.

Since a is not a 1-cutset there is a path F in $G \setminus a$ with one end y adjacent to c and an end x adjacent to some node w in $D \cup \{e, b, f, d\}$. We choose such a path F minimal with respect to this property. If x is adjacent to e or b then G contains a cycle with at least one chord (namely ab), contradicting Claim 11. So, $w \in D' = duDvf$. Note that from the minimality of F, e has no neighbor in F.

Since b is not a 1-cutset, there is in $G \setminus b$ a path H with an end z adjacent to e and an end t adjacent to some node s in $D \cup F \cup \{f, d, a, c\}$. Let $Q = acyFx$. If $s \in Q$ then $ezHtsQadD'fbe$ is a cycle with at least one chord (namely ab), contradicting Claim 11. So $s \notin Q$. Let $Q' = adD'w \setminus w$ (note that if $w = d$ then $Q' = a$). If $s \in Q'$ then $sQ'acyFxwD'fbezHts$ is a cycle with at least one chord (namely ab), contradicting Claim 11. So $s \notin Q'$. Hence, $s \in fD'w$.

Now we observe that the five non-induced paths $bezHts$, $bfD's$, $acyFxw$, $adD'w$, and $wD's$ together with edge ab have the same configuration as those in Claim 12, a contradiction.   ∎

We can now prove that G is strongly 2-bipartite. Indeed, we may assume that G has no 1-cutset and G contains no square by Claim 1. We may assume that G is not a

chordless cycle because C_3 is a clique, C_4 is a square, C_5, C_6 are induced subgraphs of the Petersen graph, and C_k where $k \geq 7$ is an output of our theorem. We may also assume that G is not a clique. Let us call a *branch* of a graph G any path of length at least 1, whose ends are of degree at least 3, and whose interior nodes are of degree 2. Since G is not a chordless cycle, has no 1-cutset, and is not a clique of size one or two, it is edge-wise partitioned into its branches. In particular, every node of G is of degree at least 2. No branch of G is of length 1, because such a branch is an edge of G that has both ends of degree at least 3, and then G contains either a triangle, a square, or an I, and this contradicts Claims 2, 1, or 13. We may also assume that G has no branch of length at least 3 because the ends of such a branch, say a and b, form a proper 2-cutset (note that a and b cannot be adjacent since there is no branch of length 1, and since there is no 1-cutset, for every connected component C of $G \setminus \{a, b\}$ there is an ab-path in $G[C \cup \{a, b\}]$).

So we proved that every branch of G is of length exactly 2. This implies that the set X of all nodes of G of degree 2 and the set Y of all nodes of G with degree at least 3 are stable sets. So G is strongly 2-bipartite. This proves Theorem 2.4.

4. STRUCTURE THEOREM

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node y complete to A (resp. x complete to B). Nodes x, y are called the *markers* of their respective blocks.

The *block* G_X (resp. G_Y) of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a node c adjacent to a, b. Node c is a called the *marker* of the block G_X (resp. G_Y).

A graph is *basic* if it is connected and it is either a clique, a hole of length at least 7, a strongly 2-bipartite graph, or an induced subgraph of the Petersen graph or the Heawood graph. Note that the square is not basic (but has a proper 1-join). Every C_k, $k \geq 3$, $k \neq 4$, is basic.

It is sometimes useful to prove that every graph in a class has an *extremal* decomposition, that is a decomposition such that one of the blocks is basic. With our basic classes, decompositions, and blocks, this is false for graphs in $\mathcal{C}$. The graph in Figure 5 is a counter-example. This graph has no proper 2-cutset and a unique 1-join. No block with respect to this proper 1-join is basic, but both blocks have a proper 2-cutset.

Lemma 4.1. *Let G_X and G_Y be the blocks of decomposition of G w.r.t. a 1-cutset, a proper 1-join, or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

  Proof. Suppose G_X and G_Y are the blocks of decomposition of G w.r.t. a 1-cutset or a proper 1-join. Then G_X and G_Y are induced subgraphs of G, and hence if $G \in \mathcal{C}$ then $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$. Conversely, suppose that $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$. If they are blocks
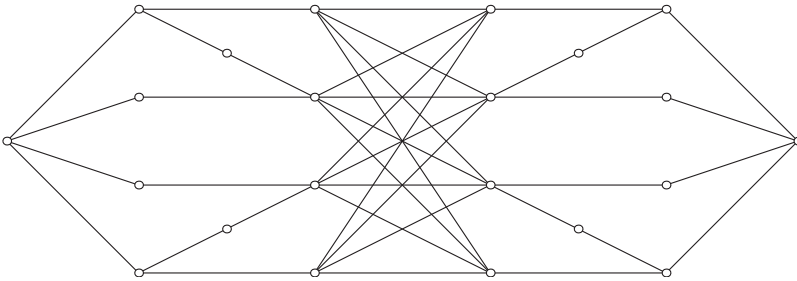
FIGURE 5. A graph in $\mathcal{C}$ with no extremal decomposition.

w.r.t. a 1-cutset, then every cycle of G belongs to G_X or G_Y, and hence $G \in \mathcal{C}$. Assume they are blocks w.r.t. a proper 1-join. Then every cycle C that has at least two nodes in X and at least two nodes in Y and has at least two nodes in A and at least two nodes in B. Since A and B are stable sets, C is either a square or it has at least two chords. It follows that every cycle of G with a unique chord is contained in G_X or G_Y (where possibly the marker node plays the role of one of the nodes of the cycle). Hence $G \in \mathcal{C}$.

Now suppose that G_X and G_Y are the blocks of decomposition of G w.r.t. a proper 2-cutset, with split (X, Y, a, b). Suppose $G \in \mathcal{C}$. Suppose w.l.o.g. that G_X contains a cycle C with a unique chord. Then C must contain c. Let P be an ab-path in $G[Y \cup \{a, b\}]$. Then $G[(V(C) \setminus \{c\}) \cup V(P)]$ is a cycle with a unique chord, a contradiction. So $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.

To prove the converse, assume that $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$, and G contains a cycle C with a unique chord. Since C cannot be contained in G_X or G_Y, it must contain a node of X and a node of Y, and hence it contains a and b. Let P_X (resp. P_Y) be the section of C in $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$). Since C contains a unique chord, w.l.o.g. P_Y is a path and P_X has a unique chord. But then $G_X[V(P_X) \cup \{c\}]$ is a cycle with a unique chord, a contradiction. ∎

Theorem 2.1 and Lemma 4.1 actually give us a complete structure theorem for class $\mathcal{C}$, i.e. every graph in $\mathcal{C}$ can be built starting from basic graphs, that can be explicitly constructed, and gluing them together by prescribed composition operations, and all graphs built this way are in $\mathcal{C}$.

Cliques, holes, and induced subgraphs of the Petersen graph or the Heawood graph can clearly be explicitly constructed. Also strongly 2-bipartite graphs can be constructed as follows. Let X and Y be node sets. We construct a bipartite graph with bipartition (X, Y) by making every node of X adjacent to two nodes of Y. Every strongly 2-bipartite graph can be constructed this way, and every graph constructed this way belongs to $\mathcal{C}$. Indeed, a graph so constructed does not have an edge both of whose endnodes are of degree at least 3, whereas a chord of a cycle has endnodes that are both of degree at least 3.

The composition operations we need are just the reverse of our decompositions and the union of two graphs. Each operation takes as input two node-disjoint graphs G_1 and G_2, and outputs a third graph G.

Operation $\mathcal{O}_0$ is the operation of taking the disjoint union of two graphs, i.e. $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Operation $\mathcal{O}_1$ is the operation that is the reverse of 1-cutset decomposition. For some node u of G_1 and some node w of G_2, G is obtained from the disjoint union of $G_1 \setminus \{u\}$ and $G_2 \setminus \{w\}$, by adding a new node v and all the edges between v and $N_{G_1}(u) \cup N_{G_2}(w)$.

Operation $\mathcal{O}_2$ is the operation that is the reverse of proper 1-join decomposition. For some node u (resp. v) of G_1 (resp. G_2) such that $N_{G_1}(u)$ (resp. $N_{G_2}(v)$) is a stable set of size at least two, G is obtained from the disjoint union of $G_1 \setminus \{u\}$ and $G_2 \setminus \{v\}$ by adding all edges between $N_{G_1}(u)$ and $N_{G_2}(v)$.

Operation $\mathcal{O}_3$ is the operation that is the reverse of proper 2-cutset decomposition. For some degree 2 node u (resp. v) of G_1 (resp. G_2) which is not a 1-cutset, with neighbors u_1 and u_2 (resp. v_1 and v_2) such that u_1 and u_2 (resp. v_1 and v_2) are non-adjacent, and $(d_{G_1}(u_1) - 1) + (d_{G_2}(v_1) - 1) \geq 3$ and $(d_{G_1}(u_2) - 1) + (d_{G_2}(v_2) - 1) \geq 3$, G is obtained from the disjoint union of $G_1 \setminus \{u, u_1, u_2\}$ and $G_2 \setminus \{v, v_1, v_2\}$ by adding new nodes w_1 and w_2 and all edges between w_1 and $(N_{G_1}(u_1) \setminus \{u\}) \cup (N_{G_2}(v_1) \setminus \{v\})$ and between w_2 and $(N_{G_1}(u_2) \setminus \{u\}) \cup (N_{G_2}(v_2) \setminus \{v\})$.

Theorem 4.2. *If $G \in \mathcal{C}$ then either G is basic or can be obtained starting from basic graphs by repeated applications of operations $\mathcal{O}_0, \ldots, \mathcal{O}_3$. Conversely, every graph obtained in this way is in $\mathcal{C}$.*

Proof. Follows from Theorem 2.1 and Lemma 4.1. ∎

5. CONSTRUCTING A DECOMPOSITION TREE

We will now construct a decomposition tree for an input graph G, and then use this tree to obtain an $\mathcal{O}(nm)$ recognition algorithm for class $\mathcal{C}$ (in descriptions of algorithms, n stands for the number of nodes and m for the number of edges). An $\mathcal{O}(n^5)$ or a slightly more involved $\mathcal{O}(n^4)$ algorithm could be obtained from first principles, but we use sophisticated algorithms from other authors, namely Dahlhaus [10] and Hopcroft and Tarjan [14, 23, 15], to get our algorithm to run in $\mathcal{O}(nm)$-time. Note that we do not use the full strength of the works of these authors since they are able to decompose fully a graph in linear time using 1-joins, or using 1-cutsets, or using 2-cutsets. But their notions of decompositions differ slightly from what we need so we just use their algorithms to find the cutsets in linear time. We leave as an open question whether it is possible to recognize graphs in $\mathcal{C}$ in $\mathcal{O}(n+m)$-time.

We could use the definition of blocks of decomposition from Section 4 to construct a decomposition tree, and use it to obtain the recognition algorithm, but such a tree cannot be used for our coloring algorithm, because for coloring we need the blocks of a square-free graph with respect to a proper 2-cutset to be also square-free. So in this section, blocks of decomposition w.r.t. a 1-cutset and a proper 1-join stay the same as in Section 4 but the blocks of decomposition w.r.t. a proper 2-cutset are redefined here below.

The *block* G_X (resp. G_Y) of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) is the graph obtained as follows:

- if there exists a node c of G such that $N(c) = \{a, b\}$, then take such a node c, and let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$;
- else G_X (resp. G_Y) is the block defined in Section 4 that is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a node c adjacent to a, b.

Node c is called the *marker* of the block G_X (resp. G_Y).

Lemma 5.1. *Let $G \in \mathcal{C}$ and suppose that G_X and G_Y are the blocks of decomposition of G w.r.t. a 1-cutset, a proper 1-join, or a proper 2-cutset. If G is connected and triangle-free then G_X and G_Y are connected and triangle-free.*

Proof. The blocks of G are clearly connected. The blocks of G with respect to a 1-cutset or a proper 1-join are induced subgraphs of G so they are triangle-free. If one of the blocks of G w.r.t. a proper 2-cutset $\{a, b\}$ contains a triangle then this triangle must contain the marker c. So the triangle must be abc and this contradicts $ab \notin E(G)$. ∎

Lemma 5.2. *Let $G \in \mathcal{C}$ and suppose that G_X and G_Y are the blocks of decomposition of G w.r.t. a proper 2-cutset with split (X, Y, a, b). If G is connected, triangle-free, square-free, Petersen-free, has no 1-cutset, and no proper 1-join, then G_X and G_Y have the same property.*

Proof. For connectivity and triangles, the lemma follows from Lemma 5.1.

For squares, suppose that w.l.o.g. G_X contains a square C. Since G is square-free, C contains the marker node c (that is not a real node of G), and hence $C = cazbc$, for some node $z \in X$. Since c is not a real node of G, $d_G(z) > 2$ for otherwise z would have been chosen to serve as a marker. Let z' be a neighbor of z that is distinct from a and b. Note that since G is triangle-free, az' and bz' are not edges. Since z is not a 1-cutset, there exists a path P in $G[X \cup \{a, b\}]$ from z' to $\{a, b\}$. We choose z' and P subject to the minimality of P. So, w.l.o.g. $z'Pa$ is a path. Note that b is not adjacent to the neighbor of a along P since z is the unique common neighbor of a, b because G is square-free. So by minimality of P, b does not have a neighbor in P. Now let Q be a path from a to b whose interior is in Y. So, $bzz'PaQb$ is a cycle with a unique chord (namely az), a contradiction.

For the Petersen graph, it suffices to notice that if a block of G contains it, then the marker c must be in it, and this is a contradiction since c is of degree 2.

For 1-cutsets, suppose w.l.o.g. that G_X has a 1-cutset with split (A, B, v). Since G is connected and $G[X \cup \{a, b\}]$ contains an ab-path, $v \neq c$ (where c is the marker node of G_X). Suppose $v = a$. Then w.l.o.g. $b \in B$, and hence $(A, B \cup Y, a)$ is a split of a 1-cutset of G (with possibly c removed from $B \cup Y$, if c is not a real node of G), a contradiction. So $v \neq a$ and by symmetry $v \neq b$. So $v \in X \setminus \{c\}$. W.l.o.g. $\{a, b, c\} \subseteq B$. Then $(A, B \cup Y, v)$ is a split of a 1-cutset of G (with possibly c removed from $B \cup Y$, if c is not a real node of G), a contradiction.

For proper 1-joins, it suffices to notice that the blocks of G are square-free, so they cannot have a proper 1-join. ∎

Lemma 5.3. *Let G_X and G_Y be the blocks of decomposition of G w.r.t. a 1-cutset, a proper 1-join, or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

Proof. By Lemma 4.1 we may assume that G_X and G_Y are the blocks of decomposition of G w.r.t. a proper 2-cutset, with split (X, Y, a, b), and that the marker c is a real node of G. So G_X and G_Y are induced subgraphs of G. Hence, $G \in \mathcal{C}$ implies $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.

To prove the converse, assume that $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$, and G contains a cycle C with a unique chord. Since C cannot be contained in G_X or G_Y, it must contain a node of X and a node of Y, and hence it contains a and b. Let P_X (resp. P_Y) be the section of C in $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$). Since C contains a unique chord, w.l.o.g. P_Y is a path and P_X has a unique chord. Note that $c \notin V(P_X)$, since c is of degree 2 in G and it is adjacent to both a and b. Hence $G_X[V(P_X) \cup \{c\}]$ is a cycle with a unique chord, a contradiction. ∎

An algorithm of Hopcroft and Tarjan [14, 23] finds in linear time a 1-cutset of G (if any). An algorithm of Dahlhaus [10] finds in linear time a 1-join of G if any. The next lemma shows how to use this algorithm to find a proper 1-join or determine that $G \notin \mathcal{C}$.

Lemma 5.4. *Let G be a graph that is not a clique and has no 1-cutset. Assume G has a 1-join. If this 1-join is not proper, then $G \notin \mathcal{C}$.*

Proof. Let (X, Y, A, B) be the split of a 1-join of G that is not proper. If $|A| = 1$ then A is a 1-cutset of G, a contradiction. So $|A| \geq 2$, and by symmetry $|B| \geq 2$. Since the 1-join is not proper, w.l.o.g. there is an edge with both ends in A. This edge together with any node of B forms a triangle, and so by Theorem 2.2 (and since G is not a clique and has no 1-cutset) $G \notin \mathcal{C}$. ∎

Recall that in a graph G two nodes a and b form a *2-cutset* if $G \setminus \{a, b\}$ is disconnected. Hopcroft and Tarjan [15] give an algorithm that finds a 2-cutset in a graph (if any) in linear time. This 2-cutset is not necessarily a proper 2-cutset (which is what we need). We now show how to find a proper 2-cutset in linear time.

Recall that if H is an induced subgraph of G and D is a set of nodes of $G \setminus H$, the *attachment of D over H* is the set of all nodes of H that have a neighbor in D.

Lemma 5.5. *There is an algorithm with the following specifications.*

Input: *A connected graph G that has no 1-cutset nor a proper 1-join, and is not basic.*
Output: *G is correctly identified as not belonging to $\mathcal{C}$, or a proper 2-cutset of G.*
Running time: $\mathcal{O}(n + m)$.

Proof. Consider the following algorithm.

Step 1: Let G_2 be the subgraph of G induced by the degree 2 nodes of G. Since G is connected, has no 1-cutset, and is not a chordless cycle (because C_k, $3 \leq k \neq 4$, is basic and C_4 admits a proper 1-join), the connected components of G_2 are paths, and for every such path P, the attachment of P over $G \setminus P$ consists of two distinct

nodes of G that are both of degree at least 3 in G. If there exists a path P in G_2 whose attachment $\{a, b\}$ over $G \setminus P$ is such that ab is an edge, then output $G \notin \mathcal{C}$ and stop.

Step 2: If there is a path P in G_2 of length at least 1 then let $\{a, b\}$ be the attachment of P over $G \setminus P$. Output $\{a, b\}$ as a proper 2-cutset of G and stop.

Step 3: Now all paths of G_2 are of length 0. Create the graph G' from $G \setminus V(G_2)$ as follows: for every path P of G_2 put an edge between the pair of nodes that are the attachment of P over $G \setminus P$. Note that if G_2 is empty, then $G = G'$. If G' has no 2-cutset, output $G \notin \mathcal{C}$ and stop.

Step 4: Find a 2-cutset $\{a, b\}$ of G'. Note that $\{a, b\}$ is also a 2-cutset of G. If ab is an edge of G, then output $G \notin \mathcal{C}$ and stop. Otherwise, output $\{a, b\}$ as a proper 2-cutset of G and stop.

Since 2-cutsets in Steps 3 and 4 can be found in time $\mathcal{O}(n+m)$ by the Hopcroft and Tarjan algorithm [15], it is clear that the above algorithm can be implemented to run in time $\mathcal{O}(n+m)$. We now prove the correctness of the algorithm.

First note that since G is not a clique and it does not have a 1-cutset, all nodes of G have degree at least 2. Suppose the algorithm stops in Step 1. So there exists a path P in G_2 whose attachment over $G \setminus P$ induces an edge ab. Since $d_G(a) \geq 3$, it follows that $V(G) \setminus (V(P) \cup \{a, b\}) \neq \emptyset$, and hence $\{a, b\}$ is a 2-cutset of G. So by Lemma 3.2, the algorithm correctly identifies G as not belonging to $\mathcal{C}$.

Suppose the algorithm stops in Step 2. By Step 1, ab is not an edge. Since $d_G(a) \geq 3$, $|V(G) \setminus (V(P) \cup \{a, b\})| \geq 2$, and since P is of length at least 1, $|V(P)| \geq 2$. Since G has no 1-cutset, there is an ab-path in $G \setminus P$. Hence $\{a, b\}$ is a proper 2-cutset of G.

Suppose the algorithm stops in Step 3. This means that G' has no 2-cutset. Since the output is $G \notin \mathcal{C}$, the only problem is when $G \in \mathcal{C}$, so let us suppose for a contradiction $G \in \mathcal{C}$. Then by Theorem 2.1, G has a proper 2-cutset $\{a, b\}$ with split (X, Y, a, b). Since $d_G(a) \geq 3$ and $d_G(b) \geq 3$, $\{a, b\} \subseteq V(G')$. If $|X \cap V(G')| \geq 1$ and $|Y \cap V(G')| \geq 1$, then $\{a, b\}$ is a 2-cutset of G', so we may assume w.l.o.g. that $X \cap V(G') = \emptyset$. Since $\{a, b\}$ is a proper 2-cutset of G, $|X| \geq 2$. So X contains two nodes u_1 and u_2 that are both of degree 2 in G. By Step 2, u_1 and u_2 are paths of G_2 of length 0. Since $\{a, b\}$ is a cutset of G, and G is connected and has no 1-cutset, it follows that both u_1 and u_2 are adjacent to both a and b. So au_1bu_2 is a square of G, so by Theorem 2.3, G must have a proper 1-join, a 1-cutset, or must be basic, in either case a contradiction.

Suppose the algorithm stops in Step 4. Let $\{a, b\}$ be a 2-cutset of G'. Then clearly $\{a, b\}$ is also a 2-cutset of G. If ab is an edge of G, then by Lemma 3.2 the algorithm correctly identifies G as not belonging to $\mathcal{C}$. So assume ab is not an edge of G. Note that for every $u \in V(G')$, $d_G(u) \geq 3$. In particular, since $a, b \in V(G')$, $d_G(a) \geq 3$ and $d_G(b) \geq 3$. Let C' be a connected component of $G' \setminus \{a, b\}$. Let u be a node of C', and let C be the connected component of $G \setminus \{a, b\}$ that contains u. Since $d_G(u) \geq 3$, it follows that $|V(C)| \geq 2$. This is true of every connected component of $G' \setminus \{a, b\}$. Also, since G is connected and has no 1-cutset, for every connected component C of $G \setminus \{a, b\}$ there is an ab-path in $G[C \cup \{a, b\}]$. Therefore $\{a, b\}$ is a proper 2-cutset of G. ∎

A *decomposition tree* of a graph G is a rooted tree T_G such that the following hold:

1. G is the root of T_G.
2. For every non-leaf node H of T_G, the children of H are the blocks of decomposition of H w.r.t. a 1-cutset, a proper 1-join, or a proper 2-cutset of H.

Lemma 5.6. *Let G be any graph and let T be a decomposition tree of G. Then T has size $\mathcal{O}(n)$.*

Proof. Note that T is finite since the children of a graph are smaller than its parent. Let T' be the subtree of T on the nodes that are graphs on at least five nodes. For any graph G we define $\varphi(G) = |V(G)| - 4$. It is easily seen that when G_X, G_Y are the blocks of G with respect to some decomposition, then $\varphi(G) \geq \varphi(G_X) + \varphi(G_Y)$. Indeed, for a 2-cutset with split (X, Y, a, b) where the marker c is not a real node of G the inequality follows from $\varphi(G) = |X| + |Y| - 2$, $\varphi(G_X) = |X| - 1$, and $\varphi(G_Y) = |Y| - 1$. For the other decompositions, the proof is similar.

Since in T' every node is a graph on at least five nodes, every node F of T' is such that $\varphi(F) \geq 1$. So the number of leaves of T' is at most $\varphi(G)$. Hence the size of T' is $\mathcal{O}(n)$. It follows that the size of T is also $\mathcal{O}(n)$, since the decomposition of the graphs that have fewer than five nodes is bounded by a constant. ∎

Decomposition trees would be sufficient for a recognition algorithm, but for coloring we need a more sophisticated kind of tree. A *proper decomposition tree* of a connected graph $G \in \mathcal{C}$ is a rooted tree T_G such that the following hold:

1. G is the root of T_G.
2. Every node of T_G is a connected graph.
3. Every leaf of T_G is basic.
4. Every non-leaf node H of T_G is of one of the following types:
   Type 1: The children of H in T_G are the blocks of decomposition w.r.t. a 1-cutset or a proper 1-join.
   Type 2: H and all its descendants are Petersen-, triangle-, square-free and have no 1-cutset and no proper 1-join. Moreover, the children of H in T_G are the blocks of decomposition w.r.t. a proper 2-cutset and every non-leaf descendant of H is of type 2.
5. If a node of T_G is a triangle-free graph then all its descendants are triangle-free graphs.

Theorem 5.7. *There is an algorithm with the following specifications.*

Input: *A connected graph G.*
Output: *G is correctly identified as not belonging to $\mathcal{C}$, or if $G \in \mathcal{C}$, a proper decomposition tree for G.*
Running time: $\mathcal{O}(mn)$.

Proof. Consider the following algorithm.

Step 1: Let G be the root of T_G.
Step 2: If all the leaves of T_G have been declared as LEAF NODE, then output T_G and stop. Otherwise, let H be a leaf of T_G that has not been declared as a LEAF NODE.

Step 3: If H is basic, declare H to be a LEAF NODE and go to Step 2.

Step 4: If H has a 1-cutset, then let the children of H be the blocks of decomposition w.r.t. this 1-cutset, and go to Step 2.

Step 5: If H has a 1-join, then check whether this 1-join is proper. If it is, then let the children of H be the blocks of decomposition by this proper 1-join, and go to Step 2. If it is not, then output $G \notin \mathcal{C}$ and stop.

Step 6: Apply algorithm from Lemma 5.5 to H. If the output is that $G \notin \mathcal{C}$ then output the same and stop. Otherwise a proper 2-cutset is found. Then let the children of H be the blocks of decomposition by this proper 2-cutset, and go to Step 2.

Note that this algorithm stops, because the children of a graph are smaller than its parent. We first prove the correctness of the algorithm. If the algorithm stops in Step 5, then by Lemma 5.4, G is correctly identified as not belonging to $\mathcal{C}$. If the algorithm stops in Step 6, then by Lemma 5.5, G is correctly identified as not belonging to $\mathcal{C}$. So we may assume that the algorithm stops in Step 2. This means that the algorithm outputs a decomposition tree T_G. By Lemma 5.3, it follows that $G \in \mathcal{C}$ since every leaf of T_G is basic. Let us check that T_G is proper.

Clearly, G is the root of T_G and every leaf of T_G is basic. Since G is connected, and by the construction of blocks of decomposition, all nodes of T_G are connected graphs. Let H be a non-leaf node of T_G. Note that H is not basic because of Step 3.

If H is Petersen-, triangle-, square-free, has no 1-cutset, and no proper 1-join, then by Theorem 2.4, H has a proper 2-cutset and is decomposed along a proper 2-cutset because of Step 6. Also, by Lemma 5.2 the children of H are also connected, Petersen-, triangle-, square-free, and have no 1-cutset and no proper 1-join. So by induction, every non-leaf descendant of H is decomposed along proper 2-cutsets and H is of type 2.

Else, H contains a triangle, a square, the Petersen graph or has a 1-cutset or a proper 1-join. By Theorems 2.2 and 2.3, H must have a 1-cutset or a proper 1-join. Note that this 1-cutset or proper 1-join is discovered by the algorithm rather than a possible proper 2-cutset. So, H is of type 1.

So every non-leaf node of T_G is of type 1 or 2, and by Lemma 5.1, if a node of T_G is a triangle-free graph then all its descendants are triangle-free graphs. We have proved that T_G is a proper decomposition tree.

We now show that the algorithm can be implemented to run in time $\mathcal{O}(nm)$. Testing whether a graph is a clique in Step 3 relies only on a check of the degrees: H is a clique if and only if every node has degree $n-1$, so this can be done in time $\mathcal{O}(n+m)$. To decide whether a graph is strongly 2-bipartite, we also check the degrees to be sure that nodes of degree 2 and nodes of degree at least 3 form stable sets. We still have to check that H is square-free, but this can be done by running the $\mathcal{O}(n+m)$ algorithm of Dahlhaus [10] for 1-joins because at this step H contains a square if and only if H has a 1-join.

To find a 1-cutset in Step 4, we use the $\mathcal{O}(n+m)$ algorithm of Hopcroft and Tarjan [14, 23]. To find a 1-join in Step 5, we use the $\mathcal{O}(n+m)$ algorithm of Dahlhaus [10]. By Lemma 5.5, Step 6 can be implemented to run in time $\mathcal{O}(n+m)$. Now we note that when the algorithm stops, it has computed a decomposition tree (that will be output or not when $G \notin \mathcal{C}$), and the number of steps processed by the algorithm is bounded

by the size of this tree. By Lemma 5.6 the size of the tree is $\mathcal{O}(n)$, so we have to run $\mathcal{O}(n)$-times each of the steps, and hence the overall complexity is $\mathcal{O}(nm)$. ∎

Theorem 5.8. *There exists an $\mathcal{O}(nm)$-time algorithm that decides whether a graph is in $\mathcal{C}$.*

   Proof. Apply the $\mathcal{O}(nm)$ algorithm from Theorem 5.7. If the output is $G \notin \mathcal{C}$ then output the same. Else G has a proper decomposition tree and $G \in \mathcal{C}$ by Lemma 5.3. ∎

6. COLORING

Let us call *third color* of a graph any stable set that contains at least one node of every odd cycle. Any graph that admits a third color S is 3-colorable: give color 3 to the third color; since $G \setminus S$ contains no odd cycle, it is bipartite: color it with colors 1, 2. We shall prove by induction that any triangle-free graph in $\mathcal{C}$ has a third color. But for the sake of induction, we need to prove a stronger statement.

   Let us call *strong third color* of a graph any stable set that contains at least one node of every cycle (odd or even). By $N[v]$ we denote $\{v\} \cup N(v)$. When v is a node of a graph G, a pair of disjoint subsets (R, T) of $V(G)$ is *admissible with respect to G and v* if one of the following holds (see Fig. 6):

- $T = N(v)$ and $R = \{v\}$;
- $T = \emptyset$ and $R = N[v]$;
- v is of degree 2, $N(v) = \{u, w\}$, $T = \{u\}$, $R = \{v, w\}$;
- v is of degree 2, $N(v) = \{u, w\}$, $T = \{u\}$, $R = N[w]$;
- v is of degree 2, $N(v) = \{u, w\}$, $T = \emptyset$, $R = \{u\} \cup N[w]$.

   We say that a pair of disjoint subsets (R, T) is an *admissible pair* of G if for some $v \in V(G)$, (R, T) is admissible w.r.t. G, v. An admissible pair (R, T) should be seen as a constraint for coloring: we will look for third colors (sometimes strong, sometimes not) that must contain every node of T and no node of R. We will do this first in basic graphs, and then by induction in all triangle-free graphs of $\mathcal{C}$, thus proving that they are 3-colorable.
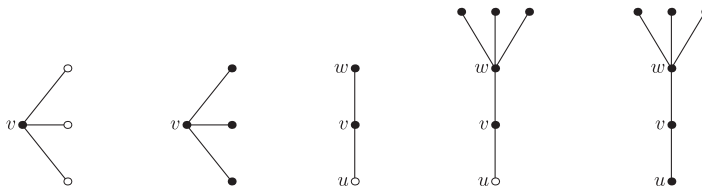

FIGURE 6. Five examples of admissible pairs (nodes of T are white, nodes of R are black).

Lemma 6.1.   *Let G be a triangle-free basic graph that is not the Petersen graph. Let (R, T) be an admissible pair of G. Then G admits a strong third color S such that $T \subseteq S$ and $S \cap R = \emptyset$. Furthermore, S can be found in time $\mathcal{O}(n+m)$.*

Proof.   Note that squares may fail to admit strong third color that satisfies our constraints (because when G is a square $R = V(G)$ is possible). But squares are not basic. The proof follows from the following claims, since it will be clear that all S's found in them can be found in time $\mathcal{O}(n+m)$.

(1) *The lemma holds when G is a chordless cycle of length at least* 7.

Because then any non-empty set of nodes is a strong third color. Since $R = V(G)$ is impossible because any path in $G[R]$ is of length at most 3, it is always possible to pick a node of G not in R. This proves (1).

(2) *The lemma holds when G is a clique, a strongly 2-bipartite graph, or is an induced subgraph of the Heawood graph.*

Note that G is bipartite (for cliques, because it is triangle-free). Let A, B be a bipartition of G. Note that A, B can be computed in linear time. Up to symmetry between A, B we may assume $T \subset A$ and $|A \cap R| \leq 2$. Let $S = A \setminus R$. So $T \subset S$ and $S \cap R = \emptyset$. Moreover, $|A \setminus S| \leq 2$. So every cycle in $G \setminus S$ contains at most two nodes of A, and since G is square-free, there is no such cycle. This proves (2).

(3) *The lemma holds when G is a proper induced subgraph of the Petersen graph.*

Note that by assumption, G is not the Petersen graph. We use our notation for the Petersen graph Π. So $V(G) \subsetneq V(\Pi) = \{a_1, \ldots, a_5, b_1, \ldots, b_5\}$. Let v be a node of G and (R, T) be admissible with respect to G, v. We may assume $v = a_1$ since the Petersen graph is vertex-transitive. Note that $a_1 \in V(G)$.

Suppose $T = N(a_1)$ and $R = \{a_1\}$. Then we put $Q = \{a_2, a_5, b_1\}$ and we observe that $\Pi \setminus Q$ is a C_6 plus an isolated node. But some node $z \neq a_1$ of Π is not a node of G. If z is in the C_6, then $S = Q \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$. Else z must be a neighbor of a_1, say a_5 up to symmetry. So, $S = (Q \cup \{a_4\}) \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

Suppose $T = \emptyset$ and $R = N[v]$. Then we put $Q = \{a_3, b_3, b_5\}$ and we observe that $\Pi \setminus Q$ is a tree. So $S = Q \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

From here on we may assume that v is a node of degree 2 of G. So up to symmetry we may assume $b_1 \notin V(G)$ and $N_G(v) \subseteq \{a_2, a_5\}$.

Suppose $N(v) = \{u, w\}$, $T = \{u\}$, $R = \{v, w\}$. So w.l.o.g. $T = \{a_2\}$ and $R = \{a_1, a_5\}$. Then we put $Q = \{a_2, b_3\}$ and we observe that $\Pi \setminus Q$ is a tree. So $S = Q \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

Suppose $N(v) = \{u, w\}$, $T = \{u\}$, $R = N[w]$. So up to symmetry we may assume $T = \{a_2\}$, $R = \{a_1, a_5, a_4, b_3\} \cap V(G)$. Then we put $Q = \{a_2, b_2, b_5\}$ and we observe that as in the previous case $\Pi \setminus Q$ is a tree. So $S = Q \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

Suppose $N(v) = \{u, w\}$, $T = \emptyset$, $R = \{u\} \cup N[w]$. So up to symmetry we may assume $R = \{a_1, a_2, a_5, b_3, a_4\} \cap V(G)$. We put $Q = \{b_1, a_3, b_4\}$ and we observe that $\Pi \setminus Q$ is a tree. So $S = Q \cap V(G)$ is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$. This proves (3). ∎

Lemma 6.2. *Let G be the Petersen graph and (R, T) be an admissible pair of G. Then G admits a third color S (possibly not strong) such that $T \subseteq S$ and $S \cap R = \emptyset$. Furthermore, S can be found in time $\mathcal{O}(1)$.*

Proof. We use our notation for the Petersen graph: $V(G) = \{a_1, \ldots, a_5, b_1, \ldots, b_5\}$. Let v be a node of G and (R, T) be admissible with respect to G, v. We may assume $v = a_1$ since the Petersen graph is vertex-transitive. Since v has degree 3, we just have to study the following two cases:

Suppose $T = N(a_1)$ and $R = \{a_1\}$. Then we put $S = T$ and we observe that $G \setminus S$ is a C_6 plus an isolated node. So S is a third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$. Note that in this case there exists no strong third color that satisfies our constraints.

Suppose $T = \emptyset$ and $R = N[v]$. Then we put $S = \{a_3, b_3, b_5\}$ and we observe that $G \setminus S$ is a tree. So S is a third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$. ∎

Lemma 6.3. *Let G be a non-basic, connected, triangle-free, square-free, and Petersen-free graph in $\mathcal{C}$ that has no 1-cutset and no proper 1-join. Let (R, T) be an admissible pair of G. Then G admits a strong third color S such that $T \subseteq S$ and $S \cap R = \emptyset$. Furthermore, S is obtained in time $\mathcal{O}(1)$ from well-chosen strong third colors of blocks of G w.r.t. a proper 2-cutset of G.*

Proof. By Theorem 2.4, G has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset of G. Let v be a node of G and (R, T) an admissible pair with respect to G, v. We now show that G admits a strong third color S such that $T \subseteq S$ and $S \cap R = \emptyset$. We use induction on the blocks of decomposition G_X and G_Y w.r.t. this proper 2-cutset, as defined in Section 5. Note that by Lemma 5.2, G_X and G_Y are connected, triangle-free, square-free, Pertersen-free, and contain no 1-cutset and no proper 1-join.

Here below, when we write "by induction," we mean that either we use inductively Lemma 6.3 for a smaller graph (when this smaller graph is not basic), or that we use Lemma 6.1 (when this smaller graph is basic). By symmetry it is enough to consider the following three cases.

Case 1. $v = a$.

Since a is not of degree 2, either $T = N(a)$ and $R = \{a\}$, or $T = \emptyset$ and $R = N[a]$.

Suppose that $T = N(a)$. By induction, there is a strong third color S_X of G_X (resp. S_Y of G_Y) such that $N_{G_X}(a) \subseteq S_X$ (resp. $N_{G_Y}(a) \subseteq S_Y$). So marker node $c \in S_X \cap S_Y$, and hence neither a nor b belongs to $S_X \cup S_Y$. Therefore $S = S_X \cup S_Y$ with possibly c removed if c is not a real node of G is a stable set of G such that $T \subseteq S$ and $S \cap R = \emptyset$ (since $R = \{a\}$). Let H be a cycle of G. If H contains a, then it must contain a node of $N(a) = T$, and hence it contains a node of S. So assume that H does not contain a. Since H does not contain a, w.l.o.g. $V(H) \subseteq X \cup \{b\}$ and does not contain c. Hence H is a cycle of G_X that does not contain c. Since S_X is a strong third color of G_X, a node of H belongs to $S_X \setminus \{c\}$, and hence to S. Therefore S is a strong third color of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

Now suppose that $R = N[a]$. Note that since c is of degree 2 in G_X, $(N_{G_X}[a], \{b\})$ is an admissible pair w.r.t. G_X, c, and hence by induction, there exists a strong third color S_X of G_X such that $N_{G_X}[a] \cap S_X = \emptyset$ (in particular, $c \notin S_X$) and $b \in S_X$. Similarly, there exists a strong third color S_Y of G_Y such that $N_{G_Y}[a] \cap S_Y = \emptyset$ (in particular, $c \notin S_Y$)

and $b \in S_Y$. Clearly, $S = S_X \cup S_Y$ is a stable set such that $\emptyset = T \subseteq S$ and $S \cap R = \emptyset$. Let us check that every cycle of G contains a node of S. Let H be a cycle of G. If H contains b then we are done since $b \in S$, so w.l.o.g. $V(H) \subseteq X \cup \{a\}$, i.e. H is a cycle of G_X and, hence, since S_X is a strong third color of G_X, S_X contains a node of H, and so does S.

Case 2. v is of degree 2, $N(v) = \{u, w\}$, $v \in X$, $w = a$, and $N[w] \subseteq R$.

Note that either $u = b$ or $u \in X$, and either $u \in R$ or $u \in T$. By induction, there exists a strong third color S_X of G_X such that $N_{G_X}[w] \cap S_X = \emptyset$ and $u \in S_X$ if and only if $u \in T$. By induction, since c is of degree 2 (and hence both $(N_{G_Y}[w], \{b\})$ and $(N_{G_Y}[w] \cup \{b\}, \emptyset)$ are admissible w.r.t. G_Y, c), there exists a strong third color S_Y of G_Y such that $N_{G_Y}[w] \cap S_Y = \emptyset$ and $b \in S_Y$ if and only if $b \in S_X$. Clearly, $S = S_X \cup S_Y$ is a stable set of G such that $T \subseteq S$ and $R \cap S = \emptyset$. Since $c \notin S_X$ and $c \notin S_Y$, it is easy to see that S contains a node of every cycle of G, i.e. S is a strong third color of G.

Case 3. $T \cup R \subseteq X \cup \{a, b\}$.

By induction, there exists a strong third color S_X of G_X such that $T \subseteq S_X$ and $R \cap S_X = \emptyset$. If $c \in S_X$ and c is a real node of G, then let $T_Y = N_{G_Y}(a)$ and $R_Y = \{a\}$. Note that (R_Y, T_Y) is an admissible pair w.r.t. G_Y, a. In all other cases, let $T_Y = S_X \cap \{a, b\}$ and $R_Y = \{c\} \cup (\{a, b\} \setminus S_X)$. Note that (R_Y, T_Y) is an admissible pair w.r.t. G_Y, c. By induction, there exists a strong third color S_Y of G_Y such that $T_Y \subseteq S_Y$ and $R_Y \cap S_Y = \emptyset$. Note that $S_X \cap \{a, b\} = S_Y \cap \{a, b\}$. Furthermore, if $c \in S_X$ and c is a real node of G, then $c \in S_Y$, and in all other cases $c \notin S_Y$ because $c \in R_Y$. If $c \in S_X$ and c is a real node of G, then let $S = S_X \cup S_Y$, and otherwise let $S = (S_X \cup S_Y) \setminus \{c\}$. Clearly, S is a stable set of G such that $T \subseteq S$ and $R \cap S = \emptyset$.

Let H be a cycle of G. We now show that S contains a node of H. If H is a cycle of G_X (resp. G_Y) then S_X (resp. S_Y) contains a node of H, and hence so does S. So we may assume that H is not a cycle of G_X or G_Y. In particular H contains both a and b, a node of X and a node of Y. Let H_Y be the ab-subpath of H whose intermediate nodes belong to Y. Note that $H_Y \neq acb$, since otherwise H belongs to G_X. So $V(H_Y) \cup \{c\}$ induces a cycle of G_Y. Since S_Y is a strong third color of G_Y, it contains a node h of $V(H_Y) \cup \{c\}$. If $h \neq c$ then $h \in S \cap V(H)$. So assume that $h = c$. But then $c \notin R_Y$ so $c \in S_Y$, and hence $c \in S_X$ and it is a real node of G. Therefore, $c \in S \cap V(H)$. ■

Lemma 6.3 implies a weaker statement: the existence of a third color (possibly not strong). But we do not know how to prove this weaker result with the weaker induction hypothesis. An attempt fails at the proof of Case 3.

Lemma 6.4. *Let G be a non-basic connected triangle-free graph in $\mathcal{C}$ and (R, T) be an admissible pair of G. Then G admits a third color S such that $T \subseteq S$ and $S \cap R = \emptyset$. Furthermore, S is obtained in time $\mathcal{O}(1)$ from well-chosen third colors of blocks of G w.r.t. a 1-cutset, a proper 1-join, or a proper 2-cutset of G.*

Proof. Here below, we use the fact that every strong third color is a third color with no explicit mention. So, we may assume that G contains a square or the Petersen graph, or has a 1-cutset or a proper 1-join, for otherwise the result follows from Lemma 6.3. Hence, by Theorem 2.3, the proof follows from the following two claims.

Here below, when we write "by induction," we mean that either we use inductively Lemma 6.4 for a smaller graph (when this smaller graph is not basic), or that we use Lemma 6.1 or 6.2 (when this smaller graph is basic).

(1) *The lemma holds when G has a 1-cutset.*

Let (X, Y, z) be a split of a 1-cutset of G. Let G_X and G_Y be the blocks of decomposition w.r.t. this 1-cutset. Note that G_X and G_Y are triangle-free by Lemma 5.1.

Case 1. $X \cap (R \cup T)$ and $Y \cap (R \cup T)$ are both non-empty.

Then, $z \in R \cup T$. We put $R_X = R \cap (X \cup \{z\})$, $R_Y = R \cap (Y \cup \{z\})$, $T_X = T \cap (X \cup \{z\})$, $T_Y = T \cap (Y \cup \{z\})$. We observe that (R_X, T_X) and (R_Y, T_Y) are admissible with respect to G_X and G_Y, respectively. So by induction there exists a third color S_X of G_X such that $S_X \subseteq T_X$, $R_X \cap S_X = \emptyset$, and a third color S_Y of G_Y such that $T_Y \subseteq S_Y$, $R_Y \cap S_Y = \emptyset$. So, $S = S_X \cup S_Y$ is a third color of G such that $T \subseteq S$ and $R \cap S = \emptyset$.

Case 2. One of $X \cap (R \cup T)$ and $Y \cap (R \cup T)$ is empty.

We assume w.l.o.g. that $Y \cap (R \cup T) = \emptyset$. Hence, $R \cup T \subseteq X \cup \{z\}$. Let S_X be a third color of G_X such that $S_X \subseteq T$, $R \cap S_X = \emptyset$. If $z \in S_X$, let S_Y be a third color of G_Y such that $z \in S_Y$. Else, let S_Y be a third color of G_Y such that $z \notin S_Y$. In either case, $S = S_X \cup S_Y$ is a third color of G such that $S \subseteq T$ and $R \cap S = \emptyset$. This proves (1).

(2) *The lemma holds when G has a proper 1-join.*

Let (X, Y, A, B) be a split of a proper 1-join of G. We show that G admits a third color S such that $T \subseteq S$ and $S \cap R = \emptyset$.

Suppose that v is of degree at least 3. Then we assume w.l.o.g. $v \in X$. So $v \in V(G_X)$, $T \cap Y = B$ or $\emptyset$, and $R \cap Y = B$ or $\emptyset$.

Suppose that v is of degree 2 and $N(v) = \{u, w\}$. If $\{u, v, w\}$ is contained in X or Y, then we assume w.l.o.g. that it is contained in X. Otherwise, v must be contained in $A \cup B$, and we assume w.l.o.g. that $v \in B$, which implies that $A = \{u, w\}$ (since $|A| \geq 2$). If $v \in B$, we assume that the marker y of block G_X is v.

So in all cases $v \in V(G_X)$, $T \cap Y = B$ or $\emptyset$, $R \cap Y = B$ or $\emptyset$, and if $N(v) = \{u, w\}$ then $v \notin A$ and $u, w \in X$. If $T \cap Y = B$ then let $T_X = (T \setminus B) \cup \{y\}$, and if $T \cap Y = \emptyset$ then let $T_X = T$. If $R \cap Y = B$ then let $R_X = (R \setminus B) \cup \{y\}$, and if $R \cap Y = \emptyset$ then let $R_X = R$. Note that (R_X, T_X) is an admissible pair w.r.t. G_X, v. By induction, there exists a third color S_X of G_X such that $T_X \subseteq S_X$ and $R_X \cap S_X = \emptyset$. By induction, there exists a third color S'_Y of G_Y such that $N(x) = B \subseteq S'_Y$, and a third color S''_Y of G_Y such that $S''_Y \cap N[x] = \emptyset$. If $y \in S_x$ then let $S_Y = S'_Y$, and otherwise let $S_Y = S''_Y$. Note that $x \notin S_Y$, i.e. $S_Y \subseteq Y$. Let $S = (S_X \cap X) \cup S_Y$. Note that only one of S'_Y, S''_Y needs to be computed once S_X is known.

Clearly, S is a stable set. If $T \cap Y = \emptyset$ then $T_X = T$ and, hence, since $T_X \subseteq S_X$, $T \subseteq S$. If $T \cap Y = B$ then $y \in T_X$ and, hence, since $T_X \subseteq S_X$ (and in particular $y \in S_X$), $B \subseteq S$, and therefore $T \subseteq S$. If $R \cap Y = \emptyset$ then $R_X = R$ and, hence, since $R_X \cap S_X = \emptyset$, $R \cap S = \emptyset$. If $R \cap Y = B$ then $R_X = (R \setminus B) \cup \{y\}$ and, hence, since $R_X \cap S_X = \emptyset$, $y \notin S_X$ and so $S_Y \cap N[x] = \emptyset$, implying that $R \cap S = \emptyset$.

So it only remains to show that S contains a node of every odd cycle of G. Let H be an odd cycle of G. If $V(H) \subseteq X$, then since S_X is a third color of G_X, S_X

contains a node of H, and hence so does S. If $V(H) \subseteq Y$, then since S_Y is a third color of G_Y, S_Y contains a node of H, and hence so does S. So we may assume that H contains both a node of X and a node of Y. Hence, H is node-wise partitioned into a path of X and a path of Y of different parity. Hence if we suppose that H is minimal with respect to the property of overlapping X, Y and being odd, then either $V(H) \cap X = \{h_X\} \subseteq A$ or $V(H) \cap Y = \{h_Y\} \subseteq B$. Suppose that $V(H) \cap X = \{h_X\}$. Then $(V(H) \setminus \{h_X\}) \cup \{x\}$ induces an odd cycle H' of G_Y. Since S_Y is a third color of G_Y, S_Y contains a node h of H'. Since $x \notin S_Y$, h is a node of $V(H) \cap S$. Finally, assume that $V(H) \cap Y = \{h_Y\}$. Then $(V(H) \setminus \{h_Y\}) \cup \{y\}$ induces an odd cycle H' of G_X. Since S_X is a third color of G_X, S_X contains a node h of H'. If $h \neq y$ then h is a node of $V(H) \cap S$. So assume $h = y$. Then $y \in S_X$ and hence $B \subseteq S_Y$, and in particular $h_Y \in V(H) \cap S$. This proves (2). ∎

Our proof of Lemmas 6.3 and 6.4 suggests that for every triangle-free graph in $\mathcal{C}$ there might exist a stable set that intersects every cycle. Such a property might be of use for stronger notions of coloring (list coloring, etc.). It holds for every basic graph (even for the Petersen graph), for every square-and-Petersen-free graph by a slight variant of Lemma 6.3 and we almost proved it in general. But it is false. Let us build a counter-example G, obtained from four disjoint copies $\Pi_1, \ldots, \Pi_4$ of the Petersen graph minus one node. So Π_i contains a set X_i of three nodes of degree 2 ($i = 1, \ldots, 4$). We add all edges between X_1, X_2, between X_2, X_3, between X_3, X_4, and between X_4, X_1. Note that G can be obtained by gluing one square $S = s_1 s_2 s_3 s_4$ and four disjoint copies $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ of the Petersen graph along operation $\mathcal{O}_2$ of Theorem 4.2 as follows: let $G = S$ and for $i = 1$ to 4, replace G by itself glued through s_i with Π_i. So $G \in \mathcal{C}$ by Theorem 4.2.

We claim now that G does not contain a stable set that intersects every cycle. Indeed, if S is such a stable set then S must contain all nodes in one of the X_i's for otherwise we build a C_4 of $G \setminus S$ by choosing a node in every X_i. So $X_1 \subseteq S$ say. We suppose that Π_1 has nodes $\{a_2, \ldots, a_5, b_1, \ldots, b_5\}$ with our usual notation. So $X_1 = \{b_1, a_2, a_5\} \subseteq S$ and we observe that every node in $C = V(\Pi_1 \setminus S)$ has a neighbor in X_1. Hence $C \cap S = \emptyset$ while $G[C]$ is a cycle on six nodes, a contradiction. Note that for any node v of G, $G \setminus v$ contains a stable set that intersects every cycle. So, a characterization by forbidding induced subgraphs of the class of graphs that admit a stable set intersecting every cycle needs to consider G somehow. For this reason, we believe that such a characterization must be complicated.

Theorem 6.5. *If $G \in \mathcal{C}$, then either $\chi(G) = \omega(G)$ or $\chi(G) \leq 3$. In particular, $\chi(G) \leq \omega(G) + 1$.*

Proof. Clearly, we may assume that G is connected. If $\omega(G) \leq 2$ then $\chi(G) \leq 3$ since G contains a third color by Lemma 6.1, 6.2, or 6.4 (indeed, every non-empty graph has an admissible pair: $(N[v], \emptyset)$). If $\omega(G) \geq 3$ then by Theorem 2.2, G admits a 1-cutset. So every 2-connected component of G is either a clique or is 3-colorable. Hence $\chi(G) = \omega(G)$. ∎

Theorem 6.6. *There exists an algorithm that computes an optimal coloring of any graph in $\mathcal{C}$ in time $\mathcal{O}(nm)$.*

Proof. Let G be a graph in $\mathcal{C}$. When G has a 1-cutset, then it is easy to obtain an optimal coloring of G from optimal colorings of its blocks. So by Theorem 2.2 we may assume that our input graph is triangle-free. We may also assume that G is connected and not bipartite. Now we show how to 3-color G by finding a third color of G.

We first construct a proper decomposition tree T_G of the input graph G in time $\mathcal{O}(nm)$, by Theorem 5.7. Let v be any node of G. We associate with node G of T_G an admissible pair (R, T), say $R = N[v]$ and $T = \emptyset$, and we use Lemmas 6.1–6.4 to recursively find a third color S of G such that $T \subseteq S$ and $S \cap R = \emptyset$.

First note that all the leaves of T_G are basic, and since G is triangle-free, they are one of the graphs described in Lemma 6.1 or 6.2. So once the appropriate admissible pairs have been associated with a given leaf of T_G, Lemma 6.1 or 6.2 shows how to find the appropriate third color (or if needed, strong third colors), each in linear time in the size of the leaf.

For a non-leaf node H of type 1 of T_G, Lemma 6.4 shows how to proceed (in linear time) to find a third color of H by asking recursively for appropriately chosen third colors of its children (i.e. choosing appropriate admissible pairs to associate with its children, and once the appropriate third colors of its children are found how to put them together to find the desired third color of H). Note that in several cases, the algorithm has to compute the third colors of the children H_1, H_2 of H in a prescribed order, that is, wait for the answer for the coloring of H_1 before knowing what admissible pair is needed for the coloring of H_2.

For a non-leaf node H of type 2 of T_G, to recursively find a third color of H, we actually need to find a strong third color. This we can do by proceeding as in the proof of Lemma 6.3. Note that since H is of type 2, every non-leaf descendant of H is of type 2. Also, no descendant of H contains the Petersen graph, so all leaves under H will have a strong third color computed by Lemma 6.1.

So the processing time at each non-leaf node of T_G is $\mathcal{O}(1)$. Since by Lemma 5.6 the size of T_G is $\mathcal{O}(n)$, the sum of processing times at the leaves of T_G is $\mathcal{O}(n+m)$. So the time needed to process the tree is $\mathcal{O}(n+m)$.

Hence, the total computation time is $\mathcal{O}(nm)$.   ∎

We note that the algorithm above has complexity $\mathcal{O}(n+m)$ once the proper decomposition tree is given. So, if one can find an $\mathcal{O}(n+m)$-time algorithm for constructing a proper decomposition tree, then one gets an $\mathcal{O}(n+m)$-time coloring algorithm for graphs in $\mathcal{C}$.

7.   CLIQUES AND STABLE SETS

The problems of finding a maximum clique and a maximum stable set for a graph in $\mathcal{C}$ have a complexity that is easy to establish. Note first that any graph in $\mathcal{C}$ contains no diamond. So any edge of a graph in $\mathcal{C}$ is contained in a unique maximal clique. Hence, to find a maximum clique in polynomial time, it is enough to look for the common neighborhood of every edge. A simple implementation of this leads to an $\mathcal{O}(nm)$-time algorithm and no faster algorithm seems to be known for diamond-free graphs. Here below we give a linear time algorithm restricted to our class $\mathcal{C}$. Note in contrast that,

as proved by Maffray and Preissmann, the coloring problem is NP-hard even when restricted to triangle-free graphs [17].

Theorem 7.1. *There exists a linear time algorithm whose input is a graph in $\mathcal{C}$ and whose output is a maximum clique of G.*

Proof. It is trivial to decide in linear time whether $\omega(G)=1$. So, we assume $\omega(G) \geq 2$. Then the 2-connected components of G can be found in linear time (see [14, 23]). By Theorem 2.2, every 2-connected component is either a clique or is triangle-free. So, to find a maximum clique it suffices for each 2-connected component to test whether it is a clique or not, and to output a largest such clique (if any). If no 2-connected component is a clique then output any edge. ∎

A 2-*subdivision* is a graph obtained from any graph by subdividing twice every edge. More precisely, every edge uv of a graph G is replaced by an induced path $uabv$ where a and b are of degree 2. Let F be the resulting graph. It is easy to see that $\alpha(F) = \alpha(G) + |E(G)|$. This construction, due to Poljak, easily yields:

Theorem 7.2 (Poljak [21]). *The problem whose instance is a 2-subdivision G and an integer k and whose question is "Does G contain a stable set of size at least k" is NP-complete.*

Since every 2-subdivision is in $\mathcal{C}$, a direct consequence is:

Theorem 7.3. *Finding a maximum stable set of a graph in $\mathcal{C}$ is NP-hard.*

8.   NP-COMPLETENESS OF $\Pi_{H_{3|3}}$

It is mentioned in Section 1 that the problem $\Pi_{H_{3|3}}$ is NP-complete. To prove this, we need a slight variation on a theorem proved in [16] using a refinement of a construction due to Bienstock [1]. We remind the reader that I denotes the graph on nodes a, b, c, d, e, f with the following edges: ab, ac, ad, be, bf.

For any integer $k \geq 1$ we define the problem Π_k whose instance is described below:

1. a graph G that does not contain I;
2. two nodes of degree 2 x and y;
3. and these are such that there exists a path P_x (resp. P_y), on $2k-1$ nodes, all interior nodes of which are of degree 2, and x (resp. y) is the middle of P_x (resp. P_y). Moreover, P_x and P_y are node disjoint.

The question of Π_k is "does there exist an induced cycle of G that goes through x, y?".

Theorem 8.1 (Lévêque et al. [16]). *For all integers $k \geq 1$ the problem Π_k is NP-complete.*

Proof.   The theorem is not stated this way in [16], so we explain here how to obtain a proof from [16]. In this article, Theorem 2.3 states that problem Π, that is, problem Π_k where Condition 3 is forgotten, is NP-complete. The proof is obtained by a reduction from 3-SAT. For every integer k and every instance f of 3-SAT, an instance denoted by $G_f(k, 1, 1, 1, 3, 1, 1)$ of Π is built. It is shown that f has a truth assignment if and only if $G_f(k, 1, 1, 1, 3, 1, 1)$ contains an induced cycle that goes through x, y. It turns out that luckily $G_f(k, 1, 1, 1, 3, 1, 1)$ is an instance of Π_k (even if unluckily this is not explicitly stated in [16]), because $G_f(k, 1, 1, 1, 3, 1, 1)$ satisfies Condition 3.   ■

Theorem 8.2.   *Problem $\Pi_{H_{3|3}}$ is NP-complete.*

Proof.   We use the fact that every realization of $H_{3|3}$ contains I (note that this is false for $H_{1|1}$, $H_{2|1}$, $H_{3|1}$, $H_{2|2}$, and $H_{3|2}$). Let G, x, y be an instance of the NP-complete problem Π_3. So, G is a graph that does not contain I and x, y are two nodes of G of degree 2.

We prepare now an instance G' of $\Pi_{H_{3|3}}$ by adding an edge between x, y. We note that G' contains a unique I, induced by $N(x) \cup N(y)$. So, any realization of $H_{3|3}$ in G' must contain P_x and P_y. It follows that G contains a hole passing through x, y if and only if G' contains a realization of $H_{3|3}$.   ■

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