# Even and Odd Holes in Cap-Free Graphs

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**Abstract:** It is an old problem in graph theory to test whether a graph contains a chordless cycle of length greater than three (hole) with a specific parity (even, odd). Studying the structure of graphs without odd holes has obvious implications for Berge's strong perfect graph conjecture that states that a graph $G$ is perfect if and only if neither $G$ nor its complement contain an odd hole. Markossian, Gasparian, and Reed have proven that if neither $G$ nor its complement contain an even hole, then $G$ is $\beta$-perfect. In this article, we extend the problem of testing whether $G(V, E)$ contains a hole of a given parity to the case where each edge of $G$ has a label ***odd*** or ***even***. A subset of $E$ is odd (resp. even) if it contains an odd (resp. even) number of odd edges. Graphs for which there exists a signing (i.e., a partition of $E$ into odd and even edges) that makes every triangle odd and every hole even are called ***even-signable***. Graphs that can be signed so that every triangle is odd and every hole is odd are called ***odd-signable***. We derive from a theorem due to Truemper co-NP characterizations of even-signable and odd-

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signable graphs. A graph is ***strongly even-signable*** if it can be signed so that every cycle of length $\geq 4$ with at most one chord is even and every triangle is odd. Clearly a strongly even-signable graph is even-signable as well. Graphs that can be signed so that cycles of length four with one chord are even and all other cycles with at most one chord are odd are called ***strongly odd-signable***. Every strongly odd-signable graph is odd-signable. We give co-NP characterizations for both strongly even-signable and strongly odd-signable graphs. A ***cap*** is a hole together with a node, which is adjacent to exactly two adjacent nodes on the hole. We derive a decomposition theorem for graphs that contain no cap as induced subgraph (***cap-free*** graphs). Our theorem is analogous to the decomposition theorem of Burlet and Fonlupt for Meyniel graphs, a well-studied subclass of cap-free graphs. If a graph is strongly even-signable or strongly odd-signable, then it is cap-free. In fact, strongly even-signable graphs are those cap-free graphs that are even-signable. From our decomposition theorem, we derive decomposition results for strongly odd-signable and strongly even-signable graphs. These results lead to polynomial recognition algorithms for testing whether a graph belongs to one of these classes.
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## 1. INTRODUCTION

In this article, we study the structure of a class of graphs that do not contain chordless cycles of length greater than three (holes) of a specific parity (even, odd).

A graph $G$ is *perfect* if, for all induced subgraphs of $G$, the size of the largest clique is equal to the chromatic number. A long-standing conjecture of Berge [1] states that $G$ is perfect if and only if neither $G$ nor its complement contain an odd hole. (The *complement* $\bar{G}$ of $G(V,E)$ has node set $V$ and two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in $G$). Understanding the structure of graphs with no odd holes may give an important contribution to this conjecture. Also, the existence of a polynomial algorithm to test whether $G$ contains an odd hole implies a polynomial algorithm to test whether $G$ is perfect, modulo the verification of the above conjecture, and it is possible that such an algorithm may itself prove the conjecture.

Markossian, Gasparian, and Reed [12] define $\beta$-perfect graphs as follows: $G$ is $\beta$-perfect if, for every induced subgraph $H$ of $G$, we have $\chi(H) = \max\{\delta(F)+1 : F$ is a node induced subgraph of $H\}$, where $\chi(H)$ is the chromatic number of $H$, and $\delta(F)$ is the smallest node degree in $F$. $\beta$-perfect graphs do not contain even holes and Markossian, Gasparian, and Reed also show that if neither $G$ nor $\bar{G}$ contain an even hole, then $G$ is $\beta$-perfect. So the study of the structure of graphs that do not contain even holes will give a better understanding of the class of $\beta$-perfect graphs.

Bienstock [3] has shown the NP-completeness of testing the existence of a hole with a specified parity (even, odd), containing a specified node of $G$, even if $G$ is triangle-free. However, it is quite possible that polynomial algorithms to test whether $G$ contains a hole with a specified parity still exist. Note that if $G$ is triangle-free, the problem of testing whether $G$ contains an odd hole amounts to testing the bipartiteness of $G$, and the problem of testing whether $G$ contains an even hole was solved in [12]; see also [5]. Finally, consider the following analogous problems: Test whether a bipartite graph contains a hole whose length divided by two has a specified parity. Bipartite graphs containing no hole whose length divided by two is odd represent balanced 0, 1 matrices and their structure is studied in [8], where a polynomial algorithm is given to test whether a bipartite graph contains a hole whose length divided by two is odd. These results are extended to balanced 0, $\pm 1$ matrices in [4]. The structure of bipartite graphs containing no hole whose length divided by two is even is much simpler, and a polynomial algorithm is shown in [9].

## 2. EVEN-SIGNABLE AND ODD-SIGNABLE GRAPHS

A convenient setting for the study of even or odd holes in graphs is the one of *signed* graphs. $G(V, E)$ is a signed graph if the edges of $G$ are given *odd* or *even* labels. A subset of $E$ is odd (resp. even) if it contains an odd (resp. even) number of odd edges. Graphs for which there exists a *signing* (i.e., a partition of $E$ into odd and even edges) that makes every triangle odd and every hole even are *even-signable*. Graphs that can be signed so that every triangle is odd and every hole is odd are *odd-signable*. Even-signable graphs were introduced in [7].

Note that $G$ contains no odd hole if and only if $G$ is even-signable with all edges odd, and $G$ contains no even hole if and only if $G$ is odd-signable with all edges odd.

Since cuts and cycles of $G$ have even intersections, by switching the labels on all edges of a cut the parity of a cycle does not change. Since, in a connected graph $G$, any edge of a spanning tree $T$ belongs to a cut of $G$ not containing any other edge of $T$, if $G$ is signed, we can switch signs on the edges of cuts so that, in the newly signed graph, the spanning tree $T$ has a specified (arbitrary) signing.

This implies that, if a (connected) graph $G(V, E)$ is even-signable (odd-signable), one can produce such a signing as follows. Order the edges of $G, e_1, \ldots, e_n$, so that the edges of $T$ are the first in the sequence and all other edges $e_j$ have the property that $e_j$ closes a chordless cycle $H_j$ of $G$ together with edges having smaller indices. Sign the edges of $T$ arbitrarily, and label the remaining edges $e_j$ so that $H_j$ is even-signed (odd-signed).

So $G$ contains no odd hole if and only if $G$ is even-signable and the above algorithm, after labeling the edges of $T$ odd, labels odd all the remaining edges. Also, $G$ contains no even hole if and only if $G$ is odd-signable and the above algorithm, after labeling the edges of $T$ odd, labels odd all the remaining edges. Hence, a polynomial algorithm that tests whether $G$ is even-signable (odd-signable) can be used to test whether $G$ contains an odd hole (even hole).

The following theorem of Truemper [15] is fundamental in obtaining co-NP characterizations for the existence of holes with specified parities.

**Theorem 2.1.** *Let $\beta$ be a* 0, 1 *vector whose entries are in one-to-one correspondence with the chordless cycles of a graph $G$. Then there exists a subset $F$ of edges of $G$ such that $|F \cap C| \equiv \beta_C$* (mod 2) *for all chordless cycles $C$ of $G$, if and only if for every induced subgraph $G'$ of $G$ of type $H_0, H_1, H_2$, or $H_3$, there exists a subset $F'$ of edges of $G'$ such that $|F' \cap C| \equiv \beta_C$* (mod 2), *for all chordless cycles $C$ of $G'$.*

The graphs $H_0, H_1, H_2$, and $H_3$ are shown in Fig. 1. Graphs of type $H_0, H_1$, or $H_2$ are referred to as 3-*path configurations* (3*PC*s). A graph of type $H_0$ is called a $3PC(x, y)$, where node $x$ and node $y$ are connected by three paths $P_1, P_2$, and $P_3$. A subgraph of type $H_1$ is called a $3PC(xyz, u)$, where $xyz$ is a triangle and $P_1, P_2$, and $P_3$ are three paths with endnodes $x, y$, and $z$, respectively, and a common endnode $u$. A graph of type $H_2$ is called a $3PC(xyz, uvw)$, consists of two node disjoint triangles $xyz$ and $uvw$, and paths $P_1, P_2$, and $P_3$ with endnodes $x$ and $u, y$ and $v$, and $z$ and $w$, respectively. In all three cases, the nodes of $P_i \cup P_j, i \neq j$, induce a hole. This implies that all paths of $H_0$ have length greater than one, and at most one path of $H_1$ has length one.

Graphs of type $H_3$ are *wheels*. These consist of a chordless cycle $H$ together with a node called the *center* that has at least three neighbors on $H$. When the center together with the nodes of $H$ induce an odd number of triangles, the wheel is called an *odd wheel*. When the center has an even number of neighbors on $H$, the wheel is called an *even wheel*. Note that a wheel may be both odd and even. Also note that $K_4$ is a wheel that is neither odd nor even and, therefore, when the wheel $(H, v)$ is odd or even, $H$ is a hole.

In this article, we write graph $G$ *contains* graph $R$ to mean that $R$ occurs as a node induced subgraph of $G$. To obtain a co-NP characterization of even-signable graphs, let $\beta_C = 0$ for all holes $C$ in $G$, and $\beta_C = 1$ for all triangles, and apply the above theorem. Similarly, for odd-signable graphs, let $\beta_C = 1$ for all chordless cycles $C$.

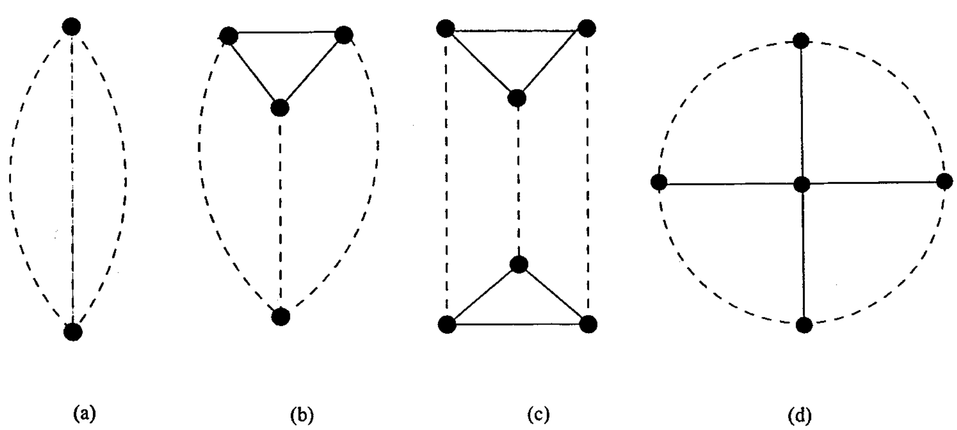


FIGURE 1. 3-path configurations and wheel.

**Theorem 2.2.** *A graph is even-signable if and only if it contains no* $3PC(xyz,u)$ *and no odd wheel.*

**Theorem 2.3.** *A graph is odd-signable if and only if it contains no* $3PC(x,y)$, *no* $3PC(xyz,uvw)$, *and no even wheel.*

***Proof.*** (Theorem 2.2) In a $3PC(xyz,u)$ we can label even all edges except $xy, yz$, and $xz$, since they form a tree $T$ in the $3PC$. Now $xy, yz$, and $xz$ must all be labeled even, since each one closes a hole with the edges of $T$. We now have an even triangle.

Assume that an odd wheel $(H,x)$ is even-signable. We arbitrarily label odd all edges having $x$ as endnode. All subpaths of the hole $H$ with the endnodes adjacent to $x$ and no intermediate node adjacent to $x$ must have an even number of odd edges, if their length is greater than one, and must be labeled odd, if they contain a unique edge, since this edge belongs to a triangle containing $x$. Thus, $H$ is signed odd.

Consider graphs $H_0$ and $H_2$. By labeling odd all edges in triangles and even all other edges, we obtain an even signing of these graphs. In wheels $(H,x)$ that are not odd, to obtain an even signing, label odd all the edges adjacent to $x$ and all edges of $H$ that belong to a triangle of $(H,x)$, and label even all the other edges of $H$. ∎

***Proof.*** (Theorem 2.3) In a $3PC(x,y)$, we can label even all edges except the two edges $xu$ and $xv$ of $P_1$ and $P_2$ having $x$ as endnode, since they form a tree $T$ in the $3PC$. Now both $xu, yv$ must be labeled odd, since each one of them closes a hole with the edges of $T$. Now $P_1$ closes an even hole with $P_2$.

In a $3PC(xyz,uvw)$, we can label $xy, xz$ odd and all other edges even except $yz, uv, vw, wu$, since they form a tree $T$ in the $3PC$. Now $yz$ must be labeled odd, since it belongs to the triangle $xyz$. This implies that edges $uv, vw, wu$ must all be labeled even. Now we have an even triangle.

In an even wheel $(H,x)$, we can arbitrarily label odd all edges having $x$ as endnode. All subpaths of the hole $H$ with the endnodes adjacent to $x$ and no intermediate node adjacent to $x$ must contain an odd number of odd edges. Thus, $H$ is made up of an even number of such subpaths and is signed even.

Consider a graph of type $H_1$. By labeling odd all edges of the triangle and even all other edges, we obtain an odd signing. For a wheel $(H,x)$ that is not even, label odd all edges having $x$ as endnode. Furthermore, on every subpath of $H$ with endnodes adjacent to $x$ and no intermediate node adjacent to $x$, label one edge odd and all others even. This gives an odd signing of $(H,x)$. ∎

## 3. STRONGLY EVEN-SIGNABLE AND STRONGLY ODD-SIGNABLE GRAPHS

A *diamond* is a cycle on four nodes with exactly one chord.

A graph is *strongly even-signable*, if it can be signed such that every triangle is odd and every cycle of length at least four with at most one chord is even. A graph is *strongly odd-signable*, if it can be signed such that every cycle with at most one chord is odd, except for the diamonds, which are even. A signing of $G$ satisfying one of the two properties is a *strong even-signing* or a *strong odd-signing* of $G$. From the definition it is clear that every strongly even-signable graph is even-signable and every strongly odd-signable graph is odd-signable. The following theorems provide co-NP characterizations of strongly even-signable and strongly odd-signable graphs.

A *cap* $(H, xyz)$ is a hole $H$ together with a node $x$ with exactly two neighbors $y$ and $z$ in $H$, and these neighbors are adjacent. Note that any signing of a cap $(H, xyz)$ with the triangle $xyz$ odd contains both an odd cycle of length $\geq 4$ with at most one chord and an even cycle with at most one chord, which is not a diamond. So if $G$ is strongly even-signable or strongly odd-signable, then $G$ contains no cap as a node-induced subgraph ($G$ is *cap-free*).

**Theorem 3.1.** *$G$ is strongly even-signable if and only if $G$ is even-signable and does not contain a cap.*

***Proof.*** Necessity follows from the above observation. To prove sufficiency of the condition, let $G$ be even-signed. If $G$ is not strongly even-signed, there exists an odd cycle $C$ with exactly one chord. The chord together with the nodes of $C$ induces two chordless cycles $C_1$ and $C_2$. Exactly one of these is odd, say $C_1$. Since $G$ contains no odd hole or even triangle, $C_1$ must be a triangle and $C_2$ must be a hole. So $C$ is a cap. ∎

The proof of the theorem also shows that, if $G$ is strongly even-signable, any even-signing of $G$ is a strong even-signing of $G$.

**Theorem 3.2.** *A graph $G$ is strongly even-signable if and only if $G$ is cap-free and does not contain an odd wheel $(H, x)$, where $x$ is adjacent to all the nodes of $H$.*

***Proof.*** Theorem 2.2 states that $G$ is even-signable if and only if $G$ contains no $3PC(xyz, u)$ and no odd wheel. If $G$ is strongly even-signable, then $G$ is cap-free. A $3PC(xyz, u)$ contains a cap and an odd wheel $(H, x)$ contains a cap if and only if $x$ is not adjacent to all the nodes of $H$. Now the theorem follows from Theorem 3.1. ∎

**Theorem 3.3.** *A graph $G$ is strongly odd-signable if and only if every cycle of $G$ with a unique chord is a diamond, $G$ contains no $3PC(x, y)$ and no even wheel $(H, x)$ with $x$ adjacent to all the nodes of $H$.*

***Proof.*** Theorem 2.3 states that $G$ is odd-signable if and only if $G$ contains no $3PC(x, y)$, no $3PC(xyz, uvw)$, and no even wheel. A $3PC(xyz, uvw)$ contains a cap, hence a cycle with a unique chord that is not a diamond. An even wheel $(H, x)$ contains a cycle with a unique chord that is not a diamond if and only if $x$ is not adjacent to all the nodes of $H$. So the above theorem is equivalent to the following:

*$G$ is strongly odd-signable if and only if every cycle of $G$ with a unique chord is a diamond and $G$ is odd-signable.*

To prove necessity, observe that, if $G$ contains a cycle with a unique chord that is not a diamond, then $G$ is not strongly odd-signable.

To prove sufficiency, consider an odd signing of the odd-signable graph $G$. If the labeling does not make it strongly odd-signable, it contains a even cycle with a unique chord that is not a diamond. ∎

It follows that, if $G$ is strongly odd-signable, any odd-signing of $G$ is a strong odd-signing of $G$.

The following corollary is a consequence of Theorem 2.3 and shows that every graph that is strongly odd-signable is a cap-free odd-signable graph, but the converse does not hold.

**Corollary 3.1.** *A cap-free graph $G$ is odd-signable if and only if $G$ contains no $3PC(x, y)$ and no triangle-free even wheel and no even wheel $(H, x)$ with $x$ adjacent to all the nodes of $H$.*

A graph $G$ is *Meyniel* if every odd length cycle of $G$, which is not a triangle, has at least two chords. These graphs were proven to be perfect by Meyniel [14]. Meyniel graphs are cap-free. In fact, they are exactly the graphs that can be strongly even-signed with all edges labeled odd. One of the most important decomposition theorems for perfect graphs is the one of Burlet and Fonlupt [2], who give a decomposition theorem and a polynomial time recognition algorithm for Meyniel graphs.

We prove a decomposition theorem for cap-free graphs. The decomposition we use is the same as the one introduced by Burlet and Fonlupt, only the indecomposable graphs will be different. This decomposition theorem can be used to obtain polynomial time algorithms to test membership in each one of the following classes:

(i) Cap-free,
(ii) Strongly even-signable or, equivalently, cap-free even-signable,
(iii) Strongly odd-signable,
(iv) Cap-free odd-signable.

## 4. AMALGAMS

A node $x$ not in $S$ is *adjacent* to a subset $S$ of $V$, if $x$ is adjacent to some node in $S$. Node $x$ is *universal* for $S$, if it is adjacent to all nodes in $S$. $x$ is *partially adjacent* to $S$, if it is adjacent to but not universal for $S$.

We describe the amalgam introduced by Burlet and Fonlupt [2].

A connected graph $G(V, E)$ contains an *amalgam* $(A, B, K)$ if $V = V_1 \cup V_2 \cup K$, where $V_1, V_2$, and $K$ are disjoint sets, $|V_1| \geq 2, |V_2| \geq 2$ and the nodes in $K$ induce a clique of $G$ (possibly $K$ is empty). Furthermore, there exist nonempty sets $A \subseteq V_1$ and $B \subseteq V_2$ such that every node in $A$ is adjacent to every node in $B$, and these are the only edges between nodes in $V_1$ and $V_2$. In addition, all nodes in $K$ are adjacent to all nodes in $A \cup B$.

Note that the removal of all the edges with one endpoint in $A$ and the other in $B$, together with all the nodes in $K$, disconnects $G$. The nodes of $K$, together with the edges between $A$ and $B$, define the amalgam $(A, B, K)$. The *blocks of the amalgam decomposition* are defined to be the graphs $G_1$ and $G_2$, where $G_1$ is the graph induced by $V_1 \cup K$ together with a node $b$ adjacent to all nodes in $A \cup K$, and similarly $G_2$ is the graph induced by $V_2 \cup K$ together with a node $a$ adjacent to all nodes in $B \cup K$. Note that $G_1$ and $G_2$ are proper induced subgraphs of $G$.

A graph is *triangulated*, if it contains no hole. A triangulated graph $G$ is obviously both strongly odd-signable and strongly even-signable. In fact, $G$ is universally signable, as defined in [6]. The structure of triangulated graphs is well studied, see e.g. [13], [11], and there are efficient recognition algorithms to test membership in this class.

A *basic cap-free graph* $G$ is either a triangulated graph or a biconnected triangle-free graph together with at most one additional node, which is adjacent to all other nodes of $G$.

In this article, we prove the following.

**Theorem 4.1.** *A cap-free graph, which is not basic, contains an amalgam.*

In the last section, we discuss the implication of this theorem for strongly even-signable and strongly odd-signable graphs.

## 5. D-STRUCTURES

Let $G(S)$ denote the subgraph of $G$ induced by the subset $S$ of $V$.

**Definition 5.1.** *A* D-structure $(C_1, C_2, K)$ *of $G$ consists of disjoint sets of nodes $C_1, C_2$, and $K$, where $|C_1| \geq 2, |C_2| \geq 2$, and the nodes of $K$ induce a clique of $G$ (possibly $K$ is empty). Furthermore, the subgraph $G(C_1)$ is connected and every node in $C_1$ is universal for $C_2 \cup K$, every node in $C_2$ is universal for $C_1 \cup K$, and there exists no node in $V \backslash (C_1 \cup C_2 \cup K)$ adjacent to both a node in $C_1$ and a node in $C_2$.*

**Lemma 5.1.** *If a cap-free graph $G$ contains a D-structure, then $G$ contains an amalgam.*

To prove this lemma, we first need to prove the following result.

**Lemma 5.2.** *Let $G(V')$ be a connected subgraph of a cap-free graph $G$, $|V'| \geq 2$. Let $z$ be a node universal for $V'$, and let $y$ be a node partially adjacent to $V'$ such that $y$ and $z$ are connected by some chordless path $P'$ in $G(V \backslash V')$. Then there exists a node $x \in V(P')$, partially adjacent to $V'$ such that, in the subpath $P$ of $P'$ from $z$ to $x$, all the nodes in $V(P) \backslash \{x\}$ are universal for $V'$.*

***Proof.*** In $P'$ pick $x$ to be the node closest to $z$ partially adjacent to $V'$. Let $P$ be the $xz$-subpath of $P'$. Let $x'$ be the node closest to $x$ in $P$ universal for $V'$. If $x'$ is not adjacent to $x$, then the subpath of $P$ connecting $x$ to $x'$, together with two adjacent nodes $u, v \in V'$, where $v$ is adjacent to $x$ and $u$ is not adjacent to $x$ forms a cap. Note that since $V'$ is connected, such a choice of nodes $u$ and $v$ is always possible. Now let $x''$ be the node of $P$ not adjacent to $V'$ closest to $x$. Pick the subpath $P''$ of $P$ containing $x''$ with only the endnodes universal for $V'$. Let $w$ be the node of $V(P)\backslash V(P'')$ adjacent to the endnode of $P''$ closest to $x$. Now $P''$ together with $w$ and a node of $V'$ adjacent to $w$ induces a cap. ∎

***Proof.*** (Lemma 5.1) Let $U$ be the set of nodes in $V\backslash(C_1 \cup C_2 \cup K)$ that are adjacent to $C_1$ and are connected to a node in $C_2$ by a path in $G(V\backslash(C_1 \cup K))$.

**Claim 1.** *Every node in $U$ is universal for $C_1$.*

***Proof.*** Assume not and let $u \in U$ be connected to $y \in C_2$ by a chordless path $P_u$ in $G(V\backslash(C_1 \cup K))$. Since $C_1$ and $C_2$ belong to a D-structure, then the length of $P_u$ is greater than one. By Lemma 5.2, we may assume that all the nodes of $P_u$, except for $u$, are universal for $C_1$. Now the node on $P_u$ adjacent to $y$ has neighbors in both $C_1$ and $C_2$, contradicting the definition of a D-structure. This completes the proof of Claim 1.

Let $K'$ contain the nodes in $K$ that are not universal for $U$ and $K'' = K\backslash K'$. Define $A = C_1, B = C_2 \cup K' \cup U$. We show that $(A, B, K'')$ is an amalgam of $G$. Claim 1 shows that every node in $B$ is universal for $A$ and, by definition of $K''$, every node in $K''$ is universal for $U$. Since $(C_1, C_2, K)$ is a D-structure, every node in $K''$ is universal for $C_1 \cup C_2 \cup K'$.

**Claim 2.** *Let $G'$ be the graph obtained from $G$ by removing all edges with one endnode in $A$ and the other in $K'$. Then in $G'(V\backslash(C_2 \cup K'' \cup U))$ no path connects a node of $K'$ and a node of $C_1 = A$.*

***Proof.*** Let $P = k, \ldots, v_k, x$ be a chordless path connecting $k \in K'$ and $x \in C_1$ and contradicting the claim. No intermediate node of $P$ is adjacent to a node in $C_2$ else, by Claim 1, $v_k$ belongs to $U$, contradicting the definition of $P$. If $P$ has length greater than 2, then the nodes of $P$ together with any node in $C_2$ induce a cap.

So $P = k, v_k, x$. Since $k$ is not universal for $U$, there exists a node $u \in U$ not adjacent to $k$. Let $P_u = x_1, \ldots, x_m$ be a chordless path connecting $u = x_1$ and a node $x_m \in C_2$ in $G(V\backslash(C_1 \cup K))$. Let $u = u_1, \ldots, u_n = x_m$ be the nodes of $P_u$ that are universal for $C_1$ with $u_i$ closer to $u$ than $u_{i+1}$. Note that all nodes $u_1, \ldots, u_{n-1}$ belong to $U$.

We now show that $u_i$ cannot be adjacent to $u_{i+1}, 1 \le i \le n-1$. Assume not and let $i$ be the highest index such that $u_i$ and $u_{i+1}$ are adjacent. If $i = n-1, u_i$ contradicts the definition of a D-structure. So $i < n-1$. Then the nodes in the subpath of $P_u$ between $u_{i+1}$ and $u_{i+2}$, together with $u_i$ and any node in $C_1$ induce a cap.

Let $x_j$ be the node of smallest index adjacent to $k$. (Since $x_m$ is adjacent to $k$, such a node exists). Since $u$ is not adjacent to $k, j > 1$. If $x_j$ is universal for $C_1$, let $x_i$ be the node of $U$ having largest index $i < j$. Now the nodes in the subpath of $P_u$ between $x_i$ and $x_j$, together with $k$ and any node of $C_1$ induce a cap. So $x_j$ is not universal for $C_1$. Let $x_i$ be the node of $V(P_u) \cap U$ having largest index $i < j$. Now the nodes in the subpath of $P_u$ between $x_i$ and $x_j$, together with $k, v_k$, and $x$ induce a cap. (Note that node $v_k$ is not adjacent to any node in $P_u$, since otherwise $v_k$ belongs to $U$, contradicting the assumption). This completes the proof of Claim 2.

The following claim shows that $(A, B, K'')$ is an amalgam of $G$.

**Claim 3.** *Let $G''$ be obtained from $G$ by removing all edges with one endnode in $A$ and the other in $B$. Then in $G''(V \backslash K'')$, no path connects a node in $A$ and a node in $B$.*

***Proof.*** Let $P = x_1, \ldots, x_n$ be a chordless path between $x_1$ in $A$ and $x_n$ in $B$ and contradicting the claim. Claim 1 shows that if $x_n \in C_2$, then $x_2 \in U$, a contradiction. Claim 2 shows $x_n \notin K'$. So $x_n \in U$ and let $P_{x_n}$ be a path connecting $x_n$ and a node in $C_2$ in $G(V \backslash (C_1 \cup K))$. Now there is a path in $G(V \backslash (C_1 \cup K))$ between $x_2$ and a node in $C_2$ only using nodes of $V(P_{x_n}) \cup V(P)$. So $x_2$ must belong to $U$, a contradiction. ∎

## 6. M-STRUCTURES

M-structures were first introduced by Burlet and Fonlupt [2] in their study of Meyniel graphs.

An induced subgraph $G(V_1)$ of $G$ is called an *M-structure* (multipartite structure) if $\bar{G}(V_1)$ contains at least two connected components each with at least two nodes. Let $W_1, \ldots, W_k$ be the node sets of these connected components. The *proper subclasses* of $G(V_1)$ are the sets $W_i$ of cardinality greater than or equal to 2. The *partition* of an M-structure is denoted by $(W_1, \ldots, W_r, K)$, where $K$ is the union of all nonproper subclasses. Note that $K$ induces a clique in $G$.

**Lemma 6.1.** *An M-structure $G(V_1)$ of $G$ is maximal with respect to node inclusion if and only if there exists no node $v \in V \backslash V_1$ such that $v$ is universal for a proper subclass of $G(V_1)$.*

***Proof.*** Let $G(V_1 \cup \{u\})$ be an M-structure. Assume node $u$ is not universal for any proper subclass of $G(V_1)$. In $\bar{G}(V_1 \cup \{u\})$ node $u$ is adjacent to at least one node in each of the proper subclasses. Thus, there exists only one proper subclass in $G(V_1 \cup \{u\})$, contradicting the assumption.

Conversely, let node $u$ be universal for some proper subclass $W_i$ of $G(V_1)$. Then $\bar{G}(V_1 \cup \{u\})$ has at least two components with more than one node, the graph induced by $W_i$, and at least one component with more than one node in $(V_1 \cup \{u\}) \backslash W_i$. ∎

The above proof yields the following.

**Corollary 6.1.** *Let $G(V_1)$ and $G(V_2)$ be M-structures with $V_1 \subseteq V_2$. Let $W_i$ and $Z_j$ be connected components of $\bar{G}(V_1)$ and $\bar{G}(V_2)$, respectively, having nonempty intersection. Then $W_i \subseteq Z_j$.*

**Lemma 6.2.** *Let $G(V_1)$ be a maximal M-structure of a cap-free graph $G$. Then a node in $V \backslash V_1$ cannot be adjacent to two proper subclasses of $G(V_1)$.*

***Proof.*** Assume node $u \in V \backslash V_1$ is adjacent to two proper subclasses: $W_1$ and $W_2$ of $G(V_1)$. Since $G(V_1)$ is maximal, by Lemma 6.1, node $u$ is not universal for either of the classes. Also, since the complement of $G(W_1)$ is connected, there must exist a pair of nodes $x_1, y_1$ adjacent in the complement, such that node $u$ is adjacent to $x_1$ but not to $y_1$. Similarly, there must exist a pair $x_2, y_2$ in $W_2$ such that $x_2, y_2$ are adjacent in the complement and node $u$ is adjacent to $x_2$ but not to $y_2$. But now $x_1, x_2, y_1, y_2$ together with node $u$ induce a cap. ∎

**Theorem 6.1.** *If $G$ is a cap-free graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then $G$ contains an amalgam.*

***Proof.*** If $G$ contains a D-structure $(C_1, C_2, K)$ then, by Lemma 5.1, $G$ contains an amalgam. So the theorem follows from the proof of the following statement:

*If $G$ is a cap-free graph containing an M-structure either with at least three proper subclasses,or with at least one proper subclass that is not a stable set, then $G$ contains a D-structure $(C_1, C_2, K)$.*

Let $G(V_1)$ be an M-structure of $G$ satisfying the above property and $G(V_2)$ a maximal M-structure with $V_1 \subseteq V_2$.

**Claim 1.** *The M-structure $G(V_2)$ either contains at least three proper subclasses or contains exactly two proper subclasses not both of which are stable sets.*

***Proof.*** If $G(V_1)$ contains a proper subclass, say $W_i$, which is not a stable set, by Corollary 6.1, there exists a proper subclass, say $Z_j$ of $G(V_2)$ such that $W_i \subseteq Z_j$. Then $Z_j$ is not a stable set. If all proper subclasses of $G(V_1)$ are stable sets, then $G(V_1)$ has at least three proper subclasses, say $W_1, W_2, \ldots, W_k$. If $G(V_2)$ has only two proper subclasses, say $Z_1, Z_2$, then by Corollary 6.1, we may assume w.l.o.g. that $W_1 \cup W_2 \subseteq Z_1$. Then $Z_1$ is not a stable set, since every node in $W_1$ is adjacent to a node in $W_2$. This completes the proof of Claim 1.

**Claim 2.** *Let $G(V_2)$ be a maximal M-structure of $G$ with partition $(W_1, W_2, K)$, where $W_1$ is not a stable set. Then $G$ contains a D-structure $(C_1, C_2, K)$.*

***Proof.*** Let $C_1$ be a connected component of $G(W_1)$ with more than one node. Let $C_2 = W_2$. Then $(C_1, C_2, K)$ is a D-structure, since by Lemma 6.2 no node of $V \backslash V_2$ is adjacent to a node in $C_1$ and a node in $C_2$, and $|C_2| \geq 2$, since $W_2$ is a proper subclass of $G(V_2)$. This completes the proof of Claim 2.

**Claim 3.** *Let $G(V_2)$ be a maximal M-structure of $G$ with at least three proper subclasses. Then $G$ contains a D-structure $(C_1, C_2, K)$.*

***Proof.*** Let $W_1, W_2, \ldots, W_l, l \geq 3$ be the proper subclass of $G(V_2)$ and let $K$ be the collection of all nonproper subclasses. Let $C_1$ be the nodes in two proper subclasses of $G(V_2)$, (note that $G(C_1)$ is a connected graph), $C_2$ be the nodes in all the other proper subclasses of $G(V_2)$. Then $(C_1, C_2, K)$ is a D-structure, since $|C_1| \geq 2, |C_2| \geq 2$, and Lemma 6.2 shows that the only nodes having neighbors in both $C_1$ and $C_2$ belong to $K$. So the proof of Claim 3 is complete. ∎

## 7. EXPANDED HOLES

An *expanded hole* consists of nonempty sets of nodes $S_1, \ldots, S_n, n \geq 4$, not all singletons, such that, for all $1 \leq i \leq n$, the graphs $G(S_i)$ are connected. Furthermore, every $s_i \in S_i$ is adjacent to $s_j \in S_j, i \neq j$, if and only if $j = i + 1$ or $j = i - 1$ (modulo $n$).

**Lemma 7.1.** *Let $G$ be a cap-free graph and let $H$ be a hole of $G$. If $s$ is a node having two adjacent neighbors in $H$, then either $s$ is universal for $H$, or $s$ together with $H$ induces an expanded hole.*

***Proof.*** Let $s$ be a node with two adjacent neighbors in $H$. If $s$ has no other neighbors on $H$, then $s$ induces a cap with $H$. Let $H = x_1, \ldots, x_n, x_1$ with node $s$ adjacent to $x_1$ and $x_n$. If $s$ is not universal for $H$, and does not induce an expanded hole together with $H$, then let $k$ be the smallest index for which $s$ is not adjacent to $x_k$. Let $l$ be the smallest index such that $l > k$ and $s$ is adjacent to $x_l$. Now node $x_{k-2}$ ($x_n$ if $k = 2$) together with the hole $s, x_{k-1}, \ldots, x_l, s$ forms a cap. ∎

**Lemma 7.2.** *Let $G$ be a cap-free graph and let $S = \cup_{i=1}^{n} S_i, n > 4$, be a maximal expanded hole in $G$ with respect to node inclusion. Either $G$ contains an M-structure with a proper subclass that is not a stable set of $G$, or all nodes that are adjacent to a node in $S_i$ and a node in $S_{i+1}$ ($S_{n+1} = S_1$) for some $i$, are universal for $S$ and induce a clique of $G$.*

***Proof.*** Let $u$ be a node adjacent to $s_1 \in S_1$ and $s_2 \in S_2$. By applying Lemma 7.1 to any hole that contains $s_1$ and $s_2$ and a node each from the sets $S_j, j > 2$, we have that $u$ is adjacent to all nodes in $S \backslash (S_1 \cup S_2)$, else the maximality of $S$ is contradicted. Now since node $u$ is adjacent to $s_1, s_2$ and is universal for all sets $S_j, j > 2$, Lemma 7.1 shows that $u$ is universal for $S_1$ and $S_2$, hence for $S$.

Let $u$ and $v$ be two nonadjacent nodes that are universal for $S$. Then $u, v$ together with $s_1 \in S_1, s_2 \in S_2$ and $s_4 \in S_4$ induces an M-structure with proper sets $W_1 = \{u, v\}$ and $W_2 = \{s_1, s_2, s_4\}$. Furthermore, $W_2$ is not a stable set of $G$. ∎

**Theorem 7.1.** *A cap-free graph that contains an expanded hole contains an amalgam.*

***Proof.*** Let $S = \cup_{i=1}^{n} S_i$ be a maximal expanded hole in $G$. First assume that $n = 4$. Then the node set $S$ induces an M-structure with proper subclasses $S_1 \cup S_3$ and $S_2 \cup S_4$. $S_2 \cup S_4$ is not a stable set because, say, $|S_2| \geq 2$ and $G(S_2)$ is connected. Hence, by Theorem 6.1, we are done.

Now assume that $n > 4$. By Lemma 5.1, it is sufficient to show that $G$ contains a D-structure $(C_1, C_2, K)$. Assume w.l.o.g. that $|S_2| \geq 2$ and let $K$ be the set of nodes that are universal for $S$. Lemma 7.2 shows that $K$ is a clique of $G$. Let $C_1 = S_2$ and $C_2 = S_1 \cup S_3$. Lemma 7.2 shows that every node that is adjacent to a node of $C_1$ and a node of $C_2$ is universal for $S$ and, hence, belongs to $K$. Therefore, $(C_1, C_2, K)$ is a D-structure. ∎

## 8. MAIN THEOREM AND ITS CONSEQUENCES

Now we are ready to prove Theorem 4.1, which we restate here for convenience.

**Theorem.** *Every connected cap-free graph that does not contain an amalgam is basic cap-free.*

***Proof.*** Assume $G$ does not contain an amalgam and is not a basic cap-free graph. Since $G$ is not triangulated, $G$ contains a nonempty biconnected triangle-free subgraph. Let $F$ be a maximal node set inducing such a biconnected triangle-free subgraph.

**Claim 1.** *Every node in $V \backslash F$ that has at least two neighbors in $F$ is universal for $F$.*

***Proof.*** Let $u$ be a node in $V \backslash F$ having at least two neighbors in $F$. The graph induced by $F \cup \{u\}$ contains a triangle $u, x, y$, else the maximality of $F$ is contradicted. Let $H$ be a hole in $G(F)$ containing $x$ and $y$. ($H$ exists since, by biconnectedness, $x$ and $y$ belong to a cycle, and since $G(F)$ contains no triangle, the smallest cycle containing $x$ and $y$ is a hole). Lemma 7.1 shows that either $u$ is universal for $H$ or forms an expanded hole with $H$. Theorem 7.1 rules out the latter possibility. Let $F' \subseteq F$ be a maximal set of nodes such that $G(F')$ contains $H$, is biconnected and such that node $u$ is universal for $F'$. If $F \neq F'$, then since $G(F)$ and $G(F')$ are biconnected, some $z \in F \backslash F'$ belongs to a hole that contains an edge of $G(F')$. Let $H'$ be such a hole. By Lemma 7.1 and Theorem 7.1, node $u$ is adjacent to all the nodes of $H'$. Let $F'' = F' \cup V(H')$. $G(F'')$ is biconnected, $u$ is universal for $F''$. Hence, $F''$ contradicts the maximality of $F'$. Hence, $u$ is universal for $F$ and the proof of Claim 1 is complete.

**Claim 2.** *Let $U$ be the set of universal nodes for $F$. Then the nodes in $U$ induce a clique of $G$.*

***Proof.*** Let $w, z \in U$ be two nonadjacent nodes of $U$ and let $v_1, \ldots, v_n, v_1$ be a hole of $G(F)$. Then nodes $w, z$ together with $v_1, v_2, v_3$, and $v_4$ induce an M-structure, either with two proper subclasses not both of which are stable if $v_1$ and $v_4$ are not adjacent, or with three proper subclasses. By Theorem 6.1, $G$ contains an amalgam. This completes the proof of Claim 2.

**Claim 3.** $V = F \cup U$.

***Proof.*** Let $S = V \backslash (F \cup U)$. By Claim 1, every node in $S$ has at most one neighbor in $F$. Let $C$ be a connected component of $G(S)$. By maximality of $F$, there is at most one node in $F$, say $y$, that has a neighbor in $C$. If such a node $y$ exists, let $C_1, \ldots, C_l$ be the connected components of $G(S)$ adjacent to $y$. Let $V_1 = C_1 \cup \cdots C_l \cup \{y\}, A = \{y\}, K = U, V_2 = V \backslash (V_1 \cup K)$, and $B$ be the set of neighbors of $y$ in $F$. Then $(A, B, K)$ is an amalgam of $G$, separating $V_1$ from $V_2$.

If no component of $G(S)$ is adjacent to a node of $F$, let $V_1 = U \cup S, A = U, V_2 = B = F$. Then $(A, B, \emptyset)$ is an amalgam of $G$. This completes the proof of Claim 3.

If $U$ contains at least two nodes, then let $V_1 = A = U, V_2 = B = F$, and $(A, B, \emptyset)$ is an amalgam of $G$. If $U$ contains at most one node, then $G$ is a basic cap-free graph. ∎

In the remainder of this section, we specialize Theorem 4.1 to:

(i) strongly even-signable graphs or, equivalently, cap-free even-signable graphs,
(ii) strongly odd-signable graphs (as observed in Section 3, these graphs are cap-free),
(iii) cap-free odd-signable graphs.

**Definition 8.1.** *A graph $G(V, E)$ is basic strongly even-signable if $G$ is one of the following*:

(i) *a triangulated graph*,
(ii) *a biconnected triangle-free graph, or*
(iii) *some $u \in V$ is universal for $V \backslash \{u\}$ and $G(V \backslash \{u\})$ is bipartite.*

**Theorem 8.1.** *A strongly even-signable* (*cap-free even-signable*) *graph that is not basic strongly even-signable contains an amalgam*.

***Proof.*** The proof follows from Theorem 4.1 and the following claim.

**Claim.** *A graph is basic cap-free and strongly even-signable if and only if it is basic strongly even-signable*.

***Proof.*** Clearly, basic strongly even-signable graphs are basic cap-free and Theorem 3.2 shows that they are strongly even signable. Conversely, let $G$ be a basic cap-free graph that is strongly even-signable and assume that $G$ is not basic strongly even-signable. Then $G$ consists of a biconnected triangle-free graph $G'$ that is not bipartite, together with a node $u$ that is universal for this graph. Since $G'$ is biconnected, nonbipartite, and triangle-free, then $G'$ contains an odd hole. This hole, together with $u$ induces an odd wheel, contradicting Theorem 3.2. ∎

**Definition 8.2.** *A graph $G(V, E)$ is basic strongly odd-signable if $G$ is one of the following*:

(i) *a triangulated graph*,
(ii) *a hole*, *or*
(iii) *some $u \in V$ is universal for $V \backslash \{u\}$ and $G(V \backslash \{u\})$ is an odd hole.*

**Theorem 8.2.** *A strongly odd-signable graph that is not basic strongly odd-signable contains an amalgam*.

***Proof.*** The proof follows from Theorem 4.1 and the following claim.

**Claim.** *A graph is basic cap-free and strongly odd-signable if and only if it is basic strongly odd-signable*.

***Proof.*** Basic strongly odd-signable graphs are basic cap-free and, by Theorem 3.3, they are strongly odd-signable. Conversely, consider a basic cap-free graph $G$

that is strongly odd-signable. If $G$ is a triangulated graph, then $G$ is basic strongly odd-signable. Otherwise, let $G$ consist of the triangle-free graph $G'$ and possibly a universal node $u$ for $G'$. Since $G'$ is biconnected triangle-free, it contains a hole $H$. Suppose that $G' \neq H$, and let $w \in V(G' \backslash H)$ be a node with at least one neighbor in $H$. If $w$ has exactly two neighbors in $H$, say $x$ and $y$, there exists a $3PC(x, y)$. If $w$ has more than two neighbors in $H$, then $w$, together with a subset of the nodes in $H$, induces a cycle with a unique chord that is not a diamond, since $G'$ is triangle-free. So $w$ has exactly one neighbor in $H$, say $x$. Since $G'$ is biconnected, there exists a path in $G' \backslash H$ from $w$ to $v$, where $v$ has a unique neighbor in $H$, say $y$. If $x$ and $y$ are adjacent, $G'$ contains a cycle with a unique chord. Otherwise, $G'$ contains a $3PC(x, y)$. If node $u$ exists and $G'$ is an even hole, then $G$ is an even wheel. ∎

**Definition 8.3.** *A graph $G(V, E)$ is basic odd-signable if $G$ is one of the following*:

(i) *a triangulated graph,*
(ii) *a biconnected triangle-free graph with no even wheel or $3PC(x, y)$, or*
(iii) *some $u \in V$ is universal for $V \backslash \{u\}$ and $G(V \backslash \{u\})$ is a biconnected triangle-free graph with no even hole.*

In [5] we study the structure of basic odd-signable graphs and we give a polynomial algorithm to test membership in this class.

**Theorem 8.3.** *A cap-free odd-signable graph that is not basic odd-signable contains an amalgam.*

***Proof.*** The proof follows from Theorem 4.1 and the following claim.

**Claim.** *A graph is basic cap-free and odd-signable if and only if it is basic odd-signable.*

***Proof.*** It is easy to verify that basic odd-signable graphs are basic cap-free and odd-signable. Conversely, consider a basic cap-free graph $G$ that is odd-signable. If $G$ is triangulated, then $G$ is basic odd-signable. Otherwise, let $G$ consist of the biconnected triangle-free graph $G'$ and possibly a universal node for $G'$. By Theorem 2.3, $G$ contains no $3PC(x, y)$ and no even wheel. So, if $G = G', G$ is basic odd-signable. If $V(G) = V(G') \cup \{u\}$, then $G'$ contains no even hole, since $G$ contains no even wheel. So, again, $G$ is basic odd-signable. ∎

## 9. RECOGNITION ALGORITHMS

We can use Theorem 4.1 to test whether a graph $G$ contains a cap as follows:

**Recognition Algorithm for Cap-Free Graphs**
**Input:** A graph $G$.
**Output:** *Yes* if graph $G$ is cap-free and *No* otherwise.
**Step 0:** Set $\mathcal{L} = \{G\}$ and $\mathcal{L}' = \emptyset$.

**Step 1:** If $\mathcal{L} = \emptyset$, go to Step 3. Otherwise remove a graph $H$ from $\mathcal{L}$ and go to Step 2.
**Step 2:** If $H$ contains an amalgam, add the blocks of the amalgam decomposition to $\mathcal{L}$ and go to Step 1. Otherwise, add $H$ to $\mathcal{L}'$ and go to Step 1.
**Step 3:** Check whether all the graphs in the list $\mathcal{L}'$ are basic cap-free graphs. If so, output *Yes*. Otherwise output *No*.
**End of Algorithm**

The correctness of the above algorithm follows from Theorem 4.1 and the following lemma.

**Lemma 9.1.** *Let $G$ contain an amalgam. Then $G$ is cap-free if and only if the blocks of the amalgam decomposition are cap-free.*

***Proof.*** The "only if" part follows, because the blocks are induced subgraphs of $G$.

Now assume that node $v$, together with hole $H$, induces a cap that is separated in the amalgam decomposition $(A, B, K)$, and let $G_1, G_2$ be the blocks of the decomposition, with node sets $V_1 \cup K \cup \{b\}$ and $V_2 \cup K \cup \{a\}$. We assume w.l.o.g. that if $H$ contains only one node of $A$ or $B$, this node is $a$ or $b$. If $H$ does not belong to $G_1$ or $G_2$, then $H = a_1, b_1, a_2, b_2$, where $a_1, a_2$ belong to $A$ and $b_1, b_2$ belong to $B$. Since the neighbors of $v$ in $H$ are adjacent, then $v \in K$. Hence, $v$ is universal for $H$ and cannot induce a cap with $H$.

So assume $H$ belongs to $G_1$ but $v$ does not. Let $v_1, v_2$ be the neighbors of $v$ in $H$. First, observe that $H$ cannot contain node $b$ and, therefore, $v_1, v_2 \in K$. It follows that $H$ contains no node of $A$ and, therefore, node $b$ together with hole $H$ is a cap of $G_1$. ∎

The above algorithm is polynomial for the following reasons:

(i) The algorithms to find amalgam decompositions are polynomial and the number of blocks is polynomial (Cornuéjols and Cunningham [10]).
(ii) We can test in polynomial time whether a graph is triangulated. The same is true of a biconnected triangle-free graph $G(V, E)$ with at most one additional universal node for $V$. So we can efficiently test whether a graph is basic cap-free.

Note that checking whether a graph $G$ contains a cap can also be done in polynomial time directly, without decomposing $G$.

**Recognition Algorithm for Strongly Even-Signable Graphs (Cap-Free Even-Signable Graphs)**
**Input:** A graph $G$.
**Output:** *Yes* if graph $G$ is strongly even-signable and *No* otherwise.
**Step 0:** Set $\mathcal{L} = \{G\}$ and $\mathcal{L}' = \emptyset$.
**Step 1:** If $\mathcal{L} = \emptyset$, go to Step 3. Otherwise remove a graph $H$ from $\mathcal{L}$ and go to Step 2.
**Step 2:** If $H$ contains an amalgam, add the blocks of the amalgam decomposition to $\mathcal{L}$ and go to Step 1. Else add $H$ to $\mathcal{L}'$ and go to Step 1.

**Step 3:** Check whether all the graphs in the list $\mathcal{L}'$ are basic strongly even-signable graphs. If so, output *Yes*. Otherwise output *No*.
**End of Algorithm**

The correctness of the above algorithm follows from Theorem 8.1 and the next lemma.

**Lemma 9.2.** *Let $G$ contain an amalgam. Then $G$ is strongly even-signable if and only if the blocks of the amalgam decomposition are strongly even-signable.*

***Proof.*** Denote the amalgam by $(A, B, K)$ and let $G_1$ and $G_2$ be the blocks of the decomposition, with node sets $V_1 \cup K \cup \{b\}$ and $V_2 \cup K \cup \{b\}$.

The "only if" part holds, because the blocks are induced subgraphs of $G$.

Conversely, assume $G_1$ and $G_2$ are strongly even-signable. Then they are cap-free, and by Lemma 9.1, so is $G$. By Theorem 3.2, it suffices to show that, if $G_1$ and $G_2$ do not contain an odd wheel $(H, x)$, where $x$ is universal for $H$, then neither does $G$. Suppose that $G$ contains such a wheel $(H, x)$. If $H$ contains a node of $V_1 \backslash A$, then $H$ is entirely contained in $G_1$. In this case, $x$ is also in $G_1$, because no node of $V_2$ can be adjacent to a node in $V_1 \backslash A$. So $H$ is contained in $A \cup B \cup K$. Since $H$ is an odd hole, it is entirely contained in $A$ or $B$, say $A$. But then $G_1$ contains node $b$, which is universal for $A$ and, hence, for $H$, a contradiction. ∎

We now turn to strongly odd-signable graphs. A *bad cycle* is a cycle with a unique chord, which is neither a cap nor a diamond. A bad cycle with chord $xy$ can be separated by an amalgam decomposition $(A, B, K)$ when both nodes $x$ and $y$ belong to $K$. So if $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are blocks of the amalgam decomposition, it is not true in general that $G_1$ and $G_2$ are strongly odd-signable if and only if $G$ is strongly odd-signable.

Note that it is possible to check in polynomial time whether some bad cycle with chord $xy$ is separated by the amalgam decomposition $(A, B, K)$ as follows: Let $U_1^{xy}$ and $U_2^{xy}$ be the neighbors of both $x$ and $y$ in $G_1$ and $G_2$, respectively.

Now $G$ contains a bad cycle that is separated by the amalgam decomposition $(A, B, K)$ if and only if, after removing edge $xy, x$ and $y$ belong to the same connected component of $G_1(V_1 \backslash U_1^{xy})$ and of $G_2(V_2 \backslash U_2^{xy})$.

A wheel $(H, x)$ is a *short* 4-*wheel* if $H$ is a 4-hole and $x$ is universal for $H$. A short 4-wheel $(H, x)$ is separated in an amalgam decomposition $(A, B, K)$ if and only if one of the following two conditions hold:

—$K$ is nonempty and both $A$ and $B$ contain at least two nonadjacent nodes.

—$A$ contains nodes $a_1, a_2, a_3$, where $a_1$ is adjacent to both $a_2$ and $a_3$ but $a_2, a_3$ are nonadjacent and $B$ contains two nonadjacent nodes (or vice versa).

A $3PC(x, y)$ is *short* if at least two of the $xy$-paths have length 2. A short $3PC(x, y)$ is separated by the amalgam decomposition $(A, B, K)$ if and only if $x$ and $y$ belong to $A$ and the intermediate nodes of the two paths of length 2 belong to $B$ (or vice versa). This happens only when:

—$A$ contains three mutually nonadjacent nodes and $B$ two nonadjacent nodes (or vice versa), or

—Both $A$ and $B$ contain two nonadjacent nodes say $a_1, a_2$ and $b_1, b_2$ and either $a_1, a_2$ belong to the same connected component of $G(V_1\backslash(A\backslash\{a_1, a_2\}))$ or $b_1, b_2$ belong to the same connected component of $G(V_2\backslash(B\backslash\{b_1, b_2\}))$.

**Recognition Algorithm for Strongly Odd-Signable Graphs**
**Input:** A graph $G$.
**Output:** *Yes* if graph $G$ is strongly odd-signable and *No* otherwise.
**Step 0:** Set $\mathcal{L} = \{G\}$ and $\mathcal{L}' = \emptyset$.
**Step 1:** If $\mathcal{L} = \emptyset$, go to Step 3. Otherwise remove a graph $H$ from $\mathcal{L}$ and go to Step 2.
**Step 2:** If $H$ does not contain an amalgam, add $H$ to $\mathcal{L}'$ and go to Step 1. Else, detect an amalgam decomposition $(A, B, K)$ with blocks $G_1$ and $G_2$. If a bad cycle, a short 4-wheel or a short $3PC(x, y)$ is separated by the decomposition $(A, B, K)$, output *No*. Else, add $G_1$ and $G_2$ to $\mathcal{L}$ and go to Step 1.
**Step 3:** Check whether all the graphs in the list $\mathcal{L}'$ are basic strongly odd-signable graphs. If so, output *Yes*. Otherwise output *No*.
**End of Algorithm**

The correctness of the above algorithm follows from Theorem 8.2 and the next lemma.

**Lemma 9.3.** *Let $G$ contain an amalgam. If no bad cycle, no short 4-wheel, and no short $3PC(x, y)$ is separated, then $G$ is strongly odd-signable if and only if the blocks of the amalgam decomposition are strongly odd-signable.*

***Proof.*** Denote the amalgam by $(A, B, K)$ and let $G_1$ and $G_2$ be the blocks of the decomposition, with node sets $V_1 \cup K \cup \{b\}$ and $V_2 \cup K \cup \{b\}$.

If $G$ is strongly odd-signable, then the blocks $G_1$ and $G_2$ also are, since they are induced subgraphs of $G$.

Conversely, assume that $G_1$ and $G_2$ are strongly odd-signable and no bad cycle, short 4-wheel or short $3PC(x, y)$ is separated. By Theorem 3.3, it suffices to show that $G$ is cap-free, contains no bad cycle, no $3PC(x, y)$ and no even wheel $(H, x)$, where $x$ is universal for $H$, knowing that these properties hold for $G_1$ and $G_2$.

Since $G_1$ and $G_2$ are cap-free, $G$ is cap-free by Lemma 9.1. Since $G_1$ and $G_2$ contain no bad cycles and, by assumption, no bad cycle is separated, $G$ contains no bad cycle.

Suppose that $G$ contains a $3PC(x, y)$ but $G_1, G_2$ do not. If $x \in V_1$ and $y \in V_2$, then either $x \in A$ or $y \in B$, but not both. Assume $y \in B$, then no node of $V_2 \cup K\backslash\{y\}$ belongs to the $3PC(x, y)$ and so the $3PC(x, y)$ is contained in $G_1$. If $x \in V_1$ and $y \in K$, then $x \in V_1\backslash A$, else $x$ and $y$ are adjacent. But again the $3PC$ is contained in $G_1$. This implies that $x$ and $y$ are both in $V_1$ or both in $V_2$, say $V_1$. At least one path of the $3PC(x, y)$ contains a node of $B$. This shows that $x, y \in A$. If only one of these paths exists, then we can assume this path to be $x, b, y$, which is also in $G_1$. So at least two of these paths exist, and we have a short $3PC$, which is separated in the amalgam decomposition.

Finally, suppose that $G$ contains an even wheel $(H, x)$. If the hole $H$ itself is separated in the amalgam decomposition, then $H = a_1, b_1, a_2, b_2$, where $a_1, a_2$ belong to $A$ and $b_1, b_2$ belong to $B$. Now obviously $v \in K$ and we have a short 4-wheel, which is separated. If $H$ is in $G_1$, then $H$ contains at most three nodes in $A \cup B$, and no node in $V_2$ can be universal for $H$. ∎

Recall that Corollary 3.1 shows that the class of strongly odd-signable graphs is properly contained in the class of odd-signable cap-free graphs. Our final algorithm tests membership in this last class.

**Recognition Algorithm for Cap-Free Odd-Signable Graphs**
**Input:** A graph $G$.
**Output:** *Yes* if graph $G$ is cap-free odd-signable and *No* otherwise.
**Step 0:** Set $\mathcal{L} = \{G\}$ and $\mathcal{L}' = \emptyset$.
**Step 1:** If $\mathcal{L} = \emptyset$, go to Step 3. Otherwise remove a graph $H$ from $\mathcal{L}$ and go to Step 2.
**Step 2:** If $H$ does not contain an amalgam, add $H$ to $\mathcal{L}'$ and go to Step 1. Else, detect an amalgam decomposition with blocks $G_1$ and $G_2$. If a short 4-wheel or a short $3PC(x, y)$ is separated by the decomposition, output *No*. Else, add $G_1$ and $G_2$ to $\mathcal{L}$ and go to Step 1.
**Step 3:** Check whether all the graphs in the list $\mathcal{L}'$ are basic odd-signable graphs. If so, output *Yes*. Otherwise output *No*.
**End of Algorithm**

The structure of triangle-free graphs that are odd-signable is studied in [5], where we give an algorithm to test membership in this class. Together with the signing algorithm of Section 2, we can, therefore, check if a triangle-free graph contains an even hole. Therefore, we can check whether a graph is basic odd-signable and accomplish Step 3.

The correctness of the above algorithm follows from Theorem 8.3 and the next lemma, whose proof is a simplification of the proof of Lemma 9.3 and is omitted.

**Lemma 9.4.** *Let $G$ contain an amalgam. If no short 4-wheel and no short $3PC(x, y)$ is separated by the decomposition, then $G$ is a cap-free odd-signable graph if and only if the blocks of the amalgam decomposition are cap-free odd-signable graphs.*

Together with the signing algorithm given in Section 2, the above algorithm recognizes in polynomial time whether a cap-free graph contains an even hole. This generalizes the result of Markossian, Gasparian and Reed [12], which recognizes in polynomial time whether a diamond-and-cap-free graph contains an even hole.

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