Decomposition of Odd-hole-free Graphs by Double Star Cutsets and 2-Joins

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Abstract

In this paper we decompose odd-hole-free graphs (graphs that do not contain as an induced subgraph a chordless cycle of odd length greater than three) with double star cutsets and 2-joins into bipartite graphs, line graphs of bipartite graphs and the complements of line graphs of bipartite graphs.

1 Introduction

In this paper, all graphs are simple. A cycle is *even* if it contains an even number of nodes, and it is *odd* otherwise. A *hole* is a chordless cycle with at least four nodes. An *odd-hole-free graph* is a graph that does not contain an odd hole. When we say that a graph G *contains* a graph H, we mean that H appears in G as an induced subgraph.

Given a graph G and a node set S, we denote by $G \setminus S$ the subgraph of G obtained by removing the nodes of S and the edges with at least one node in S. A node set $S \subset V(G)$ is a *cutset* of G if the graph $G \setminus S$ is disconnected. For $S \subseteq V(G)$, $N(S)$ denotes the set of nodes in $V(G) \setminus S$ that are adjacent to at least one node in S. A node set S is a K_m*-star* if S contains a clique C of size m and $S \subseteq C \cup N(C)$. We also refer to a K_1-star as a *star* and to a K_2-star as a *double star*.

A graph G has a *2-join*, denoted by $H_1|H_2$, if the nodes of G can be partitioned into sets H_1 and H_2 with nonempty and disjoint subsets $A_1, B_1 \subseteq H_1$, $A_2, B_2 \subseteq H_2$, such that all nodes of A_1 are adjacent to all nodes of A_2, all nodes of B_1 are adjacent to all nodes of B_2 and these are the only adjacencies between H_1 and H_2. Also, for $i = 1, 2$, $|H_i| > 2$ and if A_1 and B_1 (resp. A_2 and B_2) are both of cardinality 1, then the graph induced by H_1 (resp. H_2) is not a chordless path. 2-joins were introduced by Cornuéjols and Cunningham [9].

The main result of this paper is the following.

Theorem 1.1 *If G is an odd-hole-free graph, then G is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or G has a double star cutset or a 2-join.*

In [7] Conforti, Cornuéjols, Kapoor and Vušković obtain a polynomial time recognition algorithm for the class of even-hole-free graphs. This algorithm is based on the decomposition of even-hole-free graphs by 2-joins, double star and triple star (K_3-star) cutsets obtained in [6]. It would be of interest to try to use Theorem 1.1 to construct a polynomial time recognition algorithm for the class of odd-hole-free graphs. This problem is currently not even known to be in NP.

Odd-hole-free graphs are related to perfect graphs introduced by Berge. A graph G is *perfect* if every induced subgraph H of G has a chromatic number equal to the size of a largest clique in H. A graph is *Berge* if it contains neither an odd hole nor its complement. Every perfect graph is Berge and the *strong perfect graph conjecture* (SPGC) states that every Berge graph is perfect. Well known classes of Berge graphs are bipartite graphs, line graphs of bipartite graphs, and the complements of such graphs. It is easy to verify that these graphs are perfect. Ongoing research (in March 2001) is aimed at obtaining a decomposition theorem for Berge graphs that uses more refined cutsets that would allow for the proof of the SPGC. For example, when G is a square-free Berge graph, Conforti, Cornuéjols and Vušković

[8] showed that "double star cutset" can be replaced by "star cutset" in the statement of Theorem 1.1. Since star cutsets cannot occur in minimally imperfect graphs (Chvátal [1]) and neither can 2-joins (Cornuéjols and Cunningham [9], see also [11, 4]), it follows that the strong perfect graph conjecture holds for square-free graphs. A *skew cutset* is a cutset $S = A \cup B$ where A, B are disjoint and nonempty, and every node of A is adjacent to every node of B. Note that a star cutset is a skew cutset which itself is a double star cutset. Chvátal [1] introduced skew cutsets and conjectured that they cannot occur in a minimally imperfect graph. This conjecture implies that a decomposition theorem for Berge graphs similar to Theorem 1.1, in which "double star cutsets" are replaced by "skew cutsets", would prove the SPGC. Such a decomposition theorem and the proof of the skew cutset conjecture were recently obtained by Chudnovsky, Robertson, Seymour and Thomas [2].

1.1 Proof Outline

To obtain Theorem 1.1, we prove the following more general result. We *sign* a graph by assigning 0,1 weights to its edges in such a way that, for every triangle in the graph, the sum of the weights of its edges is odd. A graph G is *even-signable* if there is a signing of its edges so that for every hole in G, the sum of the weights of its edges is even. Clearly, every odd-hole-free graph is even-signable (assign weight 1 to all the edges).

Theorem 1.2 *If G is an even-signable graph, then G is a triangle-free graph or the line graph of a triangle-free graph or the complement of the line graph of a complete bipartite graph, or G has a double star cutset or a 2-join.*

The proof outline of Theorem 1.2 is as follows. Undefined terms will be defined later.

- Theorem 1.2 holds for graphs that contain no proper wheels and no parachutes (Section 2).
- If G contains a proper wheel that is not a beetle, then G has a double star cutset (Section 3).
- If G contains an L-parachute, then G has a double star cutset (Section 4).
- If G contains a T-parachute or a beetle, then G has a double star cutset or G contains a $3PC(\Delta,\Delta)$ with a Type t2, t2p, t4 or t5 node (Section 6).
- If G contains a $3PC(\Delta,\Delta) \neq \bar{C}_6$ with a Type t4 or t5 node, then G has a double star cutset (Section 8).
- If G contains a $3PC(\Delta,\Delta)$ with a Type t2 or t2p node, then G has a double star cutset or a 2-join (Section 9).
- If G contains a $\bar{C}_6$ with a Type t4 or t5 node, then G has a double star cutset or a 2-join, or G is the complement of the line graph of a complete bipartite graph (Section 10).

1.2 Notation and Background

Let G be a graph and H an induced subgraph of G. A node $v \notin V(H)$ is *strongly adjacent* to H, if $|N(v) \cap V(H)| \geq 2$.

By C_6 we denote a hole of length 6, and by $\bar{C}_6$ its complement.

A *path* P is a sequence of distinct nodes $x_1, \ldots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. If $n > 1$ then nodes x_1 and x_n are the *endnodes* of the path. The nodes of P that are not endnodes are called *intermediate* nodes of P. The intermediate nodes of P are also referred to as the *interior* of P. Where clear from context we write P instead of $V(P)$. Let x_i and x_l be two nodes of P, where $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the $x_i x_l$-subpath of P and is denoted by $P_{x_i x_l}$. A cycle C is a sequence of nodes $x_1, x_2, \ldots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, x_2, \ldots, x_n$ form a path and $x_1 x_n$ is an edge. The node set of a path or a cycle Q is denoted by $V(Q)$. The length of a path P is the number of edges in P and is denoted by $|P|$. Similarly the length of a cycle C is the number of edges in C and is denoted by $|C|$.

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, \ldots, x_n$ *connects* A *to* B if either $n = 1$ and x_1 has neighbors in A and B or $n > 1$ and x_1 is adjacent to at least one node in A and x_n is adjacent to at least one node in B. The path P is a *direct connection from* A *to* B if, in the subgraph induced by the node set $V(P) \cup A \cup B$, no path connecting A to B is shorter than P.

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_n$. Node x is the *center* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. A *short sector* is a sector of length 1 (i.e. it consists of one edge), and a *long sector* is a sector of length at least 2. A wheel is *odd* if it contains an odd number of short sectors. A wheel with k sectors is called a k-wheel.

A *line wheel* is a 4-wheel (H, v) that contains exactly two triangles and these two triangles have only the center v in common. A *twin wheel* is a 3-wheel containing exactly two triangles. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the nodes of H. A *triangle-free wheel* is a wheel containing no triangle. These four types of wheels are depicted in Figure 1, where solid lines represent edges and dotted lines represent paths. A *proper wheel* is a wheel that is not any of the above four types.

A $3PC(x_1 x_2 x_3, y)$ is a graph induced by three chordless paths $P^1 = x_1, \ldots, y$, $P^2 = x_2, \ldots, y$ and $P^3 = x_3, \ldots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P^i \setminus y$ and $P^j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by $\{x_1, x_2, x_3\}$. Also, at most one of the paths P^1, P^2, P^3 is an edge. We say that a graph G contains a $3PC(\Delta, .)$ if it contains a $3PC(x_1 x_2 x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$.

The following theorem is an easy consequence of a theorem of Truemper [12].

Theorem 1.3 ([5]) *A graph is even-signable if and only if it does not contain an odd wheel or a* $3PC(\Delta, .)$.

The fact that even-signable graphs do not contain odd wheels and $3PC(\Delta, .)$'s will be used throughout the paper.

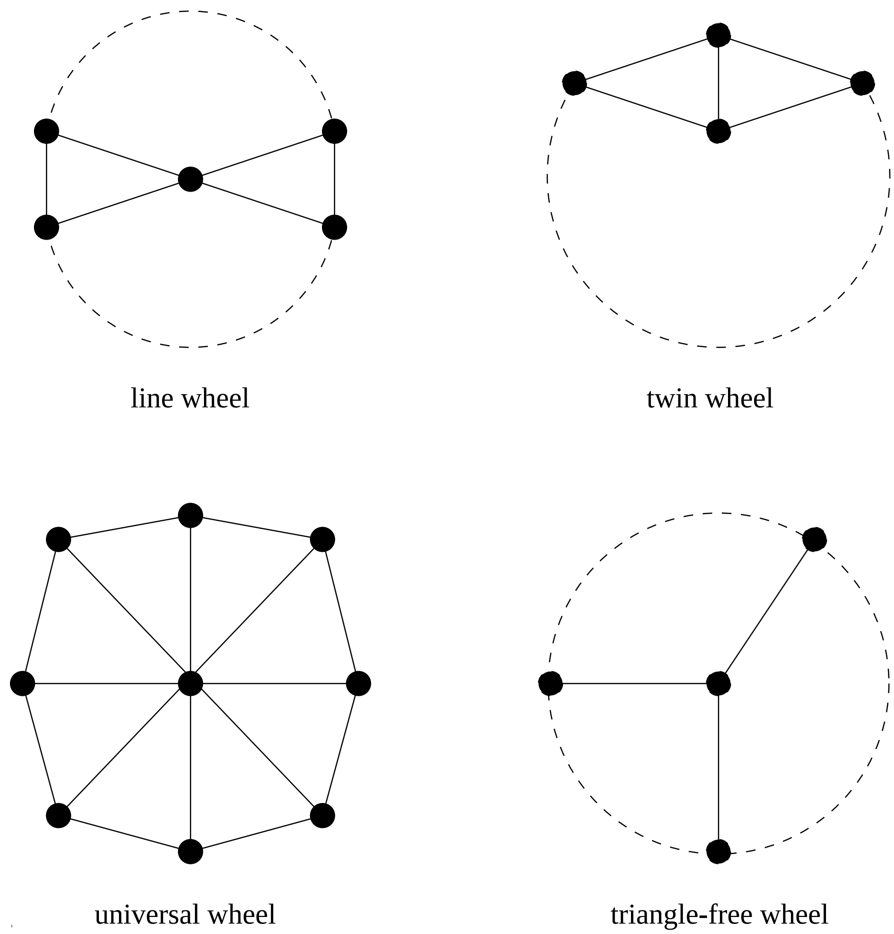

Figure 1: Wheels

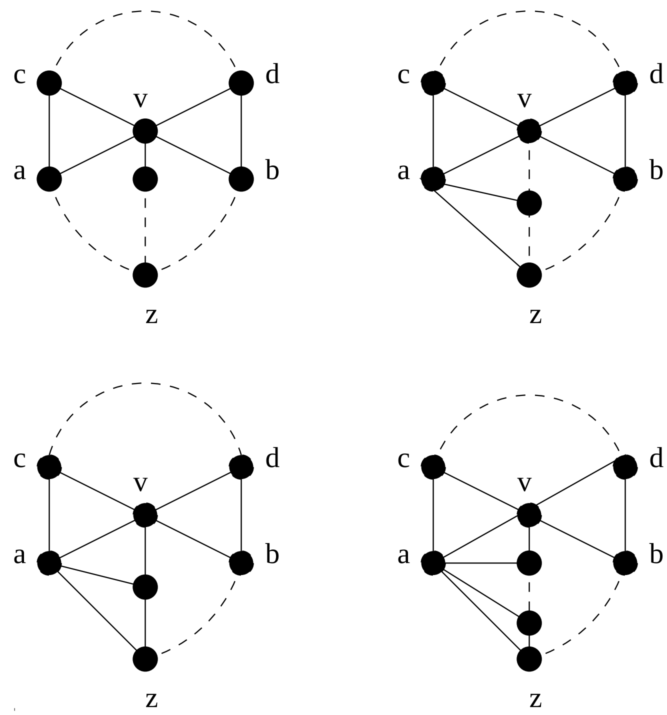

Figure 2: L-parachutes

2 WP-Free Graphs

In this section, we state a result proven in [3]. First, we need some definitions.

Definition 2.1 *An* L-parachute $LP(ca, db, v, z)$ *is a graph induced by a line wheel* (H, v) *where* $H = a, \ldots, z \ldots, b, d, \ldots, c, a$*, where* a*,* b*,* c*,* d *are the neighbors of* v *in* H*, together with a chordless path* $P = v, \ldots, z$ *of length greater than one. Furthermore, no node of* $H \setminus \{z, a\}$ *is adjacent to an intermediate node of* P*.*

Definition 2.2 *A* T-parachute $TP(t, v, a, b, z)$ *is a graph induced by a twin wheel* (H, v) *where* $H = a, t, b, \ldots, z, \ldots, a$*, where* t*,* a*,* b *are the neighbors of* v *in* H*, together with a chordless path* $P = v, \ldots, z$ *of length greater than one. Furthermore, no node of* $H \setminus \{z, a\}$ *is adjacent to an intermediate node of* P*.*

Definition 2.3 *A* parachute *is either an L-parachute or a T-parachute.*

Definition 2.4 *A graph* G *is* WP-free *if it contains neither a proper wheel nor a parachute.*

Theorem 2.5 *Let* G *be an even-signable WP-free graph that is neither a triangle-free graph nor a line graph of a triangle-free graph. Then* G *contains a double star cutset or a 2-join.*

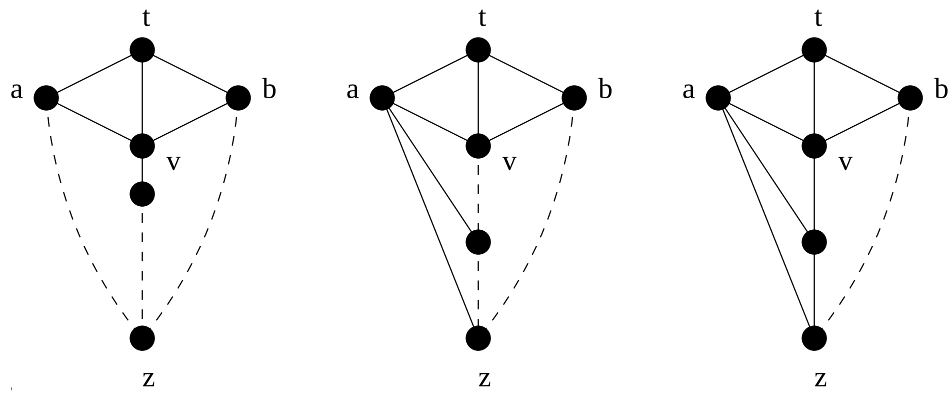

Figure 3: T-parachutes

In fact [3] proves a stronger result: "double star cutset or 2-join" in the statement of Theorem 2.5 can be replaced by "star cutset or universal 2-amalgam". Since star cutsets cannot occur in minimally imperfect graphs (Chvátal [1]) and universal 2-amalgams cannot occur in minimally imperfect Berge graphs (Conforti, Cornuéjols, Gasparyan and Vušković [4]), it follows that the strong perfect graph conjecture holds for WP-free graphs. In this paper we only need the weaker statement 2.5.

As a consequence of Theorem 2.5, it suffices to prove Theorem 1.2 when G contains a proper wheel or a parachute.

3 Proper Wheels

In this section, we prove the following theorem.

Definition 3.1 *A* beetle *is a wheel with four sectors, exactly two of which are short and are furthermore adjacent.*

Theorem 3.2 *Let G be an even-signable graph. If G contains a proper wheel that is not a beetle, then G has a double star cutset.*

To prove this theorem, we use a result of [5].

A *Mickey Mouse*, denoted by $M(xyz, H_1, H_2)$, is a graph induced by the node set $H_1 \cup H_2$ with the following properties:

- the node set $\{x, y, z\}$ induces a clique,
- H_1 is a hole that contains edge xy but does not contain node z,
- H_2 is a hole that contains edge xz but does not contain node y, and
- the node set $H_1 \cup H_2$ induces a cycle with exactly two chords, xy and xz.

In [5] we obtained the following decomposition theorem for Mickey Mouses. Note that in [5] Mickey Mouse defined as above is called a Mickey Mouse with long ears.

A node set S is an *extended star* if three nodes x, y, z of S induce a triangle and $S \subseteq N(x) \cup (N(y) \cap N(z))$. Clearly, an extended star cutset is always a double star cutset, since $S \subseteq N(x) \cup N(y)$.

Theorem 3.3 *If an even-signable graph G contains a Mickey Mouse $M(xyz, H_1, H_2)$, then $N(x) \cup (N(y) \cap N(z))$ is an extended star cutset separating nodes of H_1 from H_2.*

A *butterfly* is a wheel (H, x) with six sectors exactly two of which are long, and, if $x_1, \ldots, x_6$ are the neighbors of x in H encountered in this order, then x_1x_2, x_2x_3, x_4x_5 and x_5x_6 are edges. Denote by S_1 and S_2 the two long sectors of a butterfly (H, x) whose endnodes are x_1, x_6 and x_3, x_4 respectively.

Lemma 3.4 *Let G be an even-signable graph that does not contain a Mickey Mouse and let (H, x) be a butterfly in G. If u is strongly adjacent to (H, x) but is not adjacent to x, then u is one of the following types.*

Type a: *All the neighbors of u in H are contained in either S_1 or S_2.*

Type b: *The neighbors of u in H are contained in $S_1 \cup S_2$ and u is not of Type a.*

Type c: *u is adjacent to x_1, x_2, x_3 and the neighbors of u in H are all contained in $H \setminus x_5$, or u is adjacent to x_4, x_5, x_6 and the neighbors of u in H are all contained in $H \setminus x_2$.*

Type d: *u is adjacent to $x_1, \ldots, x_6$ and has possibly more neighbors in S_1 and S_2.*

Type e: *u is adjacent to x_2, x_5 and to no other node of H.*

Type f: *u has exactly two neighbors in H, that are furthermore adjacent and contained in $\{x_1, \ldots, x_6\}$.*

Proof: If u is adjacent to neither x_2 nor x_5, then u is of Type a or b. So w.l.o.g. assume that u is adjacent to x_2. Suppose that u is adjacent to neither x_1 nor x_3 and is not of Type e. Then u must have a neighbor in $S_1 \setminus x_1$ or $S_2 \setminus x_3$, say it has a neighbor in $S_1 \setminus x_1$. Let u_1 be the neighbor of u in S_1 that is closest to x_6 and let S_1' be the u_1x_6-subpath of S_1. If u does not have a neighbor in S_2, then the node set $\{u, x\} \cup S_1' \cup S_2$ induces a Mickey Mouse. So u must also have a neighbor in $S_2 \setminus x_3$. Let u_2 be the neighbor of u in S_2 that is closest to x_3, and let S_2' be the x_3u_2-subpath of S_2. Then the node set $\{u, x\} \cup S_1' \cup S_2'$ induces an odd wheel with center x_2. So if u is adjacent to neither x_1 nor x_3, it must be of Type e.

We now assume that u is adjacent to exactly one of x_1, x_3, say x_1. Suppose u is not of Type f. We first show that u cannot have a neighbor in S_2. Suppose it does and let u_1 (resp. u_2) be the neighbor of u in S_2 that is closest to x_3 (resp. x_4). If u_1u_2 is not an edge, then in the graph induced by $S_2 \cup \{u, x, x_2\}$ there is either a $3PC(x_2x_3x, u_1)$ (if $u_1 = u_2$) or a $3PC(x_2x_3x, u)$ (if $u_1 \neq u_2$). If u_1u_2 is an edge, the node set $S_2 \cup \{u, x, x_1\}$ induces a $3PC(u_1u_2u, x)$. Hence u does not have a neighbor in S_2. Node u must have a neighbor in $S_1 \setminus x_1$, else (H, u) is an odd wheel. Let u_1 be the neighbor of u in $S_1 \setminus x_1$ that is closest to x_6. Then the u_1x_6-subpath of S_1 together with S_2, x, x_2 and u induces a Mickey Mouse.

Now assume that u is adjacent to both x_1 and x_3. If u is not adjacent to x_5, then u is of Type c. Assume u is adjacent to x_5. By symmetry, we can assume that u is adjacent to both x_4, x_6, and so it is of Type d. □

Lemma 3.5 *Let G be an even-signable graph that does not contain Mickey Mouses. If (H, x) is a butterfly, then $S = N(x) \cup (N(x_1) \cap N(x_3)) \setminus x_2$ is a double star cutset separating x_2 from the rest of H.*

Proof: Suppose not and let $P = y_1, \ldots, y_n$ be a direct connection from x_2 to $H \setminus (S \cup x_2)$ in $G \setminus S$. By Lemma 3.4, $n > 1$, y_1 is either not strongly adjacent to H or is of Type e or f, and y_n is either not strongly adjacent to H or is of Type a, b or c (adjacent to x_4, x_5, x_6 and with at least one more neighbor in $(S_1 \cup S_2) \setminus \{x_1, x_3\}$).

First we show that x_4 and x_6 do not have a neighbor in $P \setminus y_n$. Suppose not and let y_i be the node of $P \setminus y_n$ with lowest index that is adjacent to x_4 or x_6. W.l.o.g. assume y_i is adjacent to x_6. If x_1 does not have a neighbor in $\{y_1, \ldots, y_i\}$ then $S_1 \cup \{y_1, \ldots, y_i, x, x_2\}$ induces an odd wheel with center x. If x_3 does not have a neighbor in $\{y_1, \ldots, y_i\}$ then either $T = S_2 \cup \{y_1, \ldots, y_i, x, x_2, x_6\}$ induces a Mickey Mouse (if x_4 is not adjacent to y_i) or $T \setminus x_6$ induces an odd wheel with center x (if x_4 is adjacent to y_i). So x_1 and x_3 both have a neighbor in $\{y_1, \ldots, y_i\}$. Let y_j (resp. y_k) be the node of P with lowest index adjacent to x_3 (resp. x_1). Then $j = 1$ or i, since otherwise $S_2 \cup \{y_1, \ldots, y_j, x, x_2\}$ induces a Mickey Mouse. If $j = i$ then $\{y_1, \ldots, y_i, x, x_2, x_3, x_6\}$ induces an odd wheel with center x_3. So $j = 1$ and hence $k \neq 1$. If $k \neq i$ then $S_1 \cup \{y_1, \ldots, y_k, x, x_2\}$ induces a Mickey Mouse. So $k = i$. But then $\{y_1, \ldots, y_i, x, x_1, x_2, x_6\}$ induces an odd wheel with center x_1. Therefore, x_4 and x_6 do not have a neighbor in $P \setminus y_n$.

Next we show that if y_1 is not of Type f then x_1 and x_3 do not have a neighbor in $P \setminus y_n$. Assume otherwise and let y_i be the node of $P \setminus y_n$ with lowest index adjacent to x_1 or x_3, say x_1. Then $S_1 \cup \{y_1, \ldots, y_i, x, x_2\}$ induces a Mickey Mouse. The same argument shows that if y_1 is of Type f adjacent to x_1 (resp. x_3) then x_3 (resp. x_1) does not have a neighbor in $P \setminus y_n$.

We now consider the following two cases.

Case 1: y_n is either not strongly adjacent to H or is of Type a.

W.l.o.g. y_n has a neighbor in S_1. Let u_1 (resp. u_2) be the neighbor of y_n in S_1 that is closest to x_1 (resp. x_6). Let S_1' (resp. S_1'') be the x_1u_1-subpath (resp. u_2x_6-subpath) of S_1. Node x_3 must have a neighbor in P, since otherwise $S_2 \cup P \cup S_1'' \cup \{x, x_2\}$ induces a Mickey Mouse. So y_1 is of Type f adjacent to x_3, and hence x_1 does not have a neighbor in $P \setminus y_n$. If u_1u_2 is not an edge, then $P \cup S_1' \cup S_1'' \cup \{x, x_2\}$ induces a $3PC(x_1x_2x, .)$. So u_1u_2 is an edge. Let y_k be the node of P with highest index adjacent to x_3. Then $S_1 \cup \{y_k, \ldots, y_n, x, x_3\}$ induces a $3PC(u_1u_2y_n, x)$.

Case 2: y_n is of Type b or c.

W.l.o.g. y_n has a neighbor in $S_1 \setminus x_6$. Let u_1 be the neighbor of y_n in S_1 that is closest to x_1. Let u_2 be the neighbor of y_n in S_2 that is closest to x_4 (such a neighbor always exists). Let S_1' (resp. S_2') be the x_1u_1-subpath of S_1 (resp. u_2x_4-subpath of S_2). If x_1 does not have a neighbor in $P \setminus y_n$, then $P \cup S_1' \cup S_2' \cup \{x, x_2\}$ induces a $3PC(x_1x_2x, y_n)$. Hence y_1 is of Type f adjacent to x_1, and x_3 does not have a neighbor in $P \setminus y_n$. Let u_3 (resp. u_4) be the neighbor of y_n in S_2 (resp. S_1) that is closest to x_3 (resp. x_6), and let S_2'' (resp. S_1'') be the x_3u_3-subpath of S_2 (resp. u_4x_6-subpath of S_1). If $u_3 \neq x_4$ then $S_1'' \cup P \cup S_2'' \cup \{x, x_2\}$ induces a $3PC(x_2x_3x, y_n)$. Otherwise $P \cup S_2 \cup \{x, x_2\}$ induces an odd wheel with center x. □

A *bat* is composed of a chordless path $y_1, \ldots, y_n$ and a node x such that, for some $2 < i < j < n-1$, x is adjacent to y_k if and only if $k \in \{1, i, \ldots, j, n\}$.

In the remainder of this section, when we refer to a wheel (H, x) we denote with $x_1, \ldots, x_n$ the neighbors of x in H in the order in which they appear. For $i = 1, \ldots, n$, we denote with S_i the sector of (H, x) with endnodes x_i and x_{i+1} (note $x_{n+1} = x_1$).

Lemma 3.6 *Let G be an even-signable graph that does not contain Mickey Mouses and butterflies. Let (H, x) be a wheel with a bat in G that has fewest number of sectors. Suppose that sectors $S_n, S_1, \ldots, S_k$ together with node x induce a bat where S_n and S_k are the two long sectors. If node $u \in G \setminus (H \cup x)$ is adjacent to x_2, but not to x and not to x_1, then u has no neighbors in $H \setminus \{x_2, x_3\}$.*

Proof: Since G contains no Mickey Mouse, $k \geq 3$. We first show that u has no neighbors in S_n. Suppose not and let u' (resp. u'') be the neighbor of u in S_n that is closest to x_1 (resp. x_n). Let S_n' (resp. S_n'') be the $u'x_1$-subpath (resp. $u''x_n$-subpath) of S_n. Note that $u' \neq x_n$, since otherwise $S_n \cup \{u, x, x_2\}$ induces an odd wheel with center x. Node u must have a neighbor in $H \setminus (S_n \cup x_2)$, else $(H \setminus S_n) \cup S_n'' \cup \{u, x\}$ induces an odd wheel with center x. If u is adjacent to x_i for some $i \in \{3, \ldots, n-1\}$, then $S_n' \cup \{u, x, x_2, x_i\}$ induces an odd wheel with center x_2. Otherwise, there is a shortest subpath S' of $H \setminus (S_n \cup \{x_2, x_3\})$ such that one endnode of S' is adjacent to u and the other to x, and hence $S_n' \cup S' \cup \{u, x, x_2\}$ induces an odd wheel with center x_2. Therefore, u has no neighbors in S_n.

Let x_n' be the neighbor of x_n in S_{n-1} and suppose that u has a neighbor in $H \setminus \{x_2, x_3, x_n'\}$. Then there is a shortest subpath S' of $H \setminus \{x_2, x_3, x_n'\}$ such that one endnode of S' is adjacent to u and the other to x, and hence $S_n \cup S' \cup \{u, x, x_2\}$ induces a Mickey Mouse. Therefore, u has no neighbors in $H \setminus \{x_2, x_3, x_n'\}$. Finally suppose that u is adjacent to x_n'. Then u cannot be adjacent to x_3, since otherwise (H, u) is an odd wheel. Node x_n' must be adjacent to x, else $S_n \cup \{u, x, x_2, x_n'\}$ induces an odd wheel with center x. Let H' be the hole induced by $(H \setminus S_n) \cup u$. (H', x) is a line wheel, else the choice of (H, x) is contradicted. But then (H, x) is a butterfly. □

Lemma 3.7 *If G is an even-signable graph that has a wheel with a bat, then there is a double star cutset.*

Proof: By Theorem 3.3 and Lemma 3.5, we may assume that G contains no Mickey Mouse and no butterfly. Let (H, x) be a wheel with a bat in G that has fewest number of sectors. Suppose that sectors $S_n, S_1, \ldots, S_k$ together with node x induce a bat. Since G does not contain a Mickey Mouse, $k \geq 3$. Let x_1' be the neighbor of x_1 in S_n. We show that $S = (N(x) \cup N(x_1)) \setminus \{x_2, x_4, \ldots, x_n, x_1'\}$ is a double star cutset that separates x_2 from the rest of H. Suppose not and let $P = y_1, \ldots, y_m$ be a direct connection from x_2 to $H \setminus (S \cup x_2)$ in $G \setminus S$. By Lemma 3.6, y_1 is either not strongly adjacent to H or it has exactly two neighbors in H, x_2 and x_3. So $m \geq 2$.

Suppose that y_m has a neighbor in S_n. Let u' (resp. u'') be the neighbor of y_m in S_n that is closest to x_1 (resp. x_n). Let S'_n (resp. S''_n) be the $u'x_1$-subpath (resp. $u''x_n$-subpath) of S_n. If $u' = u''$ then $P \cup S_n \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, u')$. If $u'u''$ is not an edge then $P \cup S'_n \cup S''_n \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, y_m)$. So $u'u''$ is an edge. Suppose x_3 has a neighbor in P and let y_i be its neighbor in P with highest index. Then $S_n \cup \{x, x_3, y_i, \ldots, y_m\}$ induces a $3PC(y_m u' u'', x)$. So x_3 does not have a neighbor in P. Node y_m must have a neighbor in $H \setminus S_n$, since otherwise $H \cup P$ induces a $3PC(y_m u' u'', x_2)$. Hence there is a shortest subpath S' of $H \setminus S_n$ such that one endnode of S' is adjacent to y_m and the other to x. But then $S'_n \cup S' \cup P \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, y_m)$. Therefore y_m does not have a neighbor in S_n.

Node y_m must have a neighbor in $H \setminus \{x_2, x_3\}$. Let x'_n be the neighbor of x_n in S_{n-1}. If y_m has a neighbor in $H \setminus \{x_2, x_3, x'_n\}$ then there is a shortest subpath S' of $H \setminus \{x_2, x_3, x'_n\}$ such that one endnode of S' is adjacent to y_m and the other to x, and so $S_n \cup S' \cup \{x, x_2\}$ induces a Mickey Mouse. Hence x'_n is the unique neighbor of y_m in $H \setminus \{x_2, x_3\}$. Node x'_n is adjacent to x, else $S_n \cup P \cup \{x, x_2, x'_n\}$ induces an odd wheel with center x. Suppose x_3 does not have a neighbor in P. Let H' be the hole induced by $(H \setminus S_n) \cup P$. (H', x) must be a line wheel, since otherwise our choice of (H, x) is contradicted. But then (H, x) is a butterfly. Hence x_3 has a neighbor in P. Let y_i be the neighbor of x_3 in P with highest index. If $i > 1$, then $S_n \cup \{x, x_2, x_3, x'_n, y_i, \ldots, y_m\}$ induces an odd wheel with center x. Hence $i = 1$. But then $H \cup P$ induces a $3PC(x_2x_3y_1, x'_n)$. $\square$

Proof of Theorem 3.2: Assume G has no double star cutset. Then by Lemma 3.7, G has no wheel with a bat. Let (H, x) be a proper wheel that is not a beetle. Assume w.l.o.g. that S_n is a long sector and S_1 is a short sector. Since (H, x) is not a wheel with a bat, either S_n is the only long sector, or $n > 5$ and S_n and S_{n-1} are the only long sectors. Let $S = (N(x) \cup N(x_1)) \setminus \{x_2, x_n, x'_1\}$, where x'_1 is the neighbor of x_1 in S_n. We claim that S is a double star cutset that separates x_2 from $S_n \cup S_{n-1} \setminus \{x_1, x_{n-1}\}$. Let $P = y_1, \ldots, y_m$ be a direct connection from x_2 to $S_n \cup S_{n-1} \setminus \{x_1, x_{n-1}\}$ in $G \setminus S$. Let s be the neighbor of x_n in S_{n-1}.

Case 1: y_m has a neighbor in S_n.

Let u_1 (resp. u_n) be the neighbor of y_m in S_n that is closest to x_1 (resp. x_n). Let S'_n (resp. S''_n) be the u_1x_1-subpath (resp. u_nx_n-subpath) of S_n.

If $u_1 = u_n$ then $P \cup S_n \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, u_1)$. If u_1u_n is not an edge, then $P \cup S'_n \cup S''_n \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, y_m)$. Hence u_1u_n is an edge.

A node of $H \setminus (S_1 \cup S_n)$ must have a neighbor in P, since otherwise $H \cup P$ induces a $3PC(u_1u_ny_m, x_2)$. Let u be the node of $H \setminus (S_1 \cup S_n)$ that has a neighbor in P and is closest to x_3. Let y_i be the node of P with highest index adjacent to u. If $u \neq s$ there exists a chordless path S' from u to x in $H \setminus (S_1 \cup S_n)$. But then $P_{y_iy_m} \cup S_n \cup S' \cup x$ induces a $3PC(u_1u_ny_m, x)$. Hence $u = s$.

Suppose that $i = m$. Since (H, y_m) is not an odd wheel, $u_n = x_n$. If s does not have a neighbor in $P \setminus y_m$, then $P \cup S'_{n-1} \cup S'_n \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, y_m)$. So s has a neighbor in $P \setminus y_m$. Let y_j be the neighbor of s in P with lowest index. If $s \neq x_{n-1}$ then $S_n \cup P_{y_1y_j} \cup \{x, x_2, s\}$ induces an odd wheel with center x. So $s = x_{n-1}$. If $j \neq m-1$

then $P_{y_1y_j} \cup S'_n \cup \{x, x_2, x_{n-1}, y_m\}$ induces a $3PC(xx_1x_2, x_{n-1})$. So $j = m - 1$. But then $P \cup \{x, x_2, x_{n-1}, x_n\}$ induces an odd wheel with center x_{n-1}. Therefore $i \neq m$.

If $i \neq 1$ then $P_{y_iy_m} \cup H$ induces a $3PC(u_1u_ny_m, s)$. So $i = 1$. But then $S_n \cup P_{y_iy_m} \cup \{x_2, s\}$ induces a $3PC(u_1u_ny_m, y_i)$.

Case 2: y_m has no neighbors in S_n.

Then S_{n-1} is a long sector. Let u be the neighbor of y_m in S_{n-1} that is closest to x_n, and let S'_{n-1} be the ux_n-subpath of S_{n-1}. Note that by the definition of S and P, $u \neq x_{n-1}$. Then $P \cup S_n \cup S'_{n-1} \cup \{x, x_2\}$ induces a $3PC(xx_1x_2, x_n)$. □

4 L-Parachutes

In this section we assume that G is an even-signable graph. We prove the following result.

Theorem 4.1 *If G contains an L-parachute, then G has a double star cutset.*

Definition 4.2 *A* crosspath *w.r.t. a line wheel (H, x) is a chordless path $P = y_1, \ldots, y_n$ in $G \setminus (H \cup x)$ such that x is not adjacent to any node of P and one of the following holds:*

(i) $n = 1$, (H, y_1) is a line wheel, and each of the two long sectors of (H, x) contains two adjacent neighbors of y_1.

(ii) $n > 1$, no intermediate node of P has a neighbor in H, y_1 (resp. y_n) has exactly two neighbors in H that are furthermore adjacent, the neighbors of y_1 in H are contained in one long sector of (H, x) and the neighbors of y_n in H are contained in the other long sector of (H, x).

Lemma 4.3 *If G contains an L-parachute, then G contains a line wheel with no crosspath.*

Proof: Suppose G contains an L-parachute $\Pi = LP(x_1x_2, x_4x_3, x, z)$. Let P be the xz-path of $\Pi \setminus \{x_1, x_2, x_3, x_4\}$, and let H be the hole induced by $\Pi \setminus (P \setminus z)$. Let S_1 (resp. S_2) be the long sector of (H, x) with endnodes x_1, x_4 (resp. x_2, x_3). Suppose that the line wheel (H, x) has a crosspath $Q = y_1, \ldots, y_n$. W.l.o.g. y_1 has neighbors in S_1 and y_n in S_2. Let Q' be the shortest path from y_1 to x in $(P \cup Q \cup S_2) \setminus \{x_2, x_3\}$. Then $S_1 \cup Q'$ induces a $3PC(\Delta, x)$. Therefore line wheel (H, x) has no crosspath. □

Lemma 4.4 *If G contains a line wheel with no crosspath, then G has a double star cutset.*

Proof: Let (H, x) be a line wheel with no crosspath. Let x_1, x_2, x_3, x_4 be the neighbors of x in H that appear in this order when H is traversed clockwise. W.l.o.g. x_1x_2 and x_3x_4 are edges. Let S_1 (resp. S_2) be the long sectors of H with endnodes x_1, x_4 (resp. x_2, x_3). Let x'_1 be the neighbor of x_1 in S_1. Let $S = (N(x) \cup N(x_1)) \setminus \{x'_1, x_2, x_3\}$. Suppose that S is not a double star cutset and let $P = y_1, \ldots, y_n$ be a direct connection from S_1 to S_2 in $G \setminus S$. Let u_1 (resp. u_4) be the neighbor of y_1 in S_1 that is closest to x_1 (resp. x_4). Let u_2 (resp. u_3) be the neighbor of y_n in S_2 that is closest to x_2 (resp. x_3).

If $u_2 = x_3$ then $(H \cup P \cup x) \setminus x_4$ contains a $3PC(x_1x_2x, x_3)$. So $u_2 \neq x_3$. Suppose a node of $P \setminus y_1$ is adjacent to x_4 and let y_i be such a node with highest index. Then

$(H \cup P_{y_i y_n} \cup x) \setminus x_3$ contains a $3PC(x_1x_2x, x_4)$. So no node of $P \setminus y_1$ is adjacent to x_4. If $u_1 = u_4$ then $(H \cup P \cup x) \setminus x_3$ contains a $3PC(x_1x_2x, u_1)$. So $u_1 \neq u_4$. If u_1u_4 is not an edge, then there is a $3PC(x_1x_2x, y_1)$. So u_1u_4 is an edge. If $u_2 = u_3$ then $H \cup P$ induces a $3PC(y_1u_1u_4, u_2)$. So $u_2 \neq u_3$. If u_2u_3 is not an edge then $(H \cup P \cup x) \setminus x_4$ contains a $3PC(x_1x_2x, y_n)$. So u_2u_3 is an edge. But then either P is a crosspath w.r.t. (H, x), or (H, y_1) is an odd wheel. □

Theorem 4.1 follows.

By the results of Sections 2-4, it suffices to prove Theorem 1.2 when G contains a beetle or a T-parachute.

5 Nodes Adjacent to a $3PC(\Delta, \Delta)$

Given node disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, a $3PC(a_1a_2a_3, b_1b_2b_3)$ is a graph induced by three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, having no common nodes and such that the only adjacencies between the nodes of distinct paths are the edges of the two triangles. A $3PC(a_1a_2a_3, b_1b_2b_3)$ is also referred to as a $3PC(\Delta, \Delta)$.

Throughout this section we assume that G is an even-signable graph. By Σ we denote a $3PC(a_1a_2a_3, b_1b_2b_3)$ with the three paths $P^1 = P_{a_1b_1}$, $P^2 = P_{a_2b_2}$ and $P^3 = P_{a_3b_3}$. For $i = 1, 2, 3$, we denote by a_i' the neighbor of a_i in P^i and by b_i' the neighbor of b_i in P^i. For distinct $i, j \in \{1, 2, 3\}$, we denote by H_{ij} the hole induced by $P^i \cup P^j$.

Lemma 5.1 *Let G be an even-signable graph and let Σ be a $3PC(\Delta, \Delta)$. If node u is adjacent to Σ, then it is one of the following types.*

Type tj for $j = 1, 2, 3$: *Node u has exactly j neighbors in Σ and they are all contained in $\{a_1, a_2, a_3\}$ or all in $\{b_1, b_2, b_3\}$.*

Type p1: *Node u has exactly one neighbor in Σ and u is not of Type t1.*

Type p2: *Node u has exactly two neighbors in Σ, that are furthermore adjacent and are contained in P^i, for some $i \in \{1, 2, 3\}$.*

Type p3: *Node u has at least two nonadjacent neighbors in Σ, and all the neighbors of u in Σ are contained in P^i, for some $i \in \{1, 2, 3\}$.*

Type p4: *Node u has exactly four neighbors in Σ, u_1, u_2, u_3 and u_4, where u_1u_2 is an edge that belongs to some P^i, $i \in \{1, 2, 3\}$, and u_3u_4 is an edge that belongs to some P^j, $j \in \{1, 2, 3\} \setminus \{i\}$. Furthermore, u is not adjacent to both a_i and a_j, and it is not adjacent to both b_i and b_j.*

Type t2p: *For distinct indices $i, j, k \in \{1, 2, 3\}$ and for $z \in \{a, b\}$, u is adjacent to z_i and z_j, it has at least one neighbor in $P^k \setminus \{z_k\}$, and is not adjacent to any node in $(P^i \cup P^j \cup \{z_k\}) \setminus \{z_i, z_j\}$.*

Type t3p: *Node u has at least four neighbors in Σ. For some $z \in \{a, b\}$, u is adjacent to z_1, z_2 and z_3, and all the other neighbors of u in Σ belong to P^i for some $i \in \{1, 2, 3\}$.*

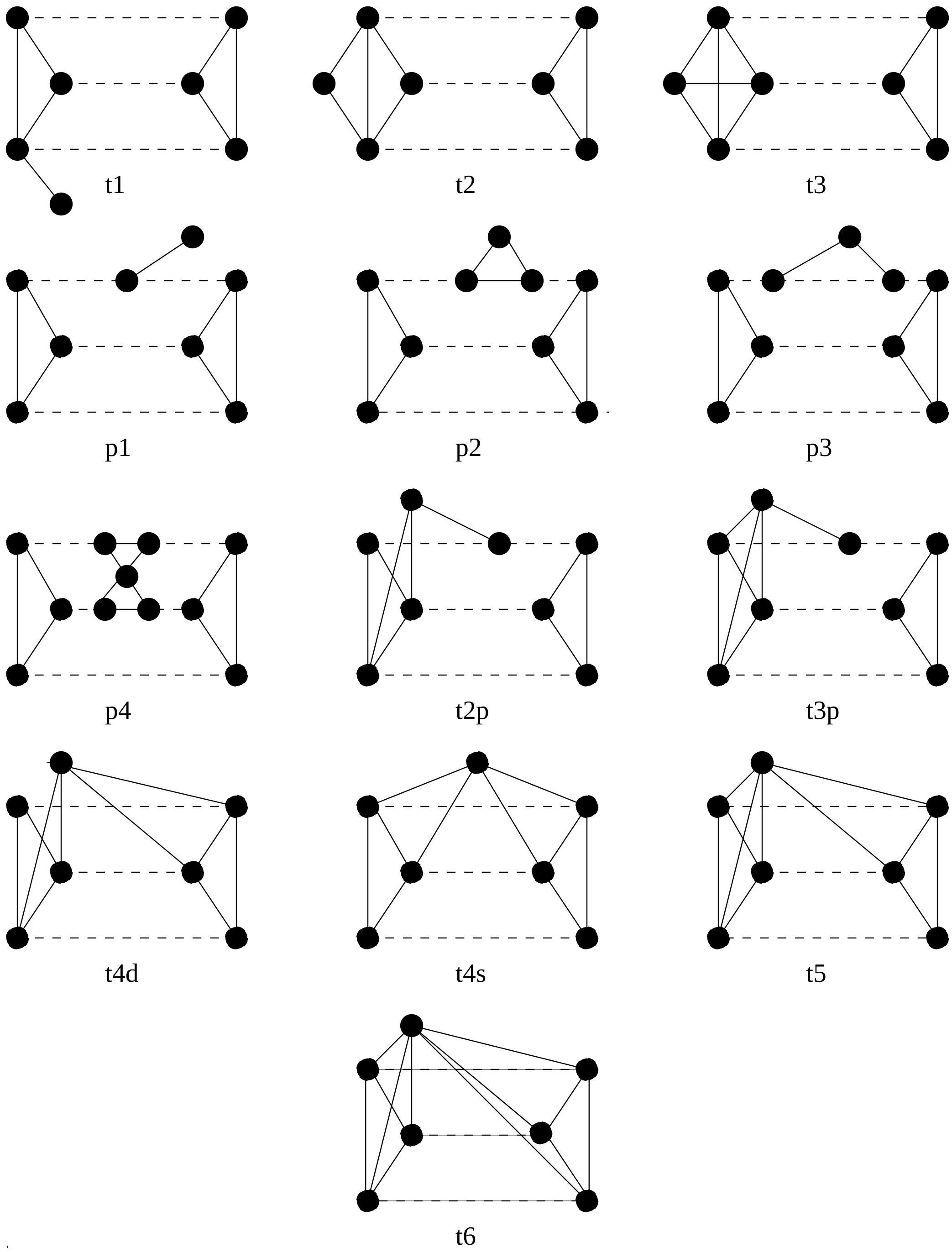

Figure 4: The different types of nodes adjacent to a $3PC(\Delta, \Delta)$

Type t4d: *For some distinct* $i, j \in \{1,2,3\}$, $N(u)\cap\{a_1,a_2,a_3,b_1,b_2,b_3\} = \{a_1,a_2,a_3,b_1,b_2,b_3\}\setminus \{a_i,b_j\}$.

Type t4s: *For some* $i \in \{1,2,3\}$, $N(u) \cap \{a_1,a_2,a_3,b_1,b_2,b_3\} = \{a_1,a_2,a_3,b_1,b_2,b_3\} \setminus \{a_i,b_i\}$. *Furthermore, if G does not contain a Mickey Mouse, then for* $j \in \{1,2,3\}\setminus\{i\}$, a_jb_j *is not an edge.*

Type t4: *Node u is of Type t4d or t4s w.r.t.* Σ.

Type tj **for** $j = 5, 6$**:** *Node u is adjacent to j nodes in* $\{a_1,a_2,a_3,b_1,b_2,b_3\}$ *and possibly other nodes of* Σ.

Proof: First we show that if for some $i \in \{1,2,3\}$, $N(u)\cap\{a_1,a_2,a_3,b_1,b_2,b_3\} = \{a_1,a_2,a_3,b_1,b_2,b_3\}\setminus \{a_i,b_i\}$, then for $j \in \{1,2,3\} \setminus \{i\}$, a_jb_j is not an edge. Suppose not. Assume w.l.o.g. that $i = 3$ and a_1b_1 is an edge. Node u must have a neighbor in P^3, since otherwise $P^3 \cup \{a_1,a_2,b_1,u\}$ induces an odd wheel with center a_1. Let u_3 (resp. v_3) be the neighbor of u in P^3 that is closest to a_3 (resp. b_3). If $u_3 = v_3$ then (H_{13}, u) is an odd wheel. If u_3v_3 is not an edge then $P^3_{a_3u_3} \cup P^3_{v_3b_3} \cup \{a_1,b_1,u\}$ induces a Mickey Mouse. So u_3v_3 is an edge, and hence $P^3 \cup \{a_2,b_1,u\}$ induces a Mickey Mouse.

Assume that u is not of Type t1, p1, p2 or p3. Then, w.l.o.g. u has neighbors in both P^1 and P^2.

Case 1: u does not have a neighbor in P^3.

First assume that u has a unique neighbor in P^1 or P^2, say P^1. Let u_1 be the neighbor of u in P^1, and w.l.o.g. assume that $u_1 \neq a_1$. Let u_2 be the neighbor of u in P^2 that is closest to a_2. If $u_2 \neq b_2$, then the node set $P^1 \cup P^2_{a_2u_2} \cup P^3 \cup \{u\}$ induces a $3PC(a_1a_2a_3, u_1)$. If $u_2 = b_2$, then either u is of Type t2 or the node set $P^1_{a_1u_1} \cup P^2 \cup P^3 \cup \{u\}$ induces a $3PC(a_1a_2a_3, u_2)$.

Now assume that u has at least two neighbors in both P^1 and P^2. Let u_1 (resp. v_1) be the neighbor of u in P^1 that is closest to a_1 (resp. b_1). Let u_2 (resp. v_2) be the neighbor of u in P^2 that is closest to a_2 (resp. b_2). First suppose that both u_1v_1 and u_2v_2 are edges. If u is adjacent to both a_1 and a_2, then $P^2 \cup P^3 \cup \{u, a_1\}$ induces an odd wheel with center a_2. So u is not adjacent to both a_1 and a_2, and similarly u is not adjacent to both b_1 and b_2. Hence u is of Type p4. Now assume w.l.o.g. that u_1v_1 is not an edge. If u is not adjacent to all four of the nodes a_1, a_2, b_1 and b_2, then either $P^1_{a_1u_1} \cup P^1_{v_1b_1} \cup P^2_{a_2u_2} \cup P^3 \cup \{u\}$ or $P^1_{a_1u_1} \cup P^1_{v_1b_1} \cup P^2_{v_2b_2} \cup P^3 \cup \{u\}$ induces a $3PC(\Delta, u)$. So u is adjacent to a_1, a_2, b_1 and b_2, and hence it is of Type t4s.

Case 2: u has a neighbor in P^3.

For $i \in \{1,2,3\}$, let u_i (resp. v_i) be the neighbor of u in P^i that is closest to a_i (resp. b_i). If u is adjacent to at most one node in $\{a_1,a_2,a_3\}$ and at most one node in $\{b_1,b_2,b_3\}$, then the node set $P^1_{v_1b_1} \cup P^2_{v_2b_2} \cup P^3_{v_3b_3} \cup \{u\}$ induces a $3PC(b_1b_2b_3, u)$. So assume w.l.o.g. that u is adjacent to b_1 and b_2. If u does not have a neighbor in $(P^1 \cup P^2) \setminus \{b_1,b_2\}$, then u is of Type t2p, t3 or t3p. So assume w.l.o.g. that $u_1 \neq b_1$. Suppose u is not of Type t4, t5 or t6. Then u is adjacent to at most one node of $\{a_1,a_2,a_3\}$. If $u_2 = b_2$ and $u_3 = b_3$, then u is of Type t3p. Otherwise, $P^1_{a_1u_1} \cup P^2_{a_2u_2} \cup P^3_{a_3u_3} \cup \{u\}$ induces a $3PC(a_1a_2a_3, u)$. □

Type t6 nodes w.r.t. Σ are further classified as follows.

Type t6a: A node u that is of Type t6 w.r.t. Σ, such that u has no neighbors in the interior of any of the paths of Σ, and either $\Sigma = \bar{C}_6$ or none of the paths of Σ is an edge.

Type t6b: A node u that is of Type t6 w.r.t. Σ, but is not of Type t6a.

Lemma 5.2 *If u is of Type t6b w.r.t. Σ, then $\Sigma \neq \bar{C}_6$ and u has a neighbor in the interior of one of the paths of Σ.*

Proof: Assume u is of Type t6b w.r.t. Σ, but u has no neighbor in the interior of any of the paths of Σ. Then w.l.o.g. a_1b_1 is an edge and a_2b_2 is not. Then $P^1 \cup P^2 \cup u$ induces an odd wheel with center u. $\square$

If node u is of Type p3, t2p or t3p w.r.t. Σ, then a subset of the node set $\Sigma \cup \{u\}$ induces a $\Sigma' = 3PC(\Delta, \Delta)$ that contains u. We say that Σ' is obtained by *substituting u into* Σ. If u is of Type t2p or t3p w.r.t. Σ, and for some $z \in \{a, b\}$ and $i \in \{1, 2, 3\}$, Σ' does not contain z_i, then we say that u is a *sibling* of z_i.

6 Beetles and T-Parachutes

Theorem 6.1 *Let G be an even-signable graph that does not contain a double star cutset. If G contains a beetle, then G contains a $3PC(\Delta, \Delta)$ with a Type t2 node. If G contains a T-parachute, then G contains a $3PC(\Delta, \Delta)$ with a Type t2, t2p, t4 or t5 node.*

Proof: By Theorem 3.2, every proper wheel of G is a beetle.

Suppose G contains a beetle or a T-parachute. For a beetle $\Pi = (H, v)$, we denote the neighbors of v on H by a, t, b and z, where at and bt are edges. For a T-parachute $\Pi = TP(t, v, a, b, z)$, we denote by (H, v) the twin wheel of Π. In both cases, we denote by P the path of Π from v to z that uses no edge of H, and by H_{za} and H_{zb} the subpaths of H from z to a and from z to b that do not contain t. Let C be the hole of Π containing b, v, z. Let $S = (N(v) \cup N(b)) \setminus \{t, m, b'\}$, where m is the neighbor of v in P and b' is the neighbor of b in H_{zb}. Let $Q = x_1, \ldots, x_n$ be a direct connection from t to $\Pi \setminus \{a, b, v, t\}$ in $G \setminus S$.

If x_n has no neighbor in C then, since x_n must have a neighbor in $H_{za} \setminus a$, $(\Pi \setminus a) \cup Q$ contains a $3PC(bvt, z)$. So x_n has a neighbor in C. If x_n has exactly one neighbor p in C, then $C \cup Q \cup t$ contains a $3PC(bvt, p)$. If x_n has two nonadjacent neighbors in C, then $C \cup Q \cup t$ contains a $3PC(bvt, x_n)$. So x_n has exactly two neighbors in C and they are adjacent. Then $C \cup Q \cup t$ induces a $\Sigma = 3PC(\Delta, \Delta)$. By Lemma 5.1, a is of Type t2, t2p, t4 or t5 w.r.t. Σ. When (H, v) is a beetle, both neighbors of x_n are in H_{zb}. It follows from Lemma 5.1 that a is of Type t2. $\square$

7 Crosspaths and Attachments

Throughout this section we assume that G is an even-signable graph that contains a $\Sigma = 3PC(\Delta, \Delta)$ and does not contain a Mickey Mouse.

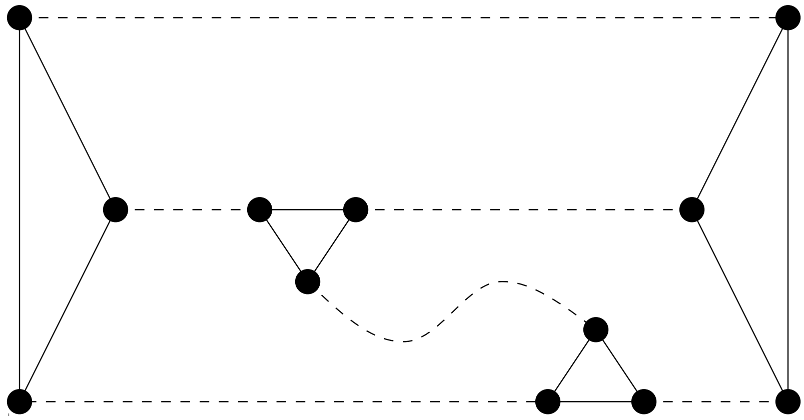

Figure 5: Crosspath

7.1 Crosspaths

Definition 7.1 *A* crosspath *w.r.t.* $\Sigma = 3PC(\Delta, \Delta)$ *is a chordless path* $P = x_1, \ldots, x_n$ *in* $G \setminus \Sigma$ *that satisfies one of the following:*

- $n = 1$ *and* x_1 *is of Type p4 w.r.t.* Σ*, or*
- $n > 1$*,* x_1 *and* x_n *are of Type p2 w.r.t.* Σ*, with neighbors in different paths of* Σ*, and no intermediate node of* P *has a neighbor in* Σ*.*

If x_1 *or* x_n *has neighbors in a path* P^i *of* Σ*, we say that* P *is a* P^i-crosspath *w.r.t.* Σ*.*

Lemma 7.2 *Let* $\Sigma = 3PC(\Delta, \Delta)$ *and let* $P = x_1, \ldots, x_n$*,* $n > 1$*, be a chordless path in* $G \setminus \Sigma$*. If* $\emptyset \neq N(x_1) \cap \Sigma \subseteq P^i$*,* $\emptyset \neq N(x_n) \cap \Sigma \subseteq P^j$*,* $i \neq j$*, and no intermediate node of* P *has a neighbor in* Σ*, then* P *is a crosspath w.r.t.* Σ*.*

Proof: Suppose that there exist Σ, P satisfying the assumptions of the lemma such that P is not a crosspath w.r.t. Σ. Choose such Σ, P with shortest possible $P = x_1, \ldots, x_n$, $n > 1$. Assume w.l.o.g. that $i = 1$ and $j = 2$. Since $N(x_1) \cap \Sigma \subseteq P^1$ and $N(x_n) \cap \Sigma \subseteq P^2$, x_1 and x_n are of Type t1, p1, p2 or p3 w.r.t. Σ. Since P is not a crosspath and P^1, P^2 are symmetrical, we may assume w.l.o.g. that x_1 is of Type t1, p1 or p3 w.r.t. Σ. If x_1 is of Type p3 w.r.t. Σ, then consider the $3PC(\Delta, \Delta) = \Sigma'$ obtained by substituting x_1 into Σ. $P \setminus x_1$ is not a crosspath w.r.t. Σ'. Therefore, by the choice of Σ, P, the pair $\Sigma', P \setminus x_1$ does not satisfy the assumptions of the lemma. Thus $n - 1 = 1$. It follows from Lemma 5.1 that x_n is of Type p4 w.r.t. Σ', which is impossible as x_n has no neighbor on P^1. So we may assume that x_1 is of Type t1 or p1 w.r.t. Σ and similarly that x_n is of Type t1, p1 or p2 w.r.t. Σ. If x_n is of Type p2 w.r.t. Σ, then the node set $P^1 \cup P^2 \cup P$ induces a $3PC(\Delta, .)$. Hence x_n is also of Type t1 or p1 w.r.t. Σ. Let u_1 (resp. u_2) be the unique neighbor of x_1 (resp. x_n) in Σ. W.l.o.g. $u_1 \neq a_1$. If $u_1 = b_1$ and $u_2 = b_2$, then the node set $P \cup P^2 \cup P^3 \cup \{b_1\}$ induces a Mickey Mouse. Otherwise $u_2 \neq b_2$ w.l.o.g. and the node set $P^1 \cup P^2_{a_2 u_2} \cup P^3 \cup P$ induces a $3PC(a_1 a_2 a_3, u_1)$. $\square$

7.2 Attachments

Lemma 7.3 *Let x be a Type t1 node w.r.t. $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$, adjacent to say a_1. Suppose that $S = (N(a_1) \cup (N(a_2) \cap N(a_3))) \setminus x$ is not a cutset and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. Then no node of $P \setminus x_n$ is adjacent to a node of $\Sigma \setminus a'_1$ and one of the following holds:*

(i) x_n is of Type t1 or p1 w.r.t. Σ and its unique neighbor in Σ is in P^1,

(ii) x_n is of Type p3 w.r.t. Σ, with neighbors in P^1,

(iii) x_n is of Type t2 w.r.t. Σ, adjacent to b_2 and b_3,

(iv) x_n is of Type t2p w.r.t. Σ, adjacent to b_2 and b_3,

(v) x_n is of Type t3p w.r.t. Σ, adjacent to b_1, b_2, b_3 and with a neighbor in $P^1 \setminus b_1$,

(vi) x_n is of Type p2 w.r.t. Σ, adjacent to a'_1, and a'_1 has a neighbor in $P \setminus x_n$, or

(vii) x_n is of Type t3 w.r.t. Σ, adjacent to b_1, b_2 and b_3, $a'_1 = b_1$ and a'_1 has a neighbor in $P \setminus x_n$.

Proof: First we show that no node of $P \setminus x_n$ is adjacent to a node of $\Sigma \setminus a'_1$. Suppose not and let x_i be the node of P with lowest index adjacent to a node of $\Sigma \setminus a'_1$. By the definition of S, x_i is adjacent to exactly one of a_2 or a_3, and no other node of Σ. W.l.o.g. assume x_i is adjacent to a_2. Then $P_{x_1x_i} \cup P^2 \cup P^3 \cup \{x, a_1\}$ induces a Mickey Mouse. Hence, no node of $P \setminus x_n$ is adjacent to a node of $\Sigma \setminus a'_1$.

Node x_n cannot be of Type t4, t5 and t6 w.r.t. Σ, since all these types of nodes are in S. Suppose that x_n is of Type t1 or p1 with the unique neighbor u in Σ. If u is not in P^1, then the node set $P^2 \cup P^3 \cup P \cup x$ induces a $3PC(a_1a_2a_3, u)$. Similarly, if x_n is of Type p3, then it must satisfy (ii), else there is a $3PC(a_1a_2a_3, x_n)$. Suppose x_n is of Type p2, with neighbors u and v in Σ, and w.l.o.g. assume that u and v are not in P^3. If a'_1 has no neighbor in $P \setminus x_n$, then $P^1 \cup P^2 \cup P \cup x$ induces a $3PC(x_nuv, a_1)$. So a'_1 has a neighbor in $P \setminus x_n$. Let x_i be the node of $P \setminus x_n$ with highest index adjacent to a'_1. If x_n is not adjacent to a'_1, then $P^1 \cup P^2 \cup P_{x_ix_n}$ induces a $3PC(x_nuv, a'_1)$. Hence (vi) holds. If x_n is of Type t2 and it does not satisfy (iii), then w.l.o.g. we may assume that it is adjacent to b_1 and b_3, and hence the node set $P \cup P^2 \cup P^3 \cup x$ induces a $3PC(a_1a_2a_3, b_3)$. Suppose x_n is of Type t2p or t3p and does not satisfy (iv) or (v). Then w.l.o.g. x_n is adjacent to b_1, b_3 and it has a neighbor in $P^2 \setminus b_2$, and hence $(P \cup P^2 \cup P^3 \cup x) \setminus \{b_2\}$ contains a $3PC(a_1a_2a_3, x_n)$. Suppose x_n is of Type t3. If a'_1 does not have a neighbor in $P \setminus x_n$, then $P \cup P^1 \cup P^3 \cup x$ induces a $3PC(x_nb_1b_3, a_1)$. So a'_1 has a neighbor in $P \setminus x_n$. Suppose that $a'_1 \neq b_1$ and let x_i be the node of $P \setminus x_n$ with highest index adjacent to a'_1. Then $P_{x_ix_n} \cup P^1 \cup P^3$ induces a $3PC(x_nb_1b_3, a'_1)$. Hence (vii) holds. Finally suppose that x_n is of Type p4 with neighbors in P^i and P^j, for some $i, j \in \{1, 2, 3\}$. Let u_i (resp. v_i) be the neighbor of x_n in P^i that is closest to a_i (resp. b_i). Similarly define u_j and v_j. If $i = 2$ and $j = 3$, then the node set $P^2_{a_2u_2} \cup P^3_{a_3u_3} \cup P \cup x$ induces a $3PC(a_1a_2a_3, x_n)$. Else we may assume w.l.o.g. that $i = 1$ and $j = 2$. Then the node set $P^1_{v_1b_1} \cup P^2_{a_2u_2} \cup P^3 \cup P \cup x$ induces a $3PC(a_1a_2a_3, x_n)$. □

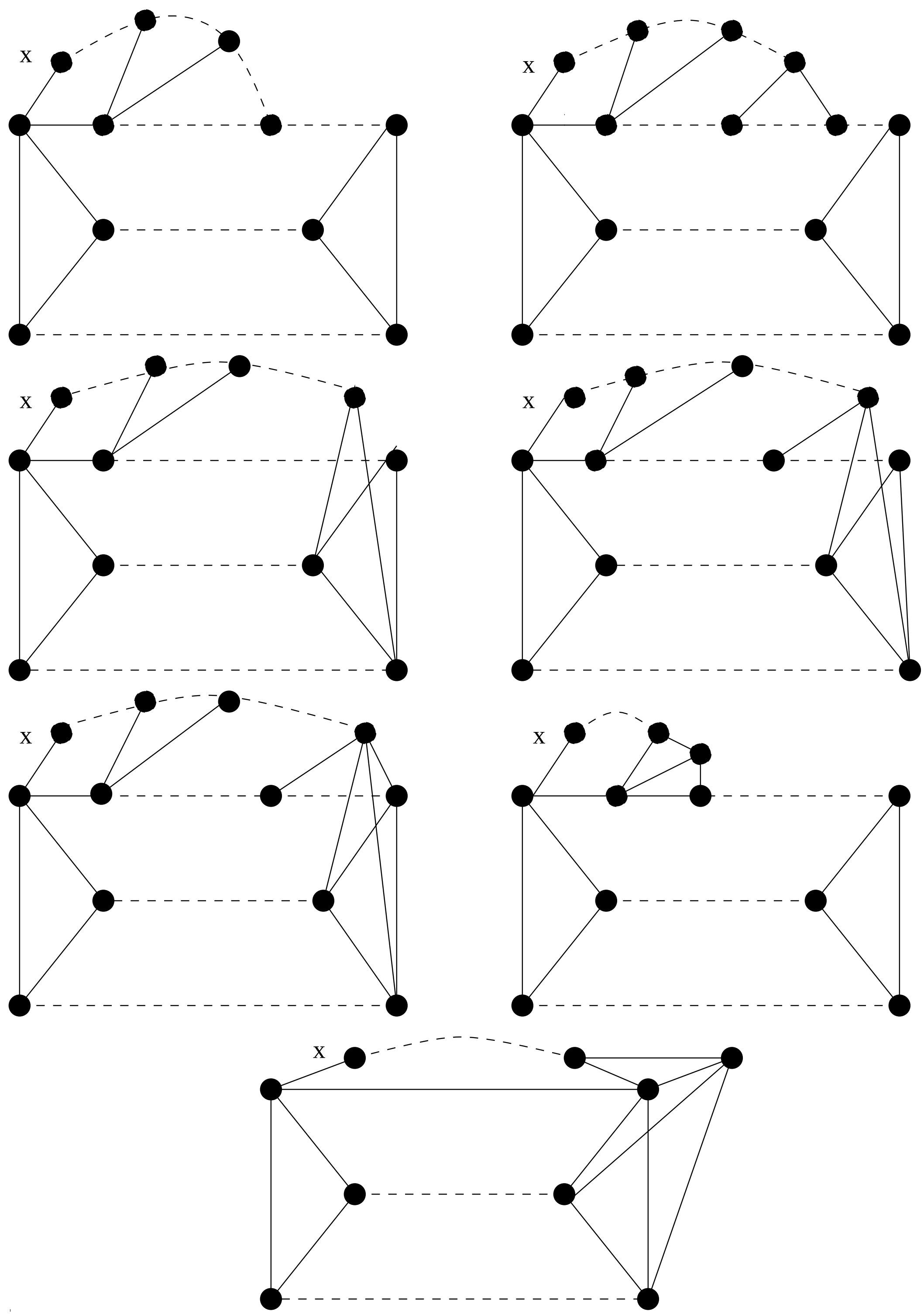

Figure 6: Attachments of a node of Type t1

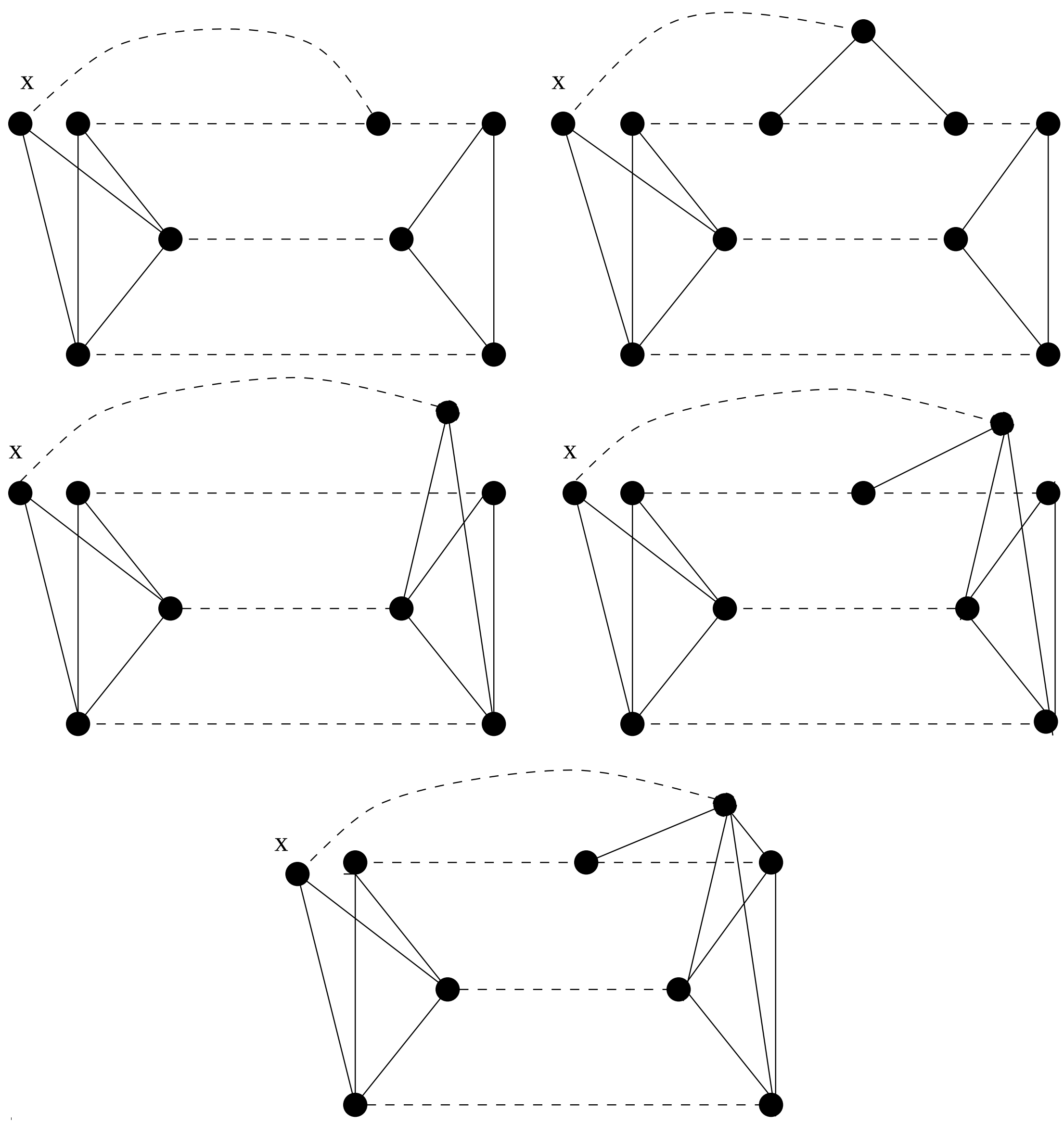

Figure 7: Attachments of a node of Type t2

Lemma 7.4 *Let x be a Type t2 node w.r.t. $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$, adjacent to say a_1 and a_3. Suppose that $S = (N(a_2) \cup (N(a_1) \cap N(a_3))) \setminus \{x, a_2'\}$ is not a cutset and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. Then no node of $P \setminus x_n$ is adjacent to a node of Σ and one of the following holds:*

(i) x_n is of Type t1 or p1 w.r.t. Σ and its unique neighbor in Σ is in P^2,

(ii) x_n is of Type p3 w.r.t. Σ, with neighbors in P^2,

(iii) x_n is of Type t2 w.r.t. Σ, adjacent to b_1 and b_3,

(iv) x_n is of Type t2p w.r.t. Σ, adjacent to b_1 and b_3, or

(v) x_n is of Type t3p w.r.t. Σ, adjacent to b_1, b_2, b_3, and with a neighbor in $P^2 \setminus b_2$.

Proof: First we show that no node of $P \setminus x_n$ has a neighbor in Σ. Suppose not. By the definition of S, the only nodes of Σ that can have a neighbor in $P \setminus x_n$ are a_1 and a_3, and no node of P is adjacent to both a_1 and a_3. Suppose that both a_1 and a_3 have a neighbor in $P \setminus x_n$. Then $P \setminus x_n$ contains a subpath P', such that one endnode of P' is adjacent to a_1, the other to a_3, and these are the only adjacencies between P' and Σ. Then $P' \cup P^1 \cup P^2 \cup a_3$ induces a Mickey Mouse. Now assume w.l.o.g. that only a_1 has a neighbor in $P \setminus x_n$. Let Q be the shortest path from x_n to a_3 in $\Sigma \cup x_n \setminus \{a_1, a_2\}$. Then $Q \cup P \cup x$ induces a hole H and (H, a_1) is a wheel. Let Q' be the shortest path from x_n to a_2 in $\Sigma \cup x_n \setminus \{a_1, a_3\}$. If $Q' \cup P \cup \{x, a_3\}$ induces a hole H', then (H', a_1) is a wheel with one more short sector than (H, a_1) and either (H, a_1) or (H', a_1) is an odd wheel. Hence H' cannot be a hole. That is, either x_n is adjacent to a_3 or the unique neighbor of x_n in Σ is a_3'. If x_n is a Type t1 or p1 node adjacent to a_3', then $P^2 \cup P^3 \cup P$ contains a $3PC(a_1a_2a_3, a_3')$. If x_n is of Type p2 or p3 adjacent to a_3, there is a contradiction to Lemma 7.2. If x_n is of Type t2p or t3p adjacent to a_3, there is a $3PC(b_1b_2x_n, a_1)$. If x_n is of Type p4 adjacent to a_3, let u and v be its two neighbors in $P^1 \cup P^2$. Then $P^1 \cup P^2 \cup P$ contains a $3PC(uvx_n, a_1)$. Therefore, no node of $P \setminus x_n$ is adjacent to a node of Σ.

Node x_n cannot be of Type t4, t5 and t6 w.r.t. Σ, since all these types of nodes are in S. Suppose x_n is of Type t1 or p1 with the unique neighbor u in Σ that is in P^1 or P^3, say in P^1. Then the node set $P^1 \cup P^3 \cup P \cup x$ induces a $3PC(xa_1a_3, u)$. Hence if x_n is of Type t1 or p1, then it must satisfy (i). Similarly, if x_n is of Type p3, then it must satisfy (ii), else there is a $3PC(xa_1a_3, x_n)$. Suppose that x_n is of Type p2, with neighbors u and v in Σ. W.l.o.g. assume that u and v are not in P^3. If x_n is not adjacent to a_1, then the node set $P^1 \cup P^2 \cup P \cup x$ induces a $3PC(x_nuv, a_1)$. So x_n is adjacent to a_1. If $n = 1$ then $P^1 \cup P^3 \cup \{x, x_1\}$ induces an odd wheel with center a_1, and otherwise $P^1 \cup P^2 \cup P \cup \{x, a_3\}$ induces an odd wheel with center a_1. If x_n is of Type t2, adjacent to b_2 and say b_1, then the node set $P^1 \cup P^2 \cup P \cup x$ induces a $3PC(x_nb_1b_2, a_1)$. So if x_n is of Type t2, then it must satisfy (iii). Similarly, if x_n is of Type t3, then there is a $3PC(x_nb_1b_2, a_1)$. If x_n is of Type t2p or t3p, and it does not satisfy (iv) or (v), then w.l.o.g. we may assume that x_n has a neighbor in $P^3 \setminus b_3$, and hence the node set $P^1 \cup P^2 \cup P \cup x$ induces a $3PC(x_nb_1b_2, a_1)$. Finally assume that x_n is of Type p4 with neighbors in P^i and P^j, for some $i, j \in \{1, 2, 3\}$. Let u_i (resp. v_i) be the neighbor of x_n in P^i that is closest to a_i (resp. b_i). Similarly define u_j and v_j. If $i = 1$ and $j = 3$, then w.l.o.g. x_n is not adjacent to a_3, and hence $P^1_{v_1b_1} \cup P^3_{v_3b_3} \cup P^2 \cup P \cup \{x, a_3\}$

induces a $3PC(b_1b_2b_3, x_n)$. Otherwise, w.l.o.g. we may assume that $i = 1$ and $j = 2$. Then the node set $P^1_{v_1b_1} \cup P^2_{v_2b_2} \cup P^3 \cup P \cup x$ induces a $3PC(b_1b_2b_3, x_n)$. □

Lemma 7.5 *Let x be a Type t3 node w.r.t. $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$, adjacent to say a_1, a_2 and a_3. Assume G has no extended star cutset, let $S = (N(a_2) \cup (N(a_1) \cap N(a_3))) \setminus \{x, a_2'\}$ and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. Then one of the following holds:*

(i) No node of Σ has a neighbor in $P \setminus x_n$ and x_n is of Type p2 or t3 w.r.t. Σ.

(ii) $n = 1$, x_n is a sibling of b_1 or b_3 w.r.t. Σ, adjacent to a_1 or a_3.

(iii) Exactly one of a_1, a_3 has a neighbor in $P \setminus x_n$, no other node of Σ has a neighbor in $P \setminus x_n$, and x_n is as described in Lemma 7.4.

Proof: First note that by the definition of S, no node of P can be of Type t4, t5 or t6 w.r.t. Σ. Also, the only nodes of Σ that can have a neighbor in $P \setminus x_n$ are a_1 and a_3, and there is no node of P adjacent to both a_1 and a_3. Suppose that both a_1 and a_3 have a neighbor in $P \setminus x_n$. Then $P \setminus x_n$ contains a subpath P' such that one endnode of P' is adjacent to a_1, the other to a_3 and these are the only adjacencies between $P \setminus x_n$ and Σ. Then $P' \cup P^1 \cup P^2 \cup a_3$ induces a Mickey Mouse. Hence, at most one of a_1, a_3 has a neighbor in $P \setminus x_n$. If a_1 or a_3 has a neighbor in $P \setminus x_n$ then by Lemma 7.4 (iii) holds.

We now assume that no node of Σ has a neighbor in $P \setminus x_n$. Suppose x_n is of Type t1 or p1 and let u be its unique neighbor in Σ. W.l.o.g. assume that u is in P^1. Then the node set $P^1 \cup P^2 \cup P \cup x$ induces a $3PC(xa_1a_2, u)$. Similarly, if x_n is of Type p3 there is a $3PC(\Delta, x_n)$. If x_n is of Type t2, with neighbors say b_1 and b_3, then the node set $P^1 \cup P^2 \cup P \cup \{x\}$ induces a $3PC(xa_1a_2, b_1)$. If x_n is a sibling of b_2, there is a $3PC(xa_1a_2, x_n)$. Suppose that x_n is a sibling of b_1. Let u be the neighbor of x_n in P^1 that is closest to a_1. If $u \neq a_1$ or $n > 1$, then $P^1_{a_1u} \cup P^2 \cup P \cup x$ induces a $3PC(xa_1a_2, x_n)$. So if x_n is of Type t2p or t3p w.r.t. Σ, then it must satisfy (ii). Finally assume that x_n is of Type p4 w.r.t. Σ. Then $P \cup \Sigma$ contains a $3PC(b_1b_2b_3, x_n)$. □

Definition 7.6 *For a node x and a path P described in Lemmas 7.3, 7.4 and 7.5, we say that the path P is an* attachment *of node x to Σ. Also, a subset of the node set $\Sigma \cup P \cup x$ induces a $3PC(\Delta, \Delta)$ that contains x. We say that this $3PC(\Delta, \Delta)$ is obtained by* substituting x and its attachment P into Σ.

Theorem 7.7 *If G is an even-signable graph that has no extended star cutset, then every node x of Type t1, t2 or t3 w.r.t. $\Sigma = 3PC(\Delta, \Delta)$ has an attachment P to Σ. Furthermore, every direct connection from x to $\Sigma \setminus S$ (for an appropriate extended star S) is an attachment.*

Proof: Follows from Theorem 3.3 and Lemmas 7.3, 7.4 and 7.5. □

8 Type t4, t5 and t6 Nodes

Theorem 8.1 *Let G be an even-signable graph that contains a $\Sigma = 3PC(\Delta, \Delta)$ and a node u such that one of the following holds:*

(i) u is of Type t4s w.r.t. Σ,

(ii) $\Sigma \neq \bar{C}_6$ and u is of Type t4d w.r.t. Σ, or

(iii) $\Sigma = \bar{C}_6$, u is of Type t4d w.r.t. Σ, say adjacent to a_1, a_2, b_1, b_3, and G does not contain two nodes v and w that are both of Type t4d w.r.t. Σ, uv and uw are not edges, v is adjacent to a_1, a_3, b_2, b_3 and w is adjacent to a_2, a_3, b_1, b_2.

Then G has a double star cutset.

Proof: Suppose G has no double star cutset. Then by Theorem 3.3, G contains no Mickey Mouse. Let $\mathcal{C}$ be the set of all ordered pairs Σ, u that satisfy (i), (ii) or (iii). Let $\Sigma, u \in \mathcal{C}$. If u is of Type t4d w.r.t. Σ, then we assume w.l.o.g. that u is adjacent to a_1, a_2, b_1, b_3. If u is of Type t4s w.r.t. Σ, then we assume w.l.o.g. that u is adjacent to a_1, a_2, b_1, b_2.

Claim 1: *If $\Sigma, u \in \mathcal{C}$ satisfy (ii), then G cannot contain nodes v and w that are of Type t4d w.r.t. Σ, such that uv and uw are not edges, v is adjacent to a_1, a_3, b_2, b_3 and w is adjacent to a_2, a_3, b_1, b_2.*

Proof of Claim 1: Suppose not. Then a_1b_1 must be an edge, since otherwise $\{a_1, a_2, a_3, b_1, u, w\}$ induces an odd wheel with center a_2. Also a_2b_2 must be an edge, since otherwise $\{a_2, b_1, b_2, b_3, u, w\}$ induces an odd wheel with center b_1. Since $\Sigma \neq \bar{C}_6$, a_3b_3 is not an edge. But then $\{a_1, a_2, a_3, b_3, u, v\}$ induces an odd wheel with center a_1. This completes the proof of Claim 1.

By Claim 1 and the hypothesis in Theorem 8.1(iii), we may assume w.l.o.g. that if $\Sigma, u \in \mathcal{C}$ and u is of Type t4d w.r.t. Σ, then there is no node v of Type t4d w.r.t. Σ such that uv is not an edge and v is adjacent to a_1, a_3, b_2, b_3.

For $\Sigma, u \in \mathcal{C}$ define the corresponding sets S as follows. If u is of Type t4d w.r.t. Σ, then let $S = (N(u) \cup N(a_2)) \setminus (\Sigma \setminus \{a_1, a_2, b_3\})$. If u is of Type t4s w.r.t. Σ, then let $S = (N(u) \cup N(a_2)) \setminus (\Sigma \setminus \{a_1, a_2, b_1, b_2\})$. Since S is not a double star cutset, there exists a direct connection $P = x_1, \ldots, x_n$ in $G \setminus S$ from $(P^1 \cup P^2) \setminus S$ to $P^3 \setminus S$. Let $\mathcal{C}'$ be a subset of $\mathcal{C}$ with the property that for all $\Sigma', u' \in \mathcal{C}'$ and all $\Sigma, u \in \mathcal{C}$, $|N(u') \cap \Sigma'| \leq |N(u) \cap \Sigma|$. Let Σ, u be chosen from $\mathcal{C}'$ so that the size of the corresponding P is minimized.

Claim 2: *No node of P is of Type t4, t5 or t6 w.r.t. Σ.*

Proof of Claim 2: By definition of S, no node of P is of Type t6 w.r.t. Σ. Suppose that some x_i is of Type t4 or t5 w.r.t. Σ. Since x_i cannot be adjacent to a_2, it must be adjacent to a_1 and a_3. If x_i is adjacent to b_1, then $\{a_1, a_2, a_3, b_1, u, x_i\}$ induces an odd wheel with center a_1. So x_i is not adjacent to b_1, and hence it is of Type t4d w.r.t. Σ, adjacent to b_2 and b_3. By the assumption following Claim 1, this cannot occur if u is of Type t4d w.r.t. Σ. Hence u is of Type t4s w.r.t. Σ, and so a_1b_1 is not an edge. But then $\{a_1, b_1, b_2, b_3, u, x_i\}$ induces an odd wheel with center b_2. This completes the proof of Claim 2.

Claim 3: *If x_i is of Type p4 w.r.t. Σ, then $i = 1$ and the neighbors of x_i in Σ are contained in $P^1 \cup P^2$.*

Proof of Claim 3: Suppose x_i is of Type p4 w.r.t. Σ. Then $i = 1$ since x_i has a neighbor in $(P^1 \cup P^2) \setminus S$.

Suppose that the neighbors of x_i in Σ are contained in $P^1 \cup P^3$. For $j = 1, 3$, let u_j (resp. v_j) be the neighbor of x_i is P^j that is closest to a_j (resp. b_j). First suppose that x_i is adjacent to a_3. Then x_i is not adjacent to a_1 and so $(\Sigma \cup x_i) \setminus P^3_{v_3 b_3}$ induces a $\Sigma' = 3PC(a_1a_2a_3, u_1v_1x_i)$. Note that $\Sigma' \neq \bar{C}_6$. Since u is adjacent to a_1, a_2, b_1 and it is not adjacent to a_3, x_i, it must be of Type t4s w.r.t. Σ'. Hence u is adjacent to u_1 and v_1. Node u cannot have neighbors in P^3, since otherwise Σ', u would contradict the choice of Σ, u. So u is of Type t4s w.r.t. Σ, and hence a_2b_2 is not an edge. But then $P^3 \cup \{a_2, b_2, u, u_1, x_i\}$ induces a $3PC(x_iu_3v_3, u)$. Therefore x_i is not adjacent to a_3.

Let $\Sigma' = 3PC(a_1a_2a_3, x_iv_3u_3)$ induced by $(\Sigma \cup x_i) \setminus P^1_{v_1b_1}$. Note that $\Sigma' \neq \bar{C}_6$. Since u is adjacent to a_1, a_2 and at least one of b_2, b_3 (i.e. it has a neighbor in the a_2v_3-path of Σ'), and it is not adjacent to a_3 and x_i, it must be of Type t4d w.r.t. Σ'. Since u is adjacent to b_1, it has fewer neighbors in Σ' than in Σ, contradicting our choice of Σ, u.

Now suppose that the neighbors of x_i in Σ are contained in $P^2 \cup P^3$. By symmetry, the above proof shows that u is of Type t4d w.r.t. Σ. For $j = 2, 3$ let u_j (resp. v_j) be the neighbor of x_i in P^j that is closest to a_j (resp. b_j). By the definition of S, x_i is not adjacent to a_2, and hence $(\Sigma \cup x_i) \setminus P^3_{v_3b_3}$ induces a $\Sigma' = 3PC(a_1a_2a_3, v_2u_2x_i)$. Note that $\Sigma' \neq \bar{C}_6$. Since u is adjacent to a_1, a_2, b_1 and it is not adjacent to a_3, x_i, it must be of Type t4s w.r.t. Σ'. Since u is adjacent to b_3, u has fewer neighbors in Σ' than in Σ, contradicting our choice of Σ, u. This completes the proof of Claim 3.

Claim 4: *If x_i is of Type t2p or t3p w.r.t. Σ, then u is of Type t4d w.r.t. Σ, $i = 1$ and x_i is a sibling of b_1.*

Proof of Claim 4: Suppose that x_i is of Type t2p or t3p w.r.t. Σ and let Σ' be obtained from Σ by substituting x_i for its sibling. By the definition of S, x_i cannot be a sibling of a_1 or a_3. Suppose that x_i is a sibling of a_2. Since u is adjacent to a_1, b_1 and exactly one node in $\{b_2, b_3\}$, and it is not adjacent to a_3, x_i, it violates Lemma 5.1 w.r.t. Σ'. Suppose x_i is a sibling of b_3. Since u is adjacent to a_1, a_2, b_1 and it is not adjacent to a_3 and x_i, it must be of Type t4s w.r.t. Σ'. So u is adjacent to b_2 and it is of Type t4s w.r.t. Σ. Since x_i has a neighbor in $P^3 \setminus b_3$, $i = n$. Since $b_1, b_2 \in S$, $n > 1$. But then Σ', u and $P' = P \setminus x_n$ contradict our choice of Σ, u and P. Suppose x_i is a sibling of b_2. Then u must be of Type t4d w.r.t. Σ', and hence w.r.t. Σ too. By the definition of S, x_i is not adjacent to a_2, and hence $\Sigma' \neq \bar{C}_6$. Also $i = 1$ and $n > 1$. But then Σ', u and $P' = P \setminus x_1$ contradict our choice of Σ, u and P. Finally suppose that x_i is a sibling of b_1. If u is of Type t4s w.r.t. Σ, then u violates Lemma 5.1 w.r.t. Σ'. So u is of Type t4d w.r.t. Σ and t2p w.r.t. Σ'. Since x_i is adjacent to b_2, $i = 1$. This completes the proof of Claim 4.

Claim 5: *No node of P is of Type p3 w.r.t. Σ.*

Proof of Claim 5: Suppose x_i is of Type p3 w.r.t. Σ. Let Σ' be obtained from Σ by substituting x_i into Σ. Note that $\Sigma \neq \bar{C}_6$. Then Σ', u and P', where $P' = x_1, \ldots, x_{i-1}$ or $P' = x_{i+1}, \ldots, x_n$, contradict our choice of Σ, u and P. This completes the proof of Claim 5.

Claim 6: *$n > 1$, x_1 is either a sibling of b_1 or it is of Type t1, p1, p2, t2, t3 or p4 w.r.t. Σ,* *and x_n is of Type t1, p1, p2, t2 or t3 w.r.t. Σ.*

Proof of Claim 6: Follows from Claims 2, 3, 4 and 5.

Claim 7: *If a_1 has a neighbor in the interior of P, then b_2 and b_3 do not.*

Proof of Claim 7: Suppose not. Let x_i and x_j be nodes of the interior of P so that x_i is adjacent to a_1, x_j is adjacent to b_2 or b_3, and no proper subpath of $P_{x_ix_j}$ has this property. By the definition of S, at most one of b_2, b_3 has a neighbor in the interior of P. Then $P^2 \cup P^3 \cup P_{x_ix_j} \cup a_1$ induces a $3PC(a_1a_2a_3, b_2)$ or a $3PC(a_1a_2a_3, b_3)$. This completes the proof of Claim 7.

By Claim 6, we now consider the following cases.

Case 1: x_n is of Type t1, p1 or p2 w.r.t. Σ.

First we show that a_1 does not have a neighbor in the interior of P. Suppose not and let x_i be the node of $P \setminus x_n$ with highest index adjacent to a_1. By Claim 7, b_2 and b_3 do not have a neighbor in the interior of P. If b_1 does not have a neighbor in $P_{x_ix_{n-1}}$, then $P_{x_ix_n}$ contradicts Lemma 7.2. So b_1 has a neighbor in $P_{x_ix_{n-1}}$. By the definition of S, u is of Type t4s w.r.t. Σ, and so a_1b_1 is not an edge. Let x_j be the node of $P_{x_ix_{n-1}}$ with highest index adjacent to b_1. Then $P_{x_jx_n}$ contradicts Lemma 7.2. Therefore a_1 does not have a neighbor in the interior of P.

Next we show that if b_1 or b_2 has a neighbor in the interior of P, then u is of Type t4s and x_n is of Type t1 w.r.t. Σ adjacent to b_3. Suppose that b_1 or b_2 has a neighbor in the interior of P. Then, by definition of S, u is of Type t4s. Suppose now that b_3 is not the unique neighbor of x_n in Σ. By definition of S, b_3 does not have a neighbor in the interior of P. Let x_i be the node of $P \setminus x_n$ with highest index adjacent to b_1 or b_2. If x_i is adjacent to exactly one of b_1, b_2, then $P_{x_ix_n}$ contradicts Lemma 7.2. Hence x_i is adjacent to both b_1 and b_2. Let $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_i)$ contained in $(\Sigma \setminus b_3) \cup P_{x_ix_n}$. Note that u is of Type t4s w.r.t. Σ'. But then Σ', u and $P_{x_1x_{i-1}}$ contradict our choice of Σ, u and P.

Case 1.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

First suppose that b_3 is the unique neighbor of x_n is Σ. Then u is of Type t4s w.r.t. Σ and so x_1 has a neighbor in $(P^1 \cup P^2) \setminus \{a_1, a_2, b_1, b_2\}$. We may assume w.l.o.g. that the neighbors of x_1 in Σ are contained in P^2. Then $(\Sigma \setminus b_2) \cup P$ contains either a $3PC(a_1a_2a_3, b_3)$ (if b_1 has no neighbors in the interior of P) or a $3PC(a_1a_2a_3, b_1)$ (otherwise). So b_3 is not the unique neighbor of x_n in Σ, and hence b_1 and b_2 do not have neighbors in the interior of P.

If b_3 has a neighbor in the interior of P, let x_i be the node of $P \setminus x_1$ with lowest index adjacent to b_3. Then $P_{x_1x_i}$ contradicts Lemma 7.2. So b_3 does not have a neighbor in the interior of P. By Lemma 7.2 applied to P, x_1 and x_n must both be of Type p2 w.r.t. Σ.

Suppose that the neighbors of x_1 in Σ are contained in P^2. Let u_2 (resp. v_2) be the neighbor of x_1 in P^2 that is closest to a_2 (resp. b_2). Let u_3 (resp. v_3) be the neighbor of x_n in P^3 that is closest to a_3 (resp. b_3). Let Σ' be the $3PC(u_2v_2x_1, u_3v_3x_n)$ induced by $P^2 \cup P^3 \cup P$. Suppose that $\Sigma' = \bar{C}_6$. Then a_2b_2 and a_3b_3 are edges, and hence u is of Type t4d w.r.t. Σ. Since u is adjacent to a_2 and b_3, and it is not adjacent to $P \cup a_3$, it violates Lemma 5.1 w.r.t. to Σ'. Hence $\Sigma' \neq \bar{C}_6$. Let $P'_{u_2u_3}$ be the u_2u_3-path of Σ', and similarly define $P'_{v_2v_3}$. Since u is adjacent to a_2, it has a neighbor in $P'_{u_2u_3} \setminus u_3$. Since u is adjacent to b_2 or b_3, it has a neighbor in $P'_{v_2v_3}$. Node u cannot be of Type t4 w.r.t. Σ', since otherwise Σ', u would contradict our choice of Σ, u. Node u cannot be of Type t2 w.r.t. Σ' since, otherwise, u is of Type t4s w.r.t. Σ and a_2b_2 is an edge, a contradiction. Also, since u has no neighbors in P, it cannot be of Type t3, t2p, t3p, t5 or t6 w.r.t. Σ'. Therefore u is of Type p4 w.r.t. Σ'. So the neighbors of u in $P'_{u_2u_3}$ are a_2 and a'_2. But then $P'_{u_2u_3} \cup P \cup \{u, a_1\}$ induces an odd wheel with center a_2.

An analogous argument holds when the neighbors of x_1 in Σ are contained in P^1.

Case 1.2: x_1 is of Type t2 or t3 w.r.t. Σ.

Then the neighbors of x_1 in Σ are contained in $\{b_1, b_2, b_3\}$. First suppose that x_1 is adjacent to b_1 and b_2. Then u is of Type t4d w.r.t. Σ, and so b_1 and b_2 do not have neighbors in the interior of P. Hence $(\Sigma \setminus b_3) \cup P$ contains a $3PC(a_1a_2a_3, b_1b_2x_1)$. Since u is adjacent to a_1, a_2, b_1, and it is not adjacent to a_3, b_2, x_1, it violates Lemma 5.1 w.r.t. Σ'. So we may assume that x_1 is adjacent to b_3, and is not adjacent to one of b_1 or b_2. Since $n > 1$, $b_3 \in S$, and so u is of Type t4d w.r.t. Σ, and hence b_1 and b_2 do not have neighbors in the interior of P. If x_1 is adjacent to b_2 then $(\Sigma \setminus b_3) \cup P$ contains a $3PC(a_1a_2a_3, b_2)$ and if x_1 is adjacent to b_1, then $(\Sigma \setminus b_3) \cup P$ contains a $3PC(a_1a_2a_3, b_1)$.

Case 1.3: x_1 is a sibling of b_1.

By Claim 4, u is of Type t4d w.r.t. Σ. So b_1 and b_2 do not have neighbors in the interior of P. Let Σ' be obtained from Σ by substituting x_1 for its sibling. Then $(\Sigma' \setminus b_3) \cup P$ contains a $3PC(a_1a_2a_3, x_1)$.

Case 1.4: x_1 is of Type p4 w.r.t. Σ.

Then $(\Sigma \cup P) \setminus \{b_1, b_2\}$ contains a $3PC(a_1a_2a_3, x_1)$.

Case 2: x_n is of Type t2 w.r.t. Σ, adjacent to a_1 and a_3.

First we show that b_1, b_2 and b_3 do not have neighbors in the interior of P. Suppose b_3 does and let x_i be the node of P with highest index adjacent to b_3. Here u must be of Type t4d w.r.t. Σ. By Claim 7, a_1 does not have a neighbor in the interior of P. If b_1 has no neighbor in $P_{x_ix_n}$, then $P^1 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(a_1a_3x_n, b_3)$. So b_1 has a neighbor in $P_{x_ix_n}$. Let x_j be the node of $P_{x_ix_n}$ with highest index adjacent to b_1. If $x_j \neq x_i$, then $P^1 \cup P^3 \cup P_{x_jx_n}$ induces a $3PC(a_1a_3x_n, b_1)$. So $x_j = x_i$. Let Σ' be the $3PC(a_1a_3x_n, b_1b_3x_i)$ induced by $P^1 \cup P^3 \cup P_{x_ix_n}$. Since u is adjacent to a_1, b_1 and b_3 but not a_3, x_n or x_i, node u violates Lemma 5.1 w.r.t. Σ'. Hence b_3 does not have a neighbor in the interior of P. Suppose b_1 has a neighbor in the interior of P and let x_i be the node of P with highest index adjacent to b_1. Here u must be of Type t4s w.r.t. Σ. If b_2 does not have a neighbor in $P_{x_ix_n}$, then $P^2 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(b_1b_2b_3, a_3)$, since a_2 has no neighbor in $P_{x_ix_n}$ by definition of S. So b_2 has a neighbor in $P_{x_ix_n}$, and by Claim 7, a_1 does not. But then $P^1 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(a_1a_3x_n, b_1)$. Therefore, b_1 does not have a neighbor in the interior of P. Finally suppose that b_2 has a neighbor in the interior of P and let x_i be the node of P with highest index adjacent to b_2. By Claim 7, a_1 does not have a neighbor in the interior of P, and hence $P^1 \cup P^3 \cup P_{x_ix_n}$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$. Since b_2 has a neighbor in the interior of P, $b_2 \in S$ and hence u is of Type t4s w.r.t. Σ. But then u is adjacent to a_1, b_1, b_2, and it is not adjacent to a_3, x_n, b_3, and hence it violates Lemma 5.1

w.r.t. Σ'.

Case 2.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

If the neighbors of x_1 in Σ are contained in P^1 then $(\Sigma \setminus a_1) \cup P$ contains a $3PC(b_1b_2b_3, a_3)$. Hence the neighbors of x_1 in Σ are contained in P^2. Suppose a_1 has a neighbor in the interior of P and let x_i be the node of P with lowest index adjacent to a_1. Then $P_{x_1x_i}$ contradicts Lemma 7.2. Hence a_1 does not have a neighbor in the interior of P. Let $\Sigma' = 3PC(a_1x_na_3, b_1b_2b_3)$ contained in $(\Sigma \setminus a_2) \cup P$. Since u is adjacent to a_1, b_1 and exactly one of b_2, b_3, and it is not adjacent to a_3 and x_n, it violates Lemma 5.1 w.r.t. Σ'.

Case 2.2: x_1 is of Type t2 or t3 w.r.t. Σ.

Then the neighbors of x_1 in Σ are contained in $\{b_1, b_2, b_3\}$. If x_1 is adjacent to b_1 and b_2, then $P^1 \cup P^2 \cup P$ contains a $3PC(b_1b_2x_1, a_1)$. Therefore x_1 is adjacent to b_3 and exactly one of b_1, b_2. If x_1 is adjacent to b_2, then $P^2 \cup P^3 \cup P$ induces a $3PC(b_2b_3x_1, a_3)$. Hence, the neighbors of x_1 in Σ are b_1 and b_3. Since $n > 1$, $b_3 \in S$ and so u is of Type t4d w.r.t. Σ. Let x_i be the neighbor of a_1 in P with lowest index. If $i \neq n$, then $P^1 \cup P^3 \cup P_{x_1x_i}$ induces a $3PC(b_1b_3x_1, a_1)$. Hence $i = n$. Then $P^1 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_1x_na_3, b_1x_1b_3)$. Since u is of Type t4d w.r.t. Σ, u is adjacent to a_1, b_1, b_3, and it is not adjacent to a_3, x_1, x_n, and hence it violates Lemma 5.1 w.r.t. Σ'.

Case 2.3: x_1 is a sibling of b_1.

Then $P^2 \cup P^3 \cup P$ induces a $3PC(x_1b_2b_3, a_3)$.

Case 2.4: x_1 is of Type p4 w.r.t. Σ.

Then $P^2 \cup P^3 \cup P$ induces a $3PC(x_1x_1'x_1'', a_3)$, where x_1' and x_1'' are the neighbors of x_1 in P^2.

Case 3: x_n is of Type t2 or t3 w.r.t. Σ, and Case 2 does not apply.

Then x_n is adjacent to b_3, and hence $b_3 \notin S$. So u is of Type t4s w.r.t. Σ, and b_3 has no neighbors in the interior of P. We now show that a_1 has no neighbors in the interior of P. Suppose it does and let x_i be the node of P with highest index adjacent to a_1. By Claim 7, b_2 has no neighbors in the interior of P. Node b_2 must be adjacent to x_n, else $P^2 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(a_1a_2a_3, b_3)$. But then $P^2 \cup P^3 \cup P_{x_ix_n}$ induces a $\Sigma' = 3PC(a_1a_2a_3, x_nb_2b_3)$. Since u is adjacent to a_1, a_2, b_2, and it is not adjacent to a_3, b_3, x_n, it violates Lemma 5.1 w.r.t. Σ'. Therefore a_1 has no neighbors in the interior of P.

Case 3.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

We may assume w.l.o.g. that the neighbors of x_1 in Σ are contained in P^1. Suppose b_2 has a neighbor in the interior of P, and let x_i be the node of P with lowest index adjacent to b_2. Then $(\Sigma \setminus b_1) \cup P_{x_1x_i}$ contains a $3PC(a_1a_2a_3, b_2)$. Hence b_2 has no neighbors in the interior of P. If b_2 is not adjacent to x_n, then $(\Sigma \setminus b_1) \cup P$ contains a $3PC(a_1a_2a_3, b_3)$. Therefore b_2 is adjacent to x_n and hence $(\Sigma \setminus b_1) \cup P$ contains a $\Sigma' = 3PC(a_1a_2a_3, x_nb_2b_3)$. Since u is adjacent to a_1, a_2, b_2, and it is not adjacent to a_3, b_3, x_n, it violates Lemma 5.1 w.r.t. Σ'.

Case 3.2: x_1 is of Type t2 or t3 w.r.t. Σ or it is a sibling of b_1.

Since u is of Type t4s w.r.t. Σ, $b_1, b_2 \in S$ and $b_3 \notin S$. By Claim 6, $n > 1$ and so this case cannot happen.

Case 3.3: x_1 is of Type p4 w.r.t. Σ.

Then $(\Sigma \cup P) \setminus \{b_1, b_2\}$ contains a $3PC(a_1a_2a_3, x_1)$. □

Theorem 8.2 *Let G be an even-signable graph that contains a $\Sigma = 3PC(\Delta, \Delta)$ and a node u such that one of the following holds:*

(i) $\Sigma \neq \bar{C}_6$ and u is of Type t5 or t6b w.r.t. Σ, or

(ii) $\Sigma = \bar{C}_6$, u is of Type t5 w.r.t. Σ and there is no node of Type t4d w.r.t. Σ.

Then G has a double star cutset.

Proof: Suppose G has no double star cutset. Then by Theorem 3.3, G has no Mickey Mouse.

Let $\mathcal{C}$ be the set of all ordered pairs Σ, u such that $\Sigma = 3PC(\Delta, \Delta) \neq \bar{C}_6$ and u is of Type t5 or t6b w.r.t. Σ, or $\Sigma = \bar{C}_6$, u is of Type t5 w.r.t. Σ and no node is of Type t4d w.r.t. Σ. If there exists $\Sigma, u \in \mathcal{C}$ such that u is of Type t5 w.r.t. Σ, then remove from $\mathcal{C}$ all Σ', u' such that u' is of Type t6 w.r.t. Σ'.

Let $\Sigma, u \in \mathcal{C}$. If u is of Type t5 w.r.t. Σ, then we assume w.l.o.g. that u is not adjacent to a_3 and that, if one of P^1, P^2 is an edge, then P^1 is an edge. If u is of Type t6b w.r.t. Σ, then we assume w.l.o.g. that u has a neighbor in the interior of P^3. For $\Sigma, u \in \mathcal{C}$ let the corresponding set $S = (N(u) \cup N(a_2)) \setminus (\Sigma \setminus \{a_2, a_3, b_1, b_2\})$. Since S is not a double star cutset, there exists a direct connection $P = x_1, \ldots, x_n$ in $G \setminus S$ from $P^1 \cup P^2$ to P^3. Choose $\Sigma, u \in \mathcal{C}$ and a corresponding P so that the size of P is minimized.

Claim 1: *No node of P is of Type t4, t5 or t6 w.r.t. Σ.*

Proof of Claim 1: By Theorem 8.1, no node can be of Type t4s w.r.t. Σ. By the definition of S, no node of P is of Type t6 w.r.t. Σ. Suppose that x_i is of Type t5 w.r.t. Σ. Then x_i is not adjacent to a_2. By our choice of Σ, u, node u is also of Type t5 w.r.t. Σ. But then $\{a_1, a_2, a_3, b_1, u, x_i\}$ induces an odd wheel with center a_1. Now suppose that x_i is of Type t4d w.r.t. Σ. Then by Theorem 8.1 $\Sigma = \bar{C}_6$, u is of Type t5 w.r.t. Σ, and hence our choice of Σ, u is contradicted. This completes the proof of Claim 1.

Claim 2: *If x_i is of Type p4 w.r.t. Σ, then $i = 1$ and the neighbors of x_i in Σ are contained in $P^1 \cup P^2$.*

Proof of Claim 2: Suppose x_i is of Type p4 w.r.t. Σ. If the neighbors of x_i in Σ are contained in $P^1 \cup P^2$ then $i = 1$.

Suppose that the neighbors of x_i in Σ are contained in $P^1 \cup P^3$. For $j = 1, 3$ let u_j (resp. v_j) be the neighbor of x_i in P^j that is closest to a_j (resp. b_j). First suppose that x_i is adjacent to a_3. Then x_i is not adjacent to a_1 and so $(\Sigma \cup x_i) \setminus P^3_{v_3b_3}$ induces a $\Sigma' = 3PC(a_1a_2a_3, u_1v_1x_i)$. Note that $\Sigma' \neq \bar{C}_6$. Suppose u is not adjacent to a_3. Since u is adjacent to a_1, a_2, b_1, and it is not adjacent to a_3, x_i, it must be of Type t4s w.r.t. Σ', a contradiction to Theorem 8.1. So u is adjacent to a_3, i.e. it is of Type t6 w.r.t. Σ, and hence it must have a neighbor in the interior of P^3. Then x_i is not adjacent to b_3 and so $(\Sigma \cup x_i) \setminus P^1_{a_1u_1}$ induces a $\Sigma'' = 3PC(x_ia_3v_3, b_1b_2b_3)$. Note that $\Sigma'' \neq \bar{C}_6$. Since u is adjacent to b_1, b_2, b_3, a_2, a_3 and it has a neighbor in the interior of P^3, and it is not adjacent to x_i, it must be of Type t5 w.r.t. Σ''. But then Σ'', u contradict our choice of Σ, u. Hence x_i is not adjacent to a_3.

Let $\Sigma' = 3PC(a_1a_2a_3, x_iv_3u_3)$ induced by $(\Sigma \cup x_i) \setminus P^1_{v_1b_1}$. Note that $\Sigma' \neq \bar{C}_6$. Suppose u is not adjacent to a_3. Since u is adjacent to a_1, a_2, b_2, and it is not adjacent to a_3, x_i, it must be of Type t4d w.r.t. Σ', a contradiction to Theorem 8.1. Hence u is adjacent to a_3, i.e. it is of Type t6 w.r.t. Σ. Then u must be of Type t3p w.r.t. Σ'. So u cannot have neighbors in $P^1_{a_1u_1} \setminus a_1$ and $P^3_{a_3u_3} \setminus a_3$. Since u is of Type t6 w.r.t. Σ, it must have a neighbor in the interior of P^3. Hence x_i is not adjacent to b_3 and so $(\Sigma \cup x_i) \setminus P^1_{a_1u_1}$ induces a $\Sigma'' = 3PC(x_iu_3v_3, b_1b_2b_3)$. Note that $\Sigma'' \neq \bar{C}_6$. Since u is adjacent to b_1, b_2, b_3, a_2 and it has a neighbor in $P^3_{v_3b_3} \setminus b_3$, and it is not adjacent to x_i, it must be of Type t5 w.r.t. Σ''. But then our choice of Σ, u is contradicted.

An analogous argument holds if the neighbors of x_i in Σ are contained in $P^2 \cup P^3$. This completes the proof of Claim 2.

Claim 3: *No node of P is of Type t2p or t3p w.r.t. Σ.*

Proof of Claim 3: Suppose that x_i is of Type t2p or t3p w.r.t. Σ and let Σ' be obtained from Σ by substituting x_i for its sibling. By definition of S, x_i cannot be a sibling of a_1 or a_3. Suppose that x_i is a sibling of a_2. Suppose u is of Type t6 w.r.t. Σ. Then P^3 is not an edge and so $\Sigma' \neq \bar{C}_6$. But then u is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. Hence u is of Type t5 w.r.t. Σ. First assume that $\Sigma \neq \bar{C}_6$. Suppose $\Sigma' = \bar{C}_6$. Then x_ib_2, a_1b_1 and a_3b_3 are all edges, and since $\Sigma \neq \bar{C}_6$, a_2b_2 is not an edge. So $\{a_1, a_2, a_3, b_2, u, x_i\}$ induces an odd wheel with center a_1. Therefore $\Sigma' \neq \bar{C}_6$. Since u is adjacent to b_1, b_2, b_3, a_1 and it is not adjacent to x_i, a_3, it must be a sibling of b_1 w.r.t. Σ'. Let Σ'' be obtained from Σ' by substituting u for b_1. Since a_2 is adjacent to a_1, a_3, u and it is not adjacent to b_3, x_i, it must be of Type t4d w.r.t. Σ''. Hence a_2b_2 is an edge and, by Theorem 8.1(ii), $\Sigma'' = \bar{C}_6$. But then a_3b_3 is also an edge. Since $\Sigma \neq \bar{C}_6$, a_1b_1 is not an edge, contradicting our assumption on node u. Hence $\Sigma = \bar{C}_6$. Let Σ'' be the $3PC(a_1x_ia_3, ub_2b_3)$ induced by $\{a_1, a_3, b_2, b_3, x_i, u\}$. Note that a_2 is of Type t4d w.r.t. Σ'', adjacent to a_1, a_3, b_2, u. We obtain a contradiction by showing that Σ'' and a_2 satisfy (iii) of Theorem 8.1. Suppose there is a node v, not adjacent to a_2, whose neighbors in Σ'' are x_i, a_3, u, b_3. Node v must be adjacent to b_1, else it violates Lemma 5.1 w.r.t. Σ'. But then $\{a_2, a_3, b_1, b_2, x_i, v\}$ induces an odd wheel with center x_i. Hence, Σ'' and a_2 satisfy (iii) of Theorem 8.1.

Now suppose that x_i is a sibling of b_2. Since x_i is not adjacent to a_2, $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of Type t5 w.r.t. Σ. But then u is of Type t4d w.r.t. Σ', a contradiction to Theorem 8.1(ii).

Next suppose that x_i is a sibling of b_1. First assume that $\Sigma \neq \bar{C}_6$. Suppose $\Sigma' = \bar{C}_6$. Then x_ia_1, a_2b_2 and a_3b_3 are all edges. Since a_3b_3 is an edge, u cannot be of Type t6 w.r.t. Σ, and so it is of Type t5 w.r.t. Σ. Since $\Sigma \neq \bar{C}_6$, a_1b_1 is not an edge. Since a_1b_1 is not an edge and a_2b_2 is an edge, our assumption on Σ and u is contradicted. Hence $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of Type t5 w.r.t. Σ. But then u is of Type t4d w.r.t. Σ', a contradiction to Theorem 8.1(ii). Hence $\Sigma = \bar{C}_6$. Then u is of Type t5 w.r.t. Σ and of Type t4d w.r.t. Σ'. We obtain a contradiction by showing that Σ' and u satisfy (iii) of Theorem 8.1. Suppose there is a node v, not adjacent to u, whose neighbors in Σ' are a_2, a_3, b_3, x_i. By Lemma 5.1, v is of Type t4d w.r.t. Σ. But then our choice of Σ, u is contradicted. Hence, Σ' and u satisfy (iii) of Theorem 8.1.

Finally suppose that x_i is a sibling of b_3. If u is of Type t5 w.r.t. Σ, then u is of Type t4s w.r.t. Σ', a contradiction to Theorem 8.1(i). Hence u is of Type t6 w.r.t. Σ. In particular $\Sigma \neq \bar{C}_6$. But then u is of Type t5 w.r.t. Σ'. So $\Sigma' = \bar{C}_6$ and there is a node v of Type t4d w.r.t. Σ', else our choice of Σ, u is contradicted. By Theorem 8.1, no node is of Type t4 w.r.t. Σ and by our choice of Σ, u, no node is of Type t5 w.r.t. Σ. So by Lemma 5.1 v must be of Type t2p w.r.t. Σ being a sibling of a_1 or a_2. Let Σ'' be obtained by substituting v into Σ. Note that $\Sigma'' \neq \bar{C}_6$. Then x_i is of Type t4d or t5 w.r.t. Σ'', contradicting Theorem 8.1 or our choice of Σ, u. This completes the proof of Claim 3.

Claim 4: *If x_i is of Type p3 w.r.t. Σ, then u is of Type t6 w.r.t. Σ, a_1b_1 and a_2b_2 are not edges, u has no neighbors in the interior of P^1 and P^2, and the neighbors of x_i in Σ are contained in P^3 (i.e. $i = n$).*

Proof of Claim 4: Suppose x_i is of Type p3 w.r.t. Σ. Let Σ' be obtained by substituting x_i into Σ. Note that $\Sigma' \neq \bar{C}_6$. So, if u is of Type t5 w.r.t. Σ, or u is of Type t6 w.r.t. Σ with a neighbor in the interior of one of the paths of Σ', then Σ', u and P', where $P' = x_1, \dots, x_{i-1}$ or $P' = x_{i+1}, \dots, x_n$, contradict our choice of Σ, u and P. Hence u is of Type t6 w.r.t. Σ, the neighbors of x_i in Σ are contained in P^3, and u has no neighbors in the interior of P^1 and P^2. Let $P'_{a_3b_3}$ be the a_3b_3-path of Σ'. If a_1b_1 is an edge, then $P'_{a_3b_3} \cup P^1 \cup u$ induces an odd wheel with center u. Hence a_1b_1 is not an edge, and similarly a_2b_2 is not an edge. This completes the proof of Claim 4.

Claim 5: *$n > 1$, x_1 is of Type t1, p1, p2 or p4 w.r.t. Σ or it is of Type t2 w.r.t. Σ adjacent to a_1 and a_3, and x_n is of Type t1, p1, p2 or p3 w.r.t. Σ, or it is of Type t2 or t3 w.r.t. Σ with neighbors in $\{b_1, b_2, b_3\}$.*

Proof of Claim 5: Follows from the definition of S and Claims 1, 2, 3 and 4.

Claim 6: *No intermediate node of P is strongly adjacent to Σ.*

Proof of Claim 6: Assume not and let x_i be an intermediate node of P with lowest index that is strongly adjacent to Σ. By the definition of S, the only nodes of Σ that can have a neighbor in the interior of P are a_3, b_1 and b_2. Hence x_i is of Type t2 w.r.t. Σ adjacent to b_1 and b_2.

First we show that at most one node of $\{a_3, b_1, b_2\}$ has a neighbor in $P_{x_2x_{i-1}}$. Suppose not. Then $P_{x_2x_{i-1}}$ contains a subpath P' such that the endnodes of P' are adjacent to distinct nodes of $\{a_3, b_1, b_2\}$ and no intermediate node of P' has a neighbor in $\{a_3, b_1, b_2\}$. If b_1 and b_2 have neighbors in P', then $P^2 \cup P^3 \cup P'$ induces a Mickey Mouse. So we may assume w.l.o.g. that one endnode of P' is adjacent to a_3 and the other to b_2. But then $P^1 \cup P^2 \cup P'$ induces a $3PC(a_1a_2a_3, b_2)$. Hence, at most one node of $\{a_3, b_1, b_2\}$ has a neighbor in $P_{x_2x_{i-1}}$.

We now show that a_3 does not have a neighbor in $P_{x_2x_{i-1}}$. Suppose it does and let x_j be the node of $P_{x_2x_{i-1}}$ with highest index adjacent to a_3. Then b_1 and b_2 do not have neighbors in $P_{x_2x_{i-1}}$, and hence $P^1 \cup P^2 \cup P_{x_jx_i}$ induces a $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_i)$. Since $i \neq j$, $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of Type t5 w.r.t. Σ, and hence it is of Type t4s w.r.t. Σ', a contradiction to Theorem 8.1. Therefore, a_3 has no neighbors in $P_{x_2x_{i-1}}$.

Suppose x_1 is of Type t1, p1 or p2 w.r.t. Σ. W.l.o.g. assume that its neighbors in Σ are contained in P^2. Then $(\Sigma \setminus b_2) \cup P_{x_1x_i}$ contains a $3PC(a_1a_2a_3, b_1)$. If x_1 is of Type t2 w.r.t. Σ, then $(\Sigma \setminus b_2) \cup P_{x_1x_i}$ contains a $3PC(a_1x_1a_3, b_1)$. Hence x_1 is of Type p4 w.r.t. Σ. Then x_1 cannot be adjacent to both b_1 and b_2, so assume w.l.o.g. that it is not adjacent to b_2. But then $(\Sigma \setminus b_1) \cup P_{x_1x_i}$ contains a $3PC(a_1a_2a_3, x_1)$. This completes the proof of Claim 6.

Claim 7: *At most one node of $\{a_3, b_1, b_2\}$ has a neighbor in the interior of P.*

Proof of Claim 7: Assume not. Then, by Claim 6, $P_{x_2x_{n-1}}$ contains a subpath P' such that the endnodes of P' are not strongly adjacent to Σ, they are adjacent to distinct nodes of $\{a_3, b_1, b_2\}$, and no intermediate node of P' is adjacent to a node of $\{a_3, b_1, b_2\}$. If the endnodes of P' are adjacent to b_1 and b_2, then $P' \cup P^2 \cup P^3$ induces a Mickey Mouse. So we may assume w.l.o.g. that the endnodes of P' are adjacent to a_3 and b_2. But then $P' \cup P^1 \cup P^2$ induces a $3PC(a_1a_2a_3, b_2)$. This completes the proof of Claim 7.

By Claim 5, we now consider the following cases.

Case 1: x_n is of Type t1, p1, p2 or p3 w.r.t. Σ.

First we show that b_1 and b_2 do not have neighbors in the interior of P. Suppose not and let x_i be the node of P with highest index adjacent to b_1 or b_2. W.l.o.g. assume that x_i is adjacent to b_2. Then, by Claim 7, a_3 and b_1 do not have neighbors in the interior of P and so Σ and $P_{x_ix_n}$ contradict Lemma 7.2. Hence, b_1 and b_2 do not have neighbors in the interior of P.

Case 1.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

By a similar argument as above, a_3 does not have a neighbor in the interior of P. By Lemma 7.2 applied to Σ and P, both x_1 and x_n must be of Type p2 w.r.t. Σ.

Suppose that the neighbors of x_1 in Σ are contained in P^1. Let u_1 (resp. v_1) be the neighbor of x_1 in P^1 that is closest to a_1 (resp. b_1). Let u_3 (resp. v_3) be the neighbor of x_n in P^3 that is closest to a_3 (resp. b_3). Let $\Sigma' = 3PC(u_1v_1x_1, u_3v_3x_n)$ induced by $P^1 \cup P^3 \cup P$. Since u is adjacent to a_1, b_1, b_3 and it is not adjacent to any node of P, it must be of Type p4 or t4s w.r.t. Σ'. If u is of Type t4s w.r.t. Σ', then Theorem 8.1(i) is contradicted. So u is of Type p4 w.r.t. Σ'. Then u must be of Type t5 w.r.t. Σ, $N(u) \cap (P^1 \cup P^3) = \{a_1, a_1', b_1, b_3\}$, and P^1 is of length greater than 2. But then $P^1 \cup P^2 \cup u$ induces a proper wheel with center u that is not a beetle.

Analogous argument holds when the neighbors of x_1 in Σ are contained in P^2.

Case 1.2: x_1 is of Type t2 w.r.t. Σ.

Then $(\Sigma \setminus a_3) \cup P$ contains a $3PC(b_1b_2b_3, a_1)$.

Case 1.3: x_1 is of Type p4 w.r.t. Σ.

Then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$ contains a $3PC(b_1b_2b_3, x_1)$.

Case 2: x_n is of Type t2 or t3 w.r.t. Σ.

Then x_n is adjacent to b_3. Suppose a_3 has a neighbor in the interior of P and let x_i be the node of P with highest index adjacent to a_3. Then, by Claim 7, b_1 and b_2 do not have a neighbor in the interior of P. If x_n is adjacent to b_1, then $P^1 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(b_1x_nb_3, a_3)$. Otherwise, $P^2 \cup P^3 \cup P_{x_ix_n}$ induces a $3PC(x_nb_2b_3, a_3)$. Therefore a_3 has no neighbors in the interior of P.

Case 2.1: x_1 is of Type t1, p1 or p2 w.r.t. Σ.

W.l.o.g. we assume that x_n is adjacent to b_1. First suppose that the neighbors of x_1 in Σ are contained in P^2. Suppose b_1 has a neighbor in the interior of P and let x_i be the node of P with lowest index adjacent to b_1. Then, by Claim 7, b_2 does not have a neighbor in the interior of P, and hence Σ and $P_{x_1x_i}$ contradict Lemma 7.2. Therefore b_1 has no neighbors in the interior of P. So $(\Sigma \setminus b_2) \cup P$ contains a $\Sigma' = 3PC(a_1a_2a_3, b_1x_nb_3)$. Note that $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', and hence our choice of Σ, u is contradicted. So u is of Type t5 w.r.t. Σ. But then u is of Type t4d w.r.t. Σ', a contradiction to Theorem 8.1(ii).

Now suppose that the neighbors of x_1 in Σ are contained in P^1. Suppose b_2 has a neighbor in the interior of P and let x_i be the node of P with lowest index adjacent to b_2. Then, by Claim 7, b_1 has no neighbors in the interior of P and hence Σ and $P_{x_1x_i}$ contradict Lemma 7.2. Therefore b_2 has no neighbors in the interior of P. If x_n is not adjacent to b_2, then $(\Sigma \setminus b_1) \cup P$ contains a $3PC(a_1a_2a_3, b_3)$. Hence x_n is adjacent to b_2. So $(\Sigma \setminus b_1) \cup P$ contains a $\Sigma' = 3PC(a_1a_2a_3, x_nb_2b_3)$. Note that $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of Type t5 w.r.t. Σ. But then u is of Type t4d w.r.t. Σ', a contradiction to Theorem 8.1(ii).

Case 2.2: x_1 is of Type t2 w.r.t. Σ.

First suppose that x_n is adjacent to b_1. We now show that b_1 cannot have a neighbor in the interior of P. Suppose not and let x_i be the node of P with lowest index adjacent to b_1. Then $P^1 \cup P^3 \cup P$ contains a $3PC(a_1x_1a_3, b_1)$. Hence b_1 has no neighbors in the interior of P, and so $P^1 \cup P^3 \cup P$ induces a $\Sigma' = 3PC(a_1x_1a_3, b_1x_nb_3)$. Since u is adjacent to a_1, b_1, b_3 and it has no neighbors in P, it must be of Type t4s w.r.t. Σ', a contradiction to Theorem 8.1(i).

Now suppose that x_n is adjacent to b_2. Node b_2 must have a neighbor in the interior of P, since otherwise $P^2 \cup P^3 \cup P$ induces a $3PC(x_nb_2b_3, a_3)$. Let x_i be the node of P with lowest index adjacent to b_2. Then, by Claim 7, b_1 has no neighbors in the interior of P, and hence $P^1 \cup P^3 \cup P_{x_1x_i} \cup b_2$ induces a $\Sigma' = 3PC(a_1x_1a_3, b_1b_2b_3)$. Note that $\Sigma' \neq \bar{C}_6$. If u is of Type t6 w.r.t. Σ, then it is of Type t5 w.r.t. Σ', contradicting our choice of Σ, u. So u is of Type t5 w.r.t. Σ, and hence it is of Type t3p w.r.t. Σ' (u being a sibling of b_1 w.r.t. Σ'). Let Σ'' be obtained from Σ' by substituting u for b_1. Note that $\Sigma'' \neq \bar{C}_6$. Since a_2 is adjacent to a_1, a_3, u and it is not adjacent to x_1, b_3, it must be of Type t4d w.r.t. Σ'', a contradiction to Theorem 8.1(ii).

Case 2.3: x_1 is of Type p4 w.r.t. Σ.

Then $(\Sigma \setminus \{b_1, b_2\}) \cup P$ contains a $3PC(a_1a_2a_3, x_1)$. □

Theorem 8.3 *Let G be an even-signable graph that contains a $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and a node u that is of Type t6a w.r.t. Σ. Assume that for some $i \in \{1,2,3\}$, there is no P^i-crosspath w.r.t. Σ, and if $\Sigma = \bar{C}_6$ then no node is of Type t4d w.r.t. Σ. Then G has a double star cutset.*

Proof: Assume there is no P^3-crosspath w.r.t. Σ, and if $\Sigma = \bar{C}_6$ then no node is of Type t4d w.r.t. Σ. Suppose G has no double star cutset. Then by Theorems 8.1 and 8.2, no node is of

Type t4s or t6b w.r.t. a $\Sigma' = 3PC(\Delta, \Delta)$, and if $\Sigma' \neq \bar{C}_6$, then no node is of Type t4d or t5 w.r.t. Σ'. In particular, no node is of Type t4d w.r.t. Σ, and hence by Theorem 8.2 no node is of Type t5 w.r.t. Σ. Let $S = (N(u) \cup N(a_2)) \setminus (\Sigma \setminus \{a_1, a_2, b_3\})$ and let $P = x_1, \ldots, x_n$ be a direct connection from $P^1 \cup P^2$ to P^3 in $G \setminus S$.

Claim 1: *No node of P is of Type t2, t2p, t3p or t6a w.r.t. Σ.*

Proof of Claim 1: By defenition of S, no node of P is of Type t6a w.r.t. Σ. If x_i is of Type t2p or t3p w.r.t. Σ, then let Σ' be obtained from Σ by substituting x_i for its sibling. If x_i is of Type t2 w.r.t. Σ, then by Theorem 7.7, x_i is attached to Σ by an attachment $Q = y_1, \ldots, y_m$. Let Σ' be obtained from Σ by substituting x_i and Q into Σ. Since u is not adjacent to x_i, it is of Type t5 or t4s w.r.t. Σ'. By Theorems 8.1 and 8.2, u is of Type t5 w.r.t. Σ', $\Sigma' = \bar{C}_6$ and there is a node v of Type t4d w.r.t. Σ'. Hence $\Sigma = \bar{C}_6$. Suppose x_i is of Type t2 w.r.t. Σ. Since $\Sigma = \bar{C}_6$, by definition of attachment, $m = 1$ and y_1 is of Type t2 w.r.t. Σ. But then $\Sigma \cup \{x_i, y_1\}$ contains a hole H of length 6 that contains x_i and y_1, such that (H, u) is a proper wheel that is not a beetle, a contradiction. So x_i is of Type t2p or t3p w.r.t. Σ. Since x_i is not adjacent to a_2, x_i cannot be a sibling of b_2, a_1 or a_3. If x_i is a sibling of a_2, then every node that is of Type t4d w.r.t. Σ' is of Type t2p w.r.t. Σ being a sibling of b_1 or b_3. If x_i is a sibling of b_1, then every node that is of Type t4d w.r.t. Σ' is of Type t2p w.r.t. Σ being a sibling of a_2 or a_3. If x_i is a sibling of b_3, then every node that is of Type t4d w.r.t. Σ' is of Type t2p w.r.t. Σ being a sibling of a_1 or a_2. Therefore Σ' and v satisfy Theorem 8.1(iii), a contradiction. This completes the proof of Claim 1.

Claim 2: *$n > 1$, x_1 is of Type t1, p1, p2, p3 or p4 w.r.t. Σ with neighbors contained in $P^1 \cup P^2$, or of Type t3 w.r.t. Σ adjacent to b_1, b_2 and b_3, and x_n is of Type t1, p1, p2 or p3 w.r.t. Σ with neighbors contained in P^3.*

Proof of Claim 2: Since there is no P^3-crosspath, if a node of P is of Type p4 w.r.t. Σ, then its neighbors are contained in $P^1 \cup P^2$. By definition of S, if a node is of Type t3 w.r.t. Σ then it is adjacent to b_1, b_2, b_3. Now the result follows from Claim 1. This completes the proof of Claim 2.

Claim 3: *No interior node of P has a neighbor in Σ.*

Proof of Claim 3: By definition of S, the only nodes of Σ that can have a neighbor in the interior of P are a_1 and b_3. First we show that not both a_1 and b_3 can have a neighbor in the interior of P. Assume not and let x_i and x_j be nodes in the interior of P adjacent to a_1 and to b_3 respectively so that the $P_{x_i x_j}$ subpath is shortest possible. Then $P_{x_i x_j} \cup P^2 \cup P^3$ induces a $3PC(a_1 a_2 a_3, b_3)$.

Now assume that b_3 has a neighbor in the interior of P. By Lemma 7.2, x_1 is of Type p4 with neighbors in $P^1 \cup P^2$, or of Type t3 adjacent to b_1, b_2, b_3. If x_1 is of Type p4, there is a $3PC(b_1 b_2 b_3, x_1)$ contained in $(P \cup \Sigma) \setminus \{a_1, a_2, a_3, x_n\}$. If x_1 is of Type t3 adjacent to b_1, b_2, b_3, then $(P \cup \Sigma) \setminus b_3$ contains a $3PC(a_1 a_2 a_3, b_1 b_2 x_1) \neq \bar{C}_6$ and u is of Type t5 w.r.t. it, a contradiction. So no interior node of P is adjacent to b_3.

Assume now that some interior node of P is adjacent to a_1. Let x_i be such a node with highest index. Then $P_{x_i x_n}$ contradicts Lemma 7.2. This completes the proof of Claim 3.

If x_1 is of Type t1, p1, p2 or p3 w.r.t. Σ, by Lemma 7.2, P is a P^3-crosspath, a contradiction. Suppose x_1 is of Type t3 w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2a_3, b_1b_2x_1)$ contained in $(\Sigma \setminus b_3) \cup P$. Note that $\Sigma' \neq \bar{C}_6$. Then u is of Type t5 w.r.t. Σ', a contradiction. So x_1 is of Type p4 w.r.t. Σ. But then $(\Sigma \setminus \{b_1, b_2, b_3\}) \cup P$ contains a $3PC(a_1a_2a_3, x_1)$. □

9 Type t2 and t2p Nodes

The main result of this section is the following.

Theorem 9.1 *Let G be an even-signable graph. If G contains a $3PC(\Delta, \Delta)$ with a Type t2 or t2p node, then G has a double star cutset or a 2-join.*

9.1 Decomposable $3PC(\Delta, \Delta)$

Definition 9.2 *A $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3) \neq \bar{C}_6$ in G is* decomposable *if there exists a node of Type t2 or t2p w.r.t. Σ, say adjacent to a_2 and a_3, but there is no P^1-crosspath w.r.t. Σ. A $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3) = \bar{C}_6$ in G is* decomposable *if there exists a node of Type t2 w.r.t. Σ, say adjacent to a_2 and a_3, but there is no P^1-crosspath w.r.t. Σ. In both cases path P^1 of Σ is called the* middle path.

Denote by H the graph induced by a decomposable $3PC(a_1a_2a_3, b_1b_2b_3)$ together with a node a_4 of Type t2 or t2p adjacent to a_2, a_3. Let $H_1 = P^1 \cup a_4$ and $H_2 = P^2 \cup P^3$. Then $H_1|H_2$ is a 2-join of H with special sets $A_1 = \{a_1, a_4\}$, $B_1 = \{b_1\}$, $A_2 = \{a_2, a_3\}$ and $B_2 = \{b_2, b_3\}$. In this section, we show that the 2-join $H_1|H_2$ of H extends to a 2-join of G. First, we prove the following results.

Lemma 9.3 *If G contains a $3PC(\Delta, \Delta)$ with a Type t2 node, then G has a double star cutset or G contains a decomposable $3PC(\Delta, \Delta)$ with a Type t2 node.*

Proof: Assume G has no double star cutset.

Connected diamonds $D(a_1a_2a_3a_4, b_1b_2b_3b_4)$ consist of two node disjoint sets $\{a_1, \ldots, a_4\}$ and $\{b_1, \ldots, b_4\}$ each of which induces a diamond such that a_1a_4 and b_1b_4 are not edges, together with four paths $P^1, \ldots, P^4$ such that for $i = 1, \ldots, 4$, P^i is an a_ib_i-path. Paths $P^1, \ldots, P^4$ are node disjoint and the only adjacencies between them are the edges of the two diamonds.

First suppose that G contains connected diamonds $D(a_1a_2a_3a_4, b_1b_2b_3b_4)$. Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ (resp. $\Sigma' = 3PC(a_4a_2a_3, b_4b_2b_3)$) induced by paths P^1, P^2 and P^3 (resp. P^4, P^2 and P^3) of D. Suppose that $P = x_1, \ldots, x_n$ is a P^1-crosspath w.r.t. Σ. W.l.o.g. x_1 has a neighbor in P^1 and x_n has a neighbor in P^2. Let u_1 and v_1 be the neighbors of x_1 in P^1. If no node of P^4 has a neighbor in P, then $P^1 \cup (P^2 \setminus b_2) \cup b_3 \cup P^4 \cup P$ contains a $3PC(x_1u_1v_1, a_2)$. So a node of P^4 has a neighbor in P. Let x_i be such a neighbor with highest index. Let v be the neighbor of x_i in P^4 that is closest to a_4. By Lemmas 5.1 and 7.2 applied to $P_{x_ix_n}$ and Σ', $P_{x_ix_n}$ is a P^4-crosspath w.r.t. Σ'. Hence $v \neq b_4$. If $i \neq 1$ then $P^4_{a_4v}, P_{x_ix_n}$ contradicts Lemma 7.4 applied to Σ. So $i = 1$. If x_1 is not adjacent to a_1, then $P^4_{a_4v}, x_1$ contradicts Lemma 7.4 applied to Σ. So x_1 is adjacent to a_1, and hence $P^1 \cup P^3 \cup P^4_{a_4v} \cup x_1$

induces a $3PC(x_1u_1v_1, a_3)$. Therefore, there is no P^1-crosspath w.r.t. Σ, and hence Σ is a decomposable $3PC(\Delta, \Delta)$.

Now we may assume that G does not contain connected diamonds.

Let $\mathcal{C}$ be the set of all pairs Σ, u where $\Sigma = 3PC(\Delta, \Delta)$ and u is of Type t2 w.r.t. Σ. Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and a_4 be a pair chosen from $\mathcal{C}$ so that Σ has the shortest middle path. W.l.o.g. a_4 is adjacent to a_2 and a_3. Suppose Σ is not decomposable and let $P = x_1, \ldots, x_n$ be a P^1-crosspath w.r.t. Σ. W.l.o.g. x_1 has a neighbor in P^1 and x_n in P^2. Let u_1 (resp. v_1) be the neighbor of x_1 in P^1 that is closest to a_1 (resp. b_1). Let u_2 be the neighbor of x_n on P^2 that is closest to a_2.

First suppose that $u_1 \neq a_1$. Let $\Sigma' = 3PC(a_1a_2a_3, u_1x_1v_1)$ contained in $(\Sigma \cup P) \setminus b_2$. By Lemma 5.1, a_4 is of Type t2 w.r.t. Σ'. Since Σ' has a shorter middle path than Σ, this contradicts our choice of Σ. Therefore, $u_1 = a_1$.

By Theorem 7.7, let $Q = y_1, \ldots, y_m$ be an attachment of a_4 to Σ. Let Σ' be obtained by substituting a_4 and Q into Σ. Suppose a_4 has a neighbor in P and let x_i be its neighbor in P with highest index. If $i = 1$ then $P^1 \cup P^3 \cup \{a_4, x_1\}$ induces a $3PC(x_1u_1v_1, a_3)$, and otherwise $a_4, P_{x_ix_n}$ contradicts Lemma 7.4 applied to Σ. So a_4 does not have a neighbor in P. Next we show that no node of $Q \setminus y_m$ is adjacent to or coincident with a node of P. Suppose not and let y_i be the node of Q with lowest index adjacent to a node of P, and let x_j be the node of P with highest index adjacent to y_i. If $j = 1$, then $P^1 \cup P^3 \cup Q_{y_1y_i} \cup \{a_4, x_1\}$ induces a $3PC(x_1u_1v_1, a_3)$. If $j > 1$, then $a_4, Q_{y_1y_i}, P_{x_jx_n}$ violates Lemma 7.4 applied to Σ. So no node of $Q \setminus y_m$ is adjacent to or coincident with a node of P.

Assume y_m is of Type t1, p1 or p3 w.r.t. Σ. We show that y_m does not have a neighbor in P. Suppose not and let x_i be the neighbor of y_m in P with highest index. Then $P_{x_ix_n}$ contradicts Lemma 7.2 applied to Σ' unless $i = 1$ and y_m is of Type p1 adjacent to a_1'. But then there is a $3PC(a_2a_3a_4, x_1a_1'y_m)$ and a_1 is a strongly adjacent node of Type t4s relative to it, a contradiction to Theorem 8.1. Therefore y_m does not have a neighbor in P. Let v be the neighbor of y_m in $P^1 \setminus a_1$ that is closest to a_1'. Let H be the hole $P \cup P^1_{a_1'v} \cup P^2_{a_2u_2} \cup Q \cup a_4$. Then (H, a_1) is an odd wheel.

Therefore, y_m is of Type t2, t2p or t3p w.r.t. Σ. We show that y_m does not have a neighbor in P. Assume not and let x_i be the neighbor of y_m in P with largest index. If $i = n$ and x_n is adjacent to b_2, then $P^2 \cup P^3 \cup \{x_n, y_m\}$ induces an odd wheel with center b_2. Otherwise, $P^2 \cup P_{x_ix_n} \cup Q \cup a_4$ induces a $3PC(\Delta, y_m)$. So y_m does not have a neighbor in P. If y_m is of Type t2, then $\Sigma \cup Q$ induces connected diamonds, contradicting our assumption. So y_m is of Type t2p or t3p w.r.t. Σ and hence it has a neighbor in $P^1 \setminus b_1$. Let v be the neighbor of y_m in P^1 that is closest to a_1. Note that $v \neq a_1$, by definition of attachment. Then $P^1_{a_1v} \cup P^2_{a_2u_2} \cup P \cup Q \cup a_4$ induces an odd wheel with center a_1. □

9.2 Double Star Cutsets

Lemma 9.4 *Let G be an even-signable graph that does not contain a $3PC(\Delta, \Delta)$ with a Type t2 node. Suppose that G contains a $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ where P_2 has length one and suppose that there exists a sibling u of a_2 w.r.t. Σ, i.e. node u is of Type t2p or t3p adjacent to a_1, a_3, b_2 (and possibly a_2). Then G has a double star cutset.*

Proof: Assume G has no double star cutset. Let $S = (N(a_2) \cup N(b_2)) \setminus \{u, b_1, b_3\}$ and let $P = x_1, \ldots, x_n$ be a direct connection from u to $\Sigma \setminus S$ in $G \setminus S$. By our assumption, no node is of Type t2 w.r.t. a $3PC(\Delta, \Delta)$. By Theorem 8.1, no node is of Type t4s w.r.t. Σ. By definition of S, no node of P is of Type t3, t2p, t3p, t4d, t5 or t6 w.r.t. Σ. Let Σ' be obtained by substituting u into Σ.

Assume w.l.o.g. that x_n has a neighbor in $P^3 \setminus a_3$. Then x_n is of Type t1, p1, p2, p3 or p4 w.r.t. Σ. Suppose x_n is of Type p4 w.r.t. Σ. Then $(\Sigma \cup P \cup u) \setminus \{a_1, a_2, a_3\}$ contains a $3PC(b_1b_2b_3, x_n)$. So x_n is not of Type p4 w.r.t. Σ, and hence $N(x_n) \cap \Sigma \subseteq P^3$. If x_n is adjacent to u, then x_n contradicts Lemma 5.1 applied to Σ'. So x_n is not adjacent to u and $N(x_1) \cap \Sigma \subseteq \{a_1, a_3\}$. Since no node can be of Type t2 w.r.t. Σ or Σ', $N(x_1) \cap \Sigma = \emptyset$.

Suppose a_1 has a neighbor in P and let x_i be such a neighbor with highest index. Since x_i is an interior node of P, $N(x_i) \cap \Sigma \subseteq \{a_1, a_3\}$. Since x_i cannot be of Type t2 w.r.t. Σ, a_1 is the unique neighbor of x_i in Σ. But then $P_{x_i x_n}$, or a subpath of it (if a_3 has a neighbor in $P_{x_{i+1} x_{n-1}}$), contradicts Lemma 7.2 applied to Σ. So a_1 does not have a neighbor in P.

Let v be the neighbor of x_n in P^3 that is closest to b_3. Then $P \cup P^3_{vb_3} \cup P^1 \cup \{u, b_2\}$ induces an odd wheel with center b_2. □

Definition 9.5 *A* double line wheel (H, x, y) *consists of a hole H and two nonadjacent nodes x and y such that both (H, x) and (H, y) are line wheels and $N(x) \cap H = N(y) \cap H$.*

Lemma 9.6 *If an even-signable graph G contains a double line wheel (H, x, y) such that $H \neq C_6$, then G has a double star cutset.*

Proof: Assume G contains a double line wheel (H, x, y) such that $H \neq C_6$, but G has no double star cutset. Let x_1, x_2, x_3, x_4 be the neighbors of x in H encountered in this order when H is traversed clockwise, and such that x_1x_2 and x_3x_4 are edges. Let S^1 (resp. S^2) be the sector of (H, x) with endnodes x_1 and x_4 (resp. x_2 and x_3). Let x_1' be the neighbor of x_1 in S^1, and w.l.o.g. assume that S^1 is of length greater than two. Let $S = (N(x) \cup N(x_1)) \setminus \{x_1', y\}$ and let $P = y_1, \ldots, y_n$ be a direct connection from y to $H \setminus S$ in $G \setminus S$.

Claim 1: *If x_2 has a neighbor in $P \setminus y_n$, then x_3 and x_4 do not.*

Proof of Claim 1: Suppose that both x_2 and x_4 have a neighbor in $P \setminus y_n$. Let P' be a subpath of $P \setminus y_n$ such that one endnode of P' is adjacent to x_2, the other to x_4 and no proper subpath of P' has this property. Then $P' \cup S^1 \cup \{x, x_2\}$ induces an odd wheel with center x.

Now suppose that both x_2 and x_3 have a neighbor in $P \setminus y_n$. Let P' be a subpath of $P \setminus y_n$ such that one endnode of P' is adjacent to x_2, the other to x_3 and no proper subpath of of P' has this property, and furthermore out of all such subpaths, P' contains a smallest indexed node of P. If P' contains y_1, then $P' \cup S^1 \cup \{x_2, x_3, y\}$ induces a proper wheel with center y that is not a beetle, contradicting Theorem 3.2. So P' does not contain y_1, and hence $S^1 \cup P' \cup \{x_2, x_3, y\}$ together with the subpath of P that connects y to P', induces an L-parachute, contradicting Theorem 4.1. This completes the proof of Claim 1.

Case 1: y_n has a neighbor in both S^1 and S^2.

Case 1.1: $n = 1$ and y_1 is adjacent to x_2.

If y_1 is not adjacent to x_4 then $(S^1 \setminus x_1) \cup \{x, y, y_1, x_2\}$ contains a $3PC(yy_1x_2, x_4)$. So y_1 is adjacent to x_4. Node y_1 must have a neighbor in $S^1 \setminus x_4$, else $S^1 \cup \{x, y_1, x_2\}$ induces an odd wheel with center x. Let u_1 be the neighbor of y_1 in S^1 that is closest to x_1. If u_1x_4 is not an edge, then $S^1_{u_1x_1} \cup \{x, x_4, x_2, y_1\}$ induces a $3PC(x_1x_2x, y_1)$. So u_1x_4 is an edge. Let $\Sigma' = 3PC(x_4u_1y_1, xx_1x_2)$ induced by $S^1 \cup \{x, x_2, y_1\}$. Then y is of Type t4d w.r.t. Σ'. By Theorem 8.1, $\Sigma' = \bar{C}_6$. But then S^1 is of length two, contradicting our assumption.

Case 1.2: $n \neq 1$ or y_1 is not adjacent to x_2.

Note that x_4 cannot be the unique neighbor of y_n in S^1, since otherwise y_n must have a neighbor in $S^2 \setminus x_3$ and hence $(H \setminus x_3) \cup P \cup x$ contains an odd wheel with center x. Suppose x_2 has no neighbor in $P \setminus y_n$. If y_n has a neighbor in $S^2 \setminus x_3$, then $(H \setminus \{x_3, x_4\}) \cup P \cup y$ contains a $3PC(x_1x_2y, y_n)$. Otherwise, x_3 is the unique neighbor of y_n in S^2 and hence $(H \setminus x_4) \cup \{x, y_n\}$ contains a $3PC(x_1x_2x, x_3)$. So x_2 has a neighbor in $P \setminus y_n$, and hence by Claim 1, x_3 and x_4 do not. In particular, $n > 1$.

Node x_2 cannot be the unique neighbor of y_n in S^2, since otherwise $(H \setminus x_1) \cup \{y, y_n\}$ contains an odd wheel with center y. If y_n is not adjacent to both x_3 and x_4, then $(H \cup P \cup y) \setminus x_1$ contains a $3PC(x_3x_4y, y_n)$. So y_n is adjacent to both x_3 and x_4. If x_2 does not have a neighbor in $P \setminus y_1$, then x_2 is adjacent to y_1 and hence $P \cup \{y, x, x_2, x_3\}$ induces an odd wheel with center y. So x_2 has a neighbor in $P \setminus y_1$. Since y_n has a neighbor in $S^1 \setminus x_4$, $(S^1 \setminus x_4) \cup (P \setminus y_1) \cup \{y, x_2, x_3\}$ contains a $3PC(x_1x_2y, y_n)$.

Case 2: $N(y_n) \cap H \subseteq S^1$

Suppose x_2 has a neighbor in $P \setminus y_n$. Then, by Claim 1, x_3 and x_4 do not. But then $(H \setminus x_1) \cup P \cup x$ contains an odd wheel with center x. So x_2 does not have a neighbor in $P \setminus y_n$.

If x_3 has a neighbor in $P \setminus y_n$, then $(H \setminus x_4) \cup P \cup x$ contains an odd wheel with center x. So x_3 does not have a neighbor in $P \setminus y_n$.

If y_n has a unique neighbor in S^1, then $H \cup P \cup y$ induces an L-parachute, contradicting Theorem 4.1. Suppose y_n has two nonadjacent neighbors in S^1. Let u_4 (resp. u_1) be the neighbor of y_n in S^1 that is closest to x_4 (resp. x_1). Then $S^1_{x_4u_4} \cup S^1_{u_1x_1} \cup S^2 \cup P \cup y$ induces either a proper wheel that is not a beetle (if $n = 1$) or an L-parachute (otherwise), contradicting Theorem 3.2 or 4.1. So y_n has exactly two neighbors in S^1, and they are adjacent. If $u_4 = x_4$, then $H \cup P \cup y$ induces an L-parachute, contradicting Theorem 4.1. If x_4 has no neighbor in $P \setminus y_n$, then $S^1 \cup P \cup y$ induces a $3PC(\Delta, y)$. Otherwise, $H \cup P$ contains a $3PC(\Delta, x_4)$.

Case 3: $N(y_n) \cap H \subseteq S^2$

Suppose x_4 has a neighbor in $P \setminus y_n$. Then by Claim 1, x_2 does not, and hence $(H \setminus x_3) \cup P \cup x$ contains an odd wheel with center x. So x_4 does not have a neighbor in $P \setminus y_n$. By Claim 1, at most one of x_2, x_3 has a neighbor in $P \setminus y_n$, and so by an analogous argument as in Case 2, there is either a proper wheel that is not a beetle or an L-parachute, contradicting Theorem 3.2 or 4.1. □

Lemma 9.7 *If G contains a $3PC(\Delta, \Delta) \neq \bar{C}_6$ with a Type t2p node, then either G contains a decomposable $3PC(\Delta, \Delta)$ or G has a double star cutset.*

Proof: By Lemma 9.3 we may assume that G does not contain a $3PC(\Delta, \Delta)$ with a Type t2 node. Assume G has no double star cutset. Let $\mathcal{C}$ be the set of all pairs Σ, u where $\Sigma = 3PC(\Delta, \Delta) \neq \bar{C}_6$ and u is of Type t2p w.r.t. Σ, and assume that $\mathcal{C} \neq \emptyset$. Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3), a_4$ be a pair chosen from $\mathcal{C}$ so that Σ has the shortest middle path. W.l.o.g. a_4 is adjacent to a_2 and a_3. Suppose Σ is not decomposable and let $P = x_1, \ldots, x_n$ be a P^1-crosspath w.r.t. Σ. W.l.o.g. x_1 has a neighbor in P^1 and x_n in P^2. Let u_1 (resp. v_1) be the neighbor of x_1 in P^1 that is closest to a_1 (resp. b_1).

First suppose that $u_1 \neq a_1$. Let $\Sigma' = 3PC(a_1a_2a_3, u_1x_1v_1)$ contained in $(\Sigma \cup P) \setminus b_2$. Note that $\Sigma' \neq \bar{C}_6$. By Theorems 8.1 and 8.2, a_4 cannot be of Type t4 or t5 w.r.t. Σ'. By our assumption a_4 cannot be of Type t2 w.r.t. Σ'. So by Lemma 5.1, a_4 is of Type t2p w.r.t. Σ'. Since Σ' has a shorter middle path than Σ, this contradicts our choice of Σ. Therefore, $u_1 = a_1$.

Suppose a_4 has no neighbor in P. Let H be the hole contained in $P \cup (P^1 \setminus a_1) \cup (P^2 \setminus b_2) \cup a_4$. Then (H, a_1) is an odd wheel. So a_4 has a neighbor in P. Let H be the hole contained in $(\Sigma \cup P) \setminus \{a_1, b_2\}$. By Theorem 3.2, (H, a_4) cannot be a proper wheel. Since a_4 is adjacent to a_2, a_3 and a node of $P^1 \setminus a_1$, and it is not adjacent to b_3, (H, a_4) must be a line wheel. In particular, a_4 is adjacent to a_1' and x_1. So (H, a_1, a_4) is a double line wheel. By Lemma 9.6, $H = C_6$. In particular, x_1 is adjacent to b_1, i.e. P^1 is an edge. But then Lemma 9.4 is contradicted. □

Lemma 9.8 *If $\Sigma = \bar{C}_6$ has a node of Type t4d and a node of Type t2, then G has a double star cutset.*

Proof: Let a_4 be of Type t2 w.r.t. Σ, adjacent to a_2 and a_3, let u be of Type t4d w.r.t. Σ, and assume that G has no double star cutset. By Theorem 7.7, let $Q = x_1, \ldots, x_n$ be an attachment of a_4 to Σ. Let Σ' be the $3PC(\Delta, \Delta)$ obtained by substituting a_4 and Q into Σ. By Lemma 7.4, x_n is of Type t1, p1, p3, t2, t2p or t3p w.r.t. Σ. Since $\Sigma = \bar{C}_6$ and x_n cannot be adjacent to a_1, node x_n cannot be of Type p1, p3, t2p or t3p w.r.t. Σ. Suppose that x_n is of Type t1 w.r.t. Σ. Then $\Sigma' \neq \bar{C}_6$. By Theorems 8.1 and 8.2, u cannot be of Type t4d or t5 w.r.t. Σ'. So by Lemma 5.1, u is of Type t2p w.r.t. Σ', being a sibling of b_2 or b_3. Let Σ'' be obtained by substituting u into Σ'. Note that $\Sigma'' \neq \bar{C}_6$. But then a_1 is of Type t4d w.r.t. Σ'', contradicting Theorem 8.1. Hence x_n is of Type t2 w.r.t. Σ. Note that x_n is adjacent to b_2 and b_3. By symmetry it is enough to consider the following two cases.

Case 1: $N(u) \cap \Sigma = \{a_2, a_3, b_1, b_2\}$

By Lemma 5.1, u is of Type t4d, t5 or t3p w.r.t. Σ'. Suppose that u is of Type t3p w.r.t. Σ'. Then $N(u) \cap \Sigma' = \{a_2, a_3, a_4, b_2\}$ and hence $Q \cup \{a_1, a_2, a_4, b_1, b_3, u\}$ induces an odd wheel with center u. Hence u is of Type t4d or t5 w.r.t. Σ'. By Theorems 8.1 and 8.2, $\Sigma' = \bar{C}_6$. Denote x_1 by b_4. Suppose there exists a node v not adjacent to u, such that $N(v) \cap \Sigma = \{a_1, a_2, b_1, b_3\}$. Then $\{a_1, a_2, a_4, b_1, b_3, b_4, v\}$ must induce an universal wheel with center v, and hence v is adjacent to a_4 and b_4. If u is of Type t4d w.r.t. Σ', then $\{a_2, a_3, a_4, b_1, u, v\}$ induces an odd wheel with center a_2. If u is of Type t5 w.r.t. Σ', then $\{a_4, b_2, b_3, b_4, u, v\}$ induces an odd wheel with center b_4. Therefore, such a node v cannot exist, and hence Σ and u satisfy (iii) of Theorem 8.1, a contradiction.

Case 2: $N(u) \cap \Sigma = \{a_1, a_2, b_1, b_3\}$

By Lemma 5.1, u is of Type t2p or t4d w.r.t. Σ'. Suppose u is of Type t2p w.r.t. Σ'. Then a_4 or x_n is the unique neighbor of u in $Q \cup \{a_4\}$, and hence $Q \cup \{a_1, a_2, a_4, b_1, b_3, u\}$ induces a proper wheel with center u that is not a beetle, a contradiction. So u is of Type t4d w.r.t. Σ'. By Theorem 8.1, $\Sigma' = \bar{C}_6$. Denote x_1 by b_4. Suppose there exists a node v not adjacent to u, such that $N(v) \cap \Sigma = \{a_1, a_3, b_2, b_3\}$. If v is adjacent to a_4, then $\{b_1, b_2, b_3, a_4, u, v\}$ induces an odd wheel with center b_3. So v is not adjacent to a_4. By Lemma 5.1 applied to Σ', v is adjacent to b_4. But then $\{a_1, a_2, a_4, b_1, b_3, b_4, v\}$ induces an odd wheel with center v. Therefore, such a node v cannot exist, and hence Σ' and u satisfy (iii) of Theorem 8.1, a contradiction. □

9.3 Blocking Sequences for 2-Joins

In this section, we consider an induced subgraph H of G which contains a 2-join $H_1|H_2$. We say that a 2-join $H_1|H_2$ *extends* to G if there exists a 2-join of G, $H_1'|H_2'$ with $H_1 \subseteq H_1'$ and $H_2 \subseteq H_2'$. We characterize the situation in which the 2-join of H does not extend to a 2-join of G.

Definition 9.9 *A* blocking sequence *for a 2-join $H_1|H_2$ of a subgraph H of G is a sequence of distinct nodes $x_1, \ldots, x_n$ in $G \setminus H$ with the following properties:*

1. *i) $H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$,*
   ii) $H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, and
   iii) if $n > 1$ then, for $i = 1, \ldots, n-1$, $H_1 \cup x_i|H_2 \cup x_{i+1}$ is not a 2-join of $H \cup \{x_i, x_{i+1}\}$.
2. *$x_1, \ldots, x_n$ is minimal with respect to Property 1, in the sense that no sequence $x_{j_1}, \ldots, x_{j_k}$ with $\{x_{j_1}, \ldots, x_{j_k}\} \subset \{x_1, \ldots, x_n\}$, satisfies Property 1.*

Blocking sequences with respect to a 1-join were introduced and studied by Geelen in [10]. Blocking sequences with respect to a 2-join were introduced in [6], where the following results are obtained.

Let H be an induced subgraph of G with 2-join $H_1|H_2$ and special sets A_1, B_1, A_2, B_2.

In the following remarks and lemmas, we let $S = x_1, \ldots, x_n$ be a blocking sequence for the 2-join $H_1|H_2$ of a subgraph H of G.

Remark 9.10 *$H_1|H_2 \cup u$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_1 = \emptyset, A_1$ or B_1. Similarly $H_1 \cup u|H_2$ is a 2-join in $H \cup u$ if and only if $N(u) \cap H_2 = \emptyset, A_2$ or B_2.*

Lemma 9.11 *If $n > 1$ then, for every node x_j, $j \in \{1, \ldots, n-1\}$, $N(x_j) \cap H_2 = \emptyset, A_2$ or B_2, and for every node x_j, $j \in \{2, \ldots, n\}$, $N(x_j) \cap H_1 = \emptyset, A_1$ or B_1.*

Lemma 9.12 *If $n > 1$ and $x_i x_{i+1}$ is not an edge, where $i \in \{1, \ldots, n-1\}$, then either $N(x_i) \cap H_2 = A_2$ and $N(x_{i+1}) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_{i+1}) \cap H_1 = B_1$.*

Theorem 9.13 *Let H be an induced subgraph of graph G that contains a 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G if and only if there exists no blocking sequence for $H_1|H_2$ in G.*

Lemma 9.14 *For $1 < i < n$, $H_1 \cup \{x_1, \dots, x_{i-1}\} | H_2 \cup \{x_{i+1}, \dots, x_n\}$ is a 2-join in $H \cup (S \setminus \{x_i\})$.*

Lemma 9.15 *If $x_i x_k$, $n \geq k > i + 1 \geq 2$, is an edge then either $N(x_i) \cap H_2 = A_2$ and $N(x_k) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_k) \cap H_1 = B_1$.*

Lemma 9.16 *If x_j is the node of lowest index adjacent to a node in H_2, then $x_1, \dots, x_j$ is a chordless path. Similarly, if x_j is the node of highest index adjacent to a node in H_1, then $x_j, \dots, x_n$ is a chordless path.*

Theorem 9.17 *Let G be a graph and H an induced subgraph of G with 2-join $H_1|H_2$ and special sets A_1, B_1, A_2, B_2. Let H' be an induced subgraph of G with 2-join $H'_1|H_2$ and special sets A'_1, B'_1, A_2, B_2 such that $A'_1 \cap A_1 \neq \emptyset$ and $B'_1 \cap B_1 \neq \emptyset$. If S is a blocking sequence for $H_1|H_2$ and $H'_1 \cap S \neq \emptyset$, then a proper subset of S is a blocking sequence for $H'_1|H_2$.*

9.4 2-Join Decompositions

Throughout this section we assume that G is an even-signable graph that does not contain a double star cutset. By Theorem 3.3 G does not contain a Mickey Mouse. By Theorems 3.2 and 4.1, G does not contain a proper wheel that is not a beetle or an L-parachute. By Theorems 8.1, 8.2 and 8.3, no node is of Type t4s w.r.t. a $\Sigma = 3PC(\Delta, \Delta)$, if $\Sigma \neq \bar{C}_6$ then no node is of Type t4d or t5 w.r.t. Σ, and if a node u is of Type t6 w.r.t. Σ then either $\Sigma = \bar{C}_6$ or none of the paths of Σ is an edge and u has no neighbors in the interior of any of the paths of Σ.

Lemma 9.18 *Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and let y be a Type t2 or t2p node w.r.t. Σ, adjacent to say b_2 and b_3. Then*

(i) there cannot exist a node x that is of Type t1 w.r.t. Σ adjacent to b_3 and y;

(ii) every node x of Type t2 w.r.t. Σ adjacent to b_1, b_2 is adjacent to y, and every sibling x of b_3 w.r.t. Σ is adjacent to y.

Proof: We first prove (i). If y is of Type t2p w.r.t. Σ, let Σ^y be obtained by substituting y into Σ. Otherwise, by Theorem 7.7 let $P^y = y_1, \dots, y_m$ be an attachment of y to Σ, and let Σ^y be obtained by substituting y and P^y into Σ. Assume there is a node x of Type t1 w.r.t. Σ, adjacent to b_3 and to y. By Lemma 5.1 applied to Σ^y, x is of Type t2 w.r.t. Σ^y. By Theorem 7.7, let $P^x = x_1, \dots, x_n$ be an attachment of x to Σ^y.

First we show that no node of P^1 is adjacent to or coincident with a node of $P^x \setminus x_n$. Assume not and let x_i be the node of $P^x \setminus x_n$ with lowest index that is adjacent to a node of P^1. Then $x, P^x_{x_1x_i}$ contradicts Lemma 7.2 applied to Σ. Therefore, no node of P^1 is adjacent to or coincident with a node of $P^x \setminus x_n$.

Suppose that x_n is of Type t1, p1 or p3 w.r.t. Σ^y. Then its neighbors in Σ^y are contained in P^2. By Lemma 7.3 applied to x, P^x and Σ, x_n is of Type t2 w.r.t. Σ, adjacent to a_1 and a_2, y is of Type t2 w.r.t. Σ and y_m is of Type t2, t2p or t3p w.r.t. Σ. But then $P^1 \cup P^x \cup \{x, y, b_2, b_3\}$ induces an odd wheel with center b_3.

So x_n is of Type t2, t2p or t3p w.r.t. Σ^y. So x_n is adjacent to a_3, and if it is of Type t2p or t3p w.r.t. Σ^y then it has a neighbor in $P^2 \setminus a_2$. If x_n is adjacent to a_1, then by Lemma 5.1 x_n is of Type t2, t2p or t3p w.r.t. Σ, and hence x, P^x contradicts Lemma 7.3 applied to Σ. So x_n is not adjacent to a_1. Hence y is of Type t2 w.r.t. Σ. By Lemma 5.1, x_n is of Type t1 w.r.t. Σ. But then $P^1 \cup P^x \cup \{x, y, b_2, b_3, a_3\}$ induces an odd wheel with center b_3.

Now we prove (ii). If x is of Type t2p or t3p w.r.t. Σ, let Σ' be obtained by substituting x for its sibling b_3. If x is of Type t2 w.r.t. Σ, then by Theorem 7.7, there is an attachment $Q = x_1, \dots, x_n$ of x to Σ. In this case, let Σ' be obtained by substituting x and its attachment Q into Σ. Note that $P^1 \cup P^2 \subseteq \Sigma'$. Suppose that y is not adjacent to x. Then by Lemma 5.1 applied to Σ', y is of Type t1 w.r.t. Σ' and hence of Type t2 w.r.t. Σ. By Theorem 7.7 there is an attachment $P^y = y_1, \dots, y_m$ of y to Σ. Let Σ^y be obtained by substituting y and P^y into Σ. Suppose x is of Type t2p or t3p w.r.t. Σ. If a_1 is contained in Σ^y, then x and Σ^y violate Lemma 5.1. So a_1 is not contained in Σ^y. In particular, y_m is of Type t2, t2p or t3p w.r.t. Σ. By defenition of attachment, y_m is not adjacent to b_1. But then y, P^y and Σ' contradict Lemma 7.3.

So x is of Type t2 w.r.t. Σ. Let R be a shortest path from x to y in $P^y \cup \Sigma' \setminus (P^2 \cup \{b_1, b_3\})$. Then $R \cup b_2$ induces a hole H'. If b_1 has a neighbor in $R \setminus x$, then b_1 is adjacent to a_1 and a_1 is in R, and hence (H', b_1) is an odd wheel. So b_1 has no neighbor in $R \setminus x$. Similarly b_3 has no neighbor in $R \setminus y$. But then (Rb_3b_1, b_2) is an odd wheel. □

Lemma 9.19 *Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ and let d be of Type t2 or t2p w.r.t. Σ adjacent to a_2 and a_3, or to b_2 and b_3. Assume that if d is of Type t2p w.r.t. Σ then $\Sigma \neq \bar{C}_6$. Suppose u is of Type t2 w.r.t. Σ adjacent to a_1 or b_1, or of Type t2p or t3p w.r.t. Σ being a sibling of a_2, b_2, a_3 or b_3, or of Type t1 w.r.t. Σ adjacent to a_2, b_2, a_3 or b_3. If u is of Type t2p or t3p w.r.t. Σ let Σ' be obtained by substituting u into Σ. If u is of Type t1 or t2 w.r.t. Σ, let $Q = y_1, \dots, y_m$ be its attachment to Σ (which exists by Theorem 7.7) and let Σ' be obtained by substituting u and Q into Σ. Then the following hold.*

(i) If there is no P^1-crosspath w.r.t. Σ, then there is no P^1-crosspath w.r.t. Σ'.

(ii) Node d is of the same type w.r.t. Σ' as it is w.r.t. Σ.

Proof: First we prove (i). W.l.o.g. we may assume that if u is of Type t2, t2p or t3p w.r.t. Σ then it is adjacent to a_1 and a_2, and if u is of Type t1 w.r.t. Σ then it is adjacent to a_3. Suppose there is no P^1-crosspath w.r.t. Σ, but that $P = x_1, \dots, x_n$ is a P^1-crosspath w.r.t. Σ'. Note that $P^1 \cup P^2 \subseteq \Sigma'$. Let P^3_u be the path of $\Sigma' \setminus (P^1 \cup P^2)$. W.l.o.g. x_1 has a neighbor in P^1. If a node of P^3 has a neighbor in $P \setminus x_n$, then by Lemma 5.1 and Lemma 7.2, a subpath of P is a P^1-crosspath w.r.t. Σ, a contradiction. So no node of P^3 is adjacent to or coincident with a node of $P \setminus x_n$. Suppose that x_n has a neighbor in P^2. Then by Lemma 5.1, x_n is of Type p2 or p4 w.r.t. Σ. Since P cannot be a P^1-crosspath w.r.t. Σ, $n > 1$, x_n is of Type p4 w.r.t. Σ, and $N(x_n) \cap \Sigma \subseteq P^2 \cup P^3$. But then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$ contains a $3PC(b_1b_2b_3, x_n)$. So x_n does not have a neighbor in P^2, and hence it has a neighbor in P^3_u. If x_n has a neighbor in P^3, then by Lemma 5.1 and Lemma 7.2, P is a P^1-crosspath w.r.t. Σ. So x_n does not have a neighbor in P^3, and hence the neighbors of x_n in P^3_u are contained in $P^3_u \setminus P^3$. Since x_n is of Type p2 or p4 w.r.t. Σ', x_n has a neighbor in Q. In particular, u is of Type t2 or t1 w.r.t. Σ. Let y_i be such a neighbor with highest index.

Suppose u is of Type t2 w.r.t. Σ. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then by Lemma 7.2, $P \cup Q_{y_i y_m}$ is a P^1-crosspath w.r.t. Σ. So y_m is of Type t2, t2p or t3p w.r.t. Σ, adjacent to b_1, b_2 and no node of $(P^1 \cup P^2) \setminus \{b_1, b_2\}$. If y_m is of Type t2 w.r.t. Σ, then $P \cup Q_{y_i y_m}$ contradicts Lemma 7.4 applied to Σ. So y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. But then $P \cup Q_{y_i y_{m-1}}$ contradicts Lemma 7.2 or Lemma 5.1 applied to Σ''.

Now suppose that u is of Type t1 w.r.t. Σ. If a_3' has a neighbor in $Q_{y_i y_{m-1}}$ then $Q_{y_i y_j} \cup P$ (where y_j is its neighbor in $Q_{y_i y_{m-1}}$ with lowest index) contradicts Lemma 7.2 applied to Σ. So a_3' does not have a neighbor in $Q_{y_i y_{m-1}}$. If y_m is of Type t1, p1, p2 or p3 w.r.t. Σ, then by Lemma 7.2 applied to Σ, the path $P \cup Q_{y_i y_m}$ is a P^1-crosspath w.r.t. Σ. If y_m is of Type t2 w.r.t. Σ, then $P \cup Q_{y_i y_m}$ contradicts Lemma 7.4 applied to Σ. Suppose y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. Then $P \cup Q_{y_i y_{m-1}}$ contradicts Lemma 7.3 applied to Σ''. So y_m is of Type t3 w.r.t. Σ. Hence $a_3' = b_3$ and a_3' has a neighbor in $Q \setminus y_m$. But then the shortest path from x_1 to b_3 in $P \cup (Q \setminus y_m) \cup b_3$ contradicts Lemma 7.2 applied to Σ. Therefore, there is no P^1-crosspath w.r.t. Σ'.

Now we prove (ii). First suppose that u is of Type t2, t2p or t3p w.r.t. Σ. W.l.o.g. we may assume that u is adjacent to a_1 and a_2. Suppose d is adjacent to a_2 and a_3. Then by Lemma 9.18(ii), d is adjacent to u. If d is of Type t2 w.r.t. Σ, then by Lemma 5.1, d is of Type t2 w.r.t. Σ'. So we may assume that d is of Type t2p w.r.t. Σ and that d is not of Type t2p w.r.t. Σ'. Then by Lemma 5.1, d must be of Type t4d w.r.t. Σ'. In particular, u is of Type t2 w.r.t. Σ, d is adjacent to y_m and y_m is of Type t2, t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting d into Σ. By defenition of attachment y_m is not adjacent to a_3, and hence y_m and Σ'' violate Lemma 5.1.

Now assume that d is adjacent to b_2 and b_3. Suppose u is of Type t2p or t3p w.r.t. Σ, or u is of Type t2 w.r.t. Σ and y_m is of Type t1, p1 or p3 w.r.t. Σ. By Lemma 5.1, if d is of Type t2 w.r.t. Σ, then d is of Type t2 w.r.t. Σ'. Suppose that d is of Type t2p w.r.t. Σ and that it is not of Type t2p w.r.t. Σ'. Then by Lemma 5.1, d is of Type t4d w.r.t. Σ'. By Theorem 8.1, $\Sigma' = \bar{C}_6$. In particular, P^1 and P^2 are edges. Let Σ'' be obtained by substituting d into Σ. Then u is of Type t4d or t5 w.r.t. Σ'', and hence by Theorems 8.1 and 8.2, $\Sigma'' = \bar{C}_6$. So P^3 is an edge, and hence $\Sigma = \bar{C}_6$, a contradiction. So now we may assume that u is of Type t2 w.r.t. Σ and y_m is of Type t2, t2p or t3p w.r.t. Σ. By Lemma 9.18(ii), d is adjacent to y_m. By Lemma 5.1, if d is of Type t2 w.r.t. Σ, then d is of Type t2 w.r.t. Σ'. Suppose d is of Type t2p w.r.t. Σ. Let Σ'' be obtained by substituting d into Σ. By Lemma 5.1 applied to u and Σ'', u is not adjacent to d. So by Lemma 5.1 applied to d and Σ', d is of Type t2p w.r.t. Σ'.

Now suppose that u is of Type t1 w.r.t. Σ, w.l.o.g. adjacent to a_3. Then $\Sigma' \neq \bar{C}_6$. Assume d is adjacent to a_2 and a_3. If d is of Type t2 w.r.t. Σ, then by Lemma 5.1, d is of Type t2 w.r.t. Σ'. So we may assume that d is of Type t2p w.r.t. Σ. By Theorem 8.1, d cannot be of Type t4d w.r.t. Σ', and hence by Lemma 5.1, d is of Type t2p w.r.t. Σ'. Now assume that d is adjacent to b_2 and b_3. If y_m is of Type t1, p1, p2 or p3, then by Lemma 5.1, d is of the same type w.r.t. Σ' as it is w.r.t. Σ. If y_m is of Type t2, t2p or t3p, then by Lemma 9.18(ii) y_m is adjacent to d and therefore by Lemma 5.1, d is of the same type w.r.t. Σ' as it is w.r.t. Σ. So we may assume that y_m is of Type t3 w.r.t. Σ. If y_m is adjacent to d, then by Lemma 5.1, d is of the same type w.r.t. Σ' as it is w.r.t. Σ. Assume y_m is not adajacent to d. By Lemma 5.1 applied to Σ', d is of Type t1 w.r.t. Σ' and hence of Type t2 w.r.t. Σ. By Theorem 7.7, let $P = x_1, \ldots, x_n$ be an attachment of d to Σ. Let Σ^d be obtained by substituting d and P into Σ. Let H be the hole induced by $Q \cup P^1 \cup \{u, a_3\}$. Then (H, b_3) must be a beetle. In particular, y_{m-1} and y_m are the only neighbors of b_3 in Q. Suppose that a node of $P \setminus x_n$ has a neighbor in $(Q \setminus \{y_{m-1}, y_m\}) \cup u$. Then there is a path from u to d that contradicts Lemma 7.4 applied to Σ. So no node of $P \setminus x_n$ has a neighbor in $(Q \setminus \{y_{m-1}, y_m\}) \cup u$. By Theorems 8.1 and 8.2, x_n cannot be of Type t4d or t5 w.r.t. Σ', and hence x_n is not adjacent to a node of $Q \cup u$. By Lemma 5.1 applied to Σ^d, y_{m-1} cannot have a neighbor in P. Suppose that y_m has a neighbor in P. Then $P \cup Q \cup (\Sigma \setminus (P^2 \cup b_1))$ contains a $3PC(y_{m-1}y_m b_3, a_3)$. So y_m does not have a neighbor in P. Let R be a shortest path from d to y_m in $(\Sigma \cup P \cup Q \cup \{u, d\}) \setminus (P^2 \cup \{b_1, b_3\})$. Then $R \cup \{b_2, b_3\}$ induces a proper wheel with center b_3 that is not a beetle, a contradiction. □

Lemma 9.20 *Let G be an even-signable graph that does not have a double star cutset. If G contains a $3PC(\Delta, \Delta)$ with a Type t2 node or a $3PC(\Delta, \Delta) \neq \bar{C}_6$ with a Type t2p node, then G has a 2-join.*

Proof: By Theorems 3.2 and 4.1, G contains neither a proper wheel that is not a beetle nor an L-parachute. By Theorems 8.1 and 8.2, there is no node of Type t4s or t6b w.r.t. a $3PC(\Delta, \Delta)$.

If G contains a $3PC(\Delta, \Delta)$ with a Type t2 node, then by Lemma 9.3, G contains a decomposable $3PC(\Delta, \Delta)$. If G contains a $3PC(\Delta, \Delta) \neq \bar{C}_6$ with a Type t2p node, then by Lemma 9.7, G contains a decomposable $3PC(\Delta, \Delta)$. So we may assume that G contains a decomposable $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ together with a node d of Type t2 or t2p adjacent to a_2, a_3 or to b_2, b_3. By Theorem 8.1 and Lemma 9.8, no node is of Type t4d w.r.t. Σ. By Theorem 8.2, no node is of Type t5 w.r.t. Σ. Since Σ has no P^1-crosspath, no node is of Type t6a w.r.t. Σ by Theorem 8.3. Suppose that the 2-join $H_1|H_2$ of $H = \Sigma, d$ does not extend to a 2-join of G. By Theorem 9.13, there is a blocking sequence $S = x_1, \ldots, x_n$. W.l.o.g. assume that H and S are chosen as follows. Let $\mathcal{H}$ be the set of all decomposable Σ, d. If $\mathcal{H}$ contains a Σ, d where d is of Type t2 w.r.t. Σ, then remove from $\mathcal{H}$ all Σ', d' where d' is of Type t2p w.r.t. Σ'. Choose an $H = \Sigma, d$ from $\mathcal{H}$ so that the size of the corresponding blocking sequence S is minimized.

Claim 1: *If x_i is of Type p4 w.r.t. Σ, then $N(x_i) \cap H \subseteq P^2 \cup P^3$. If x_i is of Type p1 or p2 w.r.t. Σ and $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$, then $N(x_i) \cap H \subseteq P^2 \cup P^3$.*

Proof of Claim 1: W.l.o.g. assume that d is adjacent to a_2, a_3. Suppose x_i is of Type p4 w.r.t. Σ. Since there is no P^1-crosspath w.r.t. Σ, $N(x_i) \cap \Sigma \subseteq P^2 \cup P^3$. Suppose x_i is adjacent to d. If d is of Type t2 w.r.t. Σ, then d, x_i contradicts Lemma 7.4. So d is of Type t2p w.r.t. Σ, and hence $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup \{d, x_i\}$ contains a $3PC(b_1b_2b_3, x_i)$. So x_i is not adjacent to d.

Now suppose that x_i is of Type p1 or p2 w.r.t. Σ, with neighbors in Σ w.l.o.g. contained in P^3. It is enough to show that x_i is not adjacent to d. Suppose x_i is adjacent to d. If d is of Type t2 w.r.t. Σ, then d, x_i contradicts Lemma 7.4. If d is of Type t2p w.r.t. Σ, then $(\Sigma \setminus \{a_1, a_3\}) \cup \{d, x_i\}$ contains a $3PC(b_1b_2b_3, d)$. This completes the proof of Claim 1.

Claim 2: *No node of S is of Type t2 w.r.t. Σ, or of Type t2p or t3p w.r.t. Σ being a sibling of a_2, a_3, b_2 or b_3, or of Type t1 w.r.t. Σ adjacent to a_2, a_3, b_2 or b_3.*

Proof of Claim 2: If x_i is of Type t2 w.r.t. Σ adjacent to a_2 and a_3, or to b_2 and b_3, then Σ, x_i is decomposable and by Theorem 9.17 applied to $H = \Sigma, d$ and $H' = \Sigma, x_i$, the minimality of S is contradicted. So by symmetry it is enough to consider the case when x_i is adjacent to a_1, a_2 and it is of Type t2, t2p or t3p w.r.t. Σ, or x_i is adjacent to a_3 and it is of Type t1 w.r.t. Σ. If x_i is of Type t2p or t3p w.r.t. Σ, let Σ' be obtained by substituting x_i into Σ. If x_i is of Type t1 or t2 w.r.t. Σ, then by Theorem 7.7, there is an attachment $Q = y_1, \dots, y_m$ of x_i to Σ. In this case let Σ' be obtained by substituting x_i and Q into Σ. Note that $P^1 \cup P^2 \subseteq \Sigma'$. By Lemma 9.19, there is no P^1-crosspath w.r.t. Σ', and node d is of the same type w.r.t. Σ' as it is w.r.t. Σ. If Σ', d is decomposable then by Theorem 9.17, the minimality of S is contradicted. So Σ', d is not decomposable. In particular, $\Sigma' = \bar{C}_6$ and d is of Type t2p w.r.t. Σ and Σ'. By the choice of Σ, d and by Lemma 9.3, G has no $3PC(\Delta, \Delta)$ with a Type t2 node. But then Σ' and d contradict Lemma 9.4. This completes the proof of Claim 2.

Claim 3: *No node of S is of Type p3 w.r.t. Σ with neighbors in $P^2 \cup P^3$.*

Proof of Claim 3: Suppose x_i is of Type p3 w.r.t. Σ and w.l.o.g. assume that its neighbors in Σ are contained in P^2. Let Σ' be obtained by substituting x_i into Σ. Note that $\Sigma' \neq \bar{C}_6$. By Lemma 5.1, d is of the same type w.r.t. Σ' as it is w.r.t. Σ.

Let P' be the a_2b_2-path of Σ'. Suppose $P = y_1, \dots, y_m$ is a P^1-crosspath w.r.t. Σ'. W.l.o.g. y_1 has a neighbor in P^1. If a node of $P \setminus y_m$ has a neighbor in P^2, then by Lemma 7.2, a subpath of $P \setminus y_m$ is a P^1-crosspath w.r.t. Σ, a contradiction. So no node of $P \setminus y_m$ has a neighbor in P^2. But then by Lemma 5.1 and Lemma 7.2, P is a P^1-crosspath w.r.t. Σ, a contradiction. Therefore, there is no P^1-crosspath w.r.t. Σ'.

But then by Theorem 9.17 applied to $H = \Sigma, d$ and $H' = \Sigma', d$, our choice of $H = \Sigma, d$ is contradicted. This completes the proof of Claim 3.

By Claim 2, no node of S is of Type t2 w.r.t. Σ, or of Type t2p or t3p w.r.t. Σ being a sibling of a_2, a_3, b_2 or b_3, or of Type t1 w.r.t. Σ adjacent to a_2, a_3, b_2 or b_3. By Claims 1 and 3, $n > 1$. Since $H_1 | H_2 \cup x_1$ is not a 2-join of $H \cup x_1$, x_1 has a neighbor in $P^1 \cup d$ and either (i) $N(x_1) \cap H \subseteq P^1 \cup d$, or (ii) x_1 is of Type t2p or t3p w.r.t. Σ being a sibling of a_1 or b_1, or (iii) x_1 is of Type t3 w.r.t. Σ adjacent to, say, a_1, a_2 and a_3, x_1 is not adjacent to d, and d is adjacent to a_2, a_3. Note that the case where x_1 is of Type t3 adjacent to a_1, a_2, a_3 and d where d is adjacent to b_2, b_3 cannot occur since, in this case, there is a $3PC(x_1a_1a_3, b_3)$. Since $H_1 \cup x_n | H_2$ is not a 2-join of $H \cup x_n$, x_n has a neighbor in $P^2 \cup P^3$, and it is of Type p1, p2 or p4 w.r.t. Σ. By Lemma 9.11, for $i \in \{2, \dots, n-1\}$, x_i either has no neighbor in H or $N(x_i) \cap \Sigma = \{a_1\}$ or $\{b_1\}$ or $\{a_1, a_2, a_3\}$ or $\{b_1, b_2, b_3\}$ and, furthermore, if say $N(x_i) \cap \Sigma = \{a_1\}$ or $\{a_1, a_2, a_3\}$ then x_i is adjacent to d if d is adjacent to a_2, a_3, and x_i is not adjacent to d if d is adjacent to b_2, b_3. Let x_j be the node of S with highest index adjacent to a node of H_1. By Lemma 9.16, $x_j, \dots, x_n$ is a chordless path. Note that $j < n$ and that nodes $x_{j+1}, \dots, x_{n-1}$ have no neighbors in H.

Claim 4: *Let Σ be a $3PC(a_1a_2a_3, b_1b_2b_3)$ with no P^1-crosspath. Suppose that x_j is of Type t3 w.r.t. Σ, say adjacent to b_1, b_2 and b_3, and there is a $\Sigma' = 3PC(a_1a_2t, b_1b_2x_j)$ that contains $P^1 \cup P^2$ and such that t is not of Type t3 w.r.t. Σ. Then there is no P^1-crosspath w.r.t. Σ'.*

Proof of Claim 4: Let P' be the path of $\Sigma' \setminus (P^1 \cup P^2)$. Suppose $P = y_1, \dots, y_m$ is a P^1-

crosspath w.r.t. Σ'. W.l.o.g. y_1 has a neighbor in P^1. Suppose y_m has a neighbor in P^2. Since P cannot be a P^1-crosspath w.r.t. Σ, a node of P has a neighbor in P^3. Let y_i be such a node with lowest index. If $i \neq m$ then by Lemma 7.2, $P_{y_1 y_i}$ is a P^1-crosspath w.r.t. Σ. So $i = m$ and hence y_m is of Type p4 w.r.t. Σ. But then $(\Sigma \setminus \{a_1, a_2, a_3\}) \cup P$ contains a $3PC(b_1 b_2 b_3, y_m)$. So y_m has a neighbor in P'. Suppose that $P \cup P^3 \cup P' \setminus \{x_j, t\}$ contains a path from y_1 to P^3. Then by Lemma 7.2 applied to the shortest such path, there is a P^1-crosspath w.r.t. Σ. So no such path exists and hence no node of P^3 is adjacent to or coincident with a node of $P \cup P' \setminus \{x_j, t\}$. By a similar argument, $t \neq a_3$. So t is of Type t2, t2p or t3p w.r.t. Σ. Note that $P \cup P' \setminus x_j$ contains a chordless path T from y_1 to t. If t is of Type t2 w.r.t. Σ, then T contradicts Lemma 7.4 applied to Σ. So t is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting t into Σ. Then $T \setminus t$ contradicts Lemma 7.2 applied to Σ''. This completes the proof of Claim 4.

We now consider the following cases.

Case 1: x_j is of Type t3 w.r.t. Σ.

If x_n is of Type p1 or p4 w.r.t. Σ, then $x_j, \ldots, x_n$ contradicts Lemma 7.5. So x_n is of Type p2 w.r.t. Σ. W.l.o.g. x_n has a neighbor in P^3 and d is adjacent to a_2, a_3. Suppose x_j is adjacent to b_1, b_2 and b_3. Then there is a $\Sigma' = 3PC(a_1 a_2 a_3, b_1 b_2 x_j)$ contained in $(\Sigma \setminus b_3) \cup \{x_j, \ldots, x_n\}$. Note that $\Sigma' \neq \bar{C}_6$. By Claim 4, there is no P^1-crosspath w.r.t. Σ'. By Lemma 5.1, d is of the same type w.r.t. Σ' as it is w.r.t. Σ, and hence Σ', d is a decomposable $3PC(\Delta, \Delta)$. But then, by Theorem 9.17, the minimality of S is contradicted. So x_j is adjacent to a_1, a_2 and a_3. Let $\Sigma' = 3PC(a_1 a_2 x_j, b_1 b_2 b_3)$ be contained in $(\Sigma \setminus a_3) \cup \{x_j, \ldots, x_n\}$. Note that $\Sigma' \neq \bar{C}_6$. By Claim 4, there is no P^1-crosspath w.r.t. Σ'. If d is adjacent to x_j, then by Lemma 5.1 applied to Σ', d is of the same type w.r.t. Σ' as it is w.r.t. Σ, and hence Σ', d is a decomposable $3PC(\Delta, \Delta)$ and the minimality of S is contradicted. So d is not adjacent to x_j, and hence by Lemma 5.1, d is of Type t1 w.r.t. Σ' and of Type t2 w.r.t. Σ. By Theorem 7.7, let $Q = y_1, \ldots, y_m$ be an attachment of d to Σ.

First we show that no node of Q is adjacent to or coincident with a node of $\{x_j, \ldots, x_n\}$. Suppose not and let y_k be the node of Q with highest index that has a neighbor in $\{x_j, \ldots, x_n\}$. Let x_i be the neighbor of y_k in $\{x_j, \ldots, x_n\}$ with highest index.

Suppose $i \neq j$. Consider the possibilities for Q allowed by Lemma 7.4. If y_m is of Type t1, p1 or p3 w.r.t. Σ, then by Lemma 7.2 applied to Σ, $Q_{y_k y_m}, x_i, \ldots, x_n$ is a P^1-crosspath w.r.t. Σ, a contradiction. If y_m is of Type t2 w.r.t. Σ, then $Q_{y_k y_m}, x_i, \ldots, x_n$ contradicts Lemma 7.4 applied to Σ. So y_m is of Type t2p or t3p w.r.t. Σ. Let Σ'' be obtained by substituting y_m into Σ. Then either $Q_{y_{k+1} y_{m-1}}, x_i, \ldots, x_n$ (if $k \neq m$) or $x_i, \ldots, x_n$ (otherwise) contradicts Lemma 7.3 or 5.1 applied to Σ''. Therefore, $i = j$.

If x_j is adjacent to y_m, then y_m and Σ' contradict Lemma 5.1 (since y_m cannot be adjacent to a_1 by definition of attachment). So x_j is not adjacent to y_m, i.e. $k < m$. Then $x_j, Q_{y_k y_m}$ contradicts Lemma 7.5 applied to Σ.

Therefore, no node of Q is adjacent to or coincident with a node of $\{x_j, \ldots, x_n\}$. Let $\Sigma'' = 3PC(a_2 a_3 d, \Delta)$ be obtained by substituting d and Q into Σ. Then $x_j, \ldots, x_n$ contradicts Lemma 7.4 applied to Σ''.

Case 2: x_j is of Type t1 adjacent to a_1 or b_1, or $j = 1$ and x_1 is of Type p1, p2 or p3 w.r.t. Σ.

If x_n is if Type p1 or p2 w.r.t. Σ, then by Lemma 7.2, $x_j, \ldots, x_n$ is a P^1-crosspath w.r.t. Σ. If x_n is of Type p4 w.r.t. Σ, then $(\Sigma \setminus \{b_2, b_3\}) \cup \{x_j, \ldots, x_n\}$ contains a $3PC(a_1a_2a_3, x_n)$.

Case 3: $j = 1$ and d is the unique neighbor of x_1 in H.

W.l.o.g. assume that d is adjacent to a_2, a_3. If d is of Type t2p w.r.t. Σ, let Σ' be obtained by substituting d into Σ. Then $x_1, \ldots, x_n$ contradicts Lemma 7.3 applied to Σ'. So d is of Type t2 w.r.t. Σ. Then $d, x_1, \ldots, x_n$ contradicts Lemma 7.4.

Case 4: $j = 1$ and x_1 is of Type t2p or t3p w.r.t. Σ.

W.l.o.g. x_1 is a sibling of b_1. Let Σ' be obtained by substituting x_1 into Σ. Then $x_2, \ldots, x_n$ contradicts Lemma 7.3 or 5.1 applied to Σ'. □

Theorem 9.1 follows from Lemmas 9.4 and 9.20.

Corollary 9.21 *Let G be an even-signable graph. If G contains a proper wheel, or an L-parachute, or a $3PC(\Delta, \Delta)$ with a Type t2, t2p or t4s node, or a $3PC(\Delta, \Delta) \neq \bar{C}_6$ with a Type t4d or t5 node, then G has a double star cutset or a 2-join.*

Proof: If G contains a proper wheel, the result holds by Theorem 3.2 when the wheel is not a beetle, and by Theorems 6.1 and 9.1 when the wheel is a beetle. If G contains an L-parachute, the result holds by Theorem 4.1. If G contains a $\Sigma = 3PC(\Delta, \Delta)$ with a Type t2 or t2p node, the result holds by Theorem 9.1. If Σ has a Type t4s node or if $\Sigma \neq \bar{C}_6$ has a Type t4d node, the result holds by Theorem 8.1. If G contains a $\Sigma \neq \bar{C}_6$ with a Type t5 node, then the result holds by Theorem 8.2. □

So, by Theorem 6.1, it only remains to consider the case when G contains a $\bar{C}_6$ with a Type t4d or t5 node.

10 $\bar{C}_6$ with Type t4d or t5 Nodes

In this section we prove the following two theorems.

Theorem 10.1 *If G is an even-signable graph that does not have a double star cutset nor a 2-join, then G cannot contain a $\bar{C}_6$ with a Type t5 node.*

Theorem 10.2 *Let G be an even-signable graph that does not have a double star cutset nor a 2-join. If G contains a $\bar{C}_6$ with a Type t4d node, then G is the complement of the line graph of a complete bipartite graph.*

Throughout this section we assume that G is an even-signable graph that does not have a double star cutset nor a 2-join.

Lemma 10.3 *Let $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ be a $\bar{C}_6$ in G. Then the following hold.*

(i) No node is of Type t1 or t3p w.r.t. Σ.

(ii) If there is a node of Type t4d or t5 w.r.t. Σ, then there is no node of Type t3 w.r.t. Σ.

Proof: By Theorem 9.1, no node is of Type t2 or t2p w.r.t. a $3PC(\Delta,\Delta)$. By Lemma 9.4, there cannot be a node of Type t3p w.r.t. Σ.

Suppose node u is of Type t1 w.r.t. Σ. By Theorem 7.7, there is an attachment $P = x_1,\dots,x_n$ of u to Σ. Then x_n must be of Type t3 w.r.t. Σ. W.l.o.g. assume that u is adjacent to a_3. Then $P^2 \cup P^3 \cup P \cup u$ induces a proper wheel with center b_3, contradicting Corollary 9.21. Therefore, no node is of Type t1 w.r.t. Σ, and (i) holds.

If a node is of Type t5 w.r.t. Σ, then by Theorem 8.2, there is a node of Type t4d w.r.t. Σ. So to prove (ii) we may assume that there is a node u of Type t3 w.r.t. Σ and a node v of Type t4d w.r.t. Σ. W.l.o.g. assume that u is adjacent to a_1,a_2,a_3 and v to a_1,a_2,b_1,b_3. By Theorem 7.7, let $P = x_1,\dots,x_n$ be an attachment of u to Σ. Since no node is of Type t1, t2p or t3p w.r.t. Σ, x_n must be of Type p2 or t3 w.r.t. Σ and no node of $P \setminus x_n$ is adjacent to a node of Σ. First suppose that x_n is of Type p2 w.r.t. Σ. Let Σ' be obtained by substituting u and P into Σ. Note that $\Sigma' \neq \bar{C}_6$. By Lemma 5.1, v is of Type t2p, t4d or t5 w.r.t. Σ', contradicting Theorem 8.1, Theorem 8.2 or Theorem 9.1. So x_n is of Type t3 w.r.t. Σ. Let $\Sigma' = 3PC(a_1a_2u, b_1b_2x_n)$ (resp. $\Sigma'' = 3PC(ua_2a_3, x_nb_2b_3)$) be obtained by substituting u and P into Σ. By Lemma 5.1, v is of Type t3p, t4d or t5 w.r.t. Σ'. By Theorem 9.4 v cannot be of Type t3p w.r.t. Σ'. Suppose v is of Type t4d w.r.t. Σ'. Then v is adjacent to x_n and not adjacent to u, and hence v is of Type t2p w.r.t. Σ'', contradicting Theorem 9.1. So v is of Type t5 w.r.t. Σ', i.e. it is adjacent to both u and x_n. By Theorem 8.1, there is a node w of Type t4d w.r.t. Σ that is not adjacent to v and is adjacent to a_1,a_3,b_2,b_3. By the same argument as above, w must be adjacent to both u and x_n. But then $\{a_1,b_1,b_2,u,v,w\}$ induces an odd wheel with center a_1. □

Corollary 10.4 *If there is a node of Type t4d or t5 w.r.t. $\Sigma = \bar{C}_6$, then nodes of $G \setminus \Sigma$ that have a neighbor in Σ are of Type p2, t4d, t5 or t6 w.r.t. Σ.*

Proof: Since $\Sigma = \bar{C}_6$, no node is of Type p1, p3, p4 or t4s w.r.t. Σ. By Theorem 9.1, no node is of Type t2 or t2p w.r.t. Σ. By Lemma 10.3, no node is of Type t1, t3 or t3p w.r.t. Σ. □

Proof of Theorem 10.1: Let G be an even-signable graph that does not have a double star cutset nor a 2-join. Suppose that $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3)$ is a $\bar{C}_6$ in G and x is of Type t5 w.r.t. Σ. W.l.o.g. x is not adjacent to a_3. Let $S = (N(x) \cup N(a_2)) \setminus (\Sigma \setminus \{a_2,a_3,b_1,b_2\})$. Since S is not a double star cutset, there exists a direct connection $P = x_1,\dots,x_n$ in $G \setminus S$ from a_1 to b_3.

First we show that no node of S is of Type t4d or t5 w.r.t. Σ. Suppose x_i is of Type t4d or t5 w.r.t. Σ. Since x_i cannot be adjacent to a_2, it is adjacent to a_1,a_3,b_2. If x_i is adjacent to b_1, then $\{a_1,a_2,a_3,b_1,x,x_i\}$ induces an odd wheel with center a_1. So x_i is not adjacent to b_1, and hence it is adjacent to b_3. In particular, x_i is of Type t4d w.r.t. Σ. By Theorem 8.1, there is a node u not adjacent to x_i, that is of Type t4d w.r.t. Σ adjacent to a_2,a_3,b_1,b_2. Then $\{a_1,a_3,x_i,b_1,b_2,u\}$ induces a $\Sigma' = 3PC(a_1x_ia_3, b_1b_2u)$. Since x is adjacent to a_1,b_1,b_2, and it is not adjacent to x_i and a_3, by Lemma 5.1 x must be of Type t3p w.r.t. Σ'. But then Lemma 10.3 is contradicted. Therefore no node of S is of Type t4d or t5 w.r.t. Σ.

By Corollary 10.4 and by defenition of S, x_1 is of Type p2 w.r.t. Σ adjacent to a_1 and b_1, x_n is of Type p2 w.r.t. Σ adjacent to a_3 and b_3, and no iteremediate node of P has a neighbor in Σ. Let $\Sigma' = 3PC(a_1b_1x_1, a_3b_3x_n)$ induced by $P \cup \{a_1, a_3, b_1, b_3\}$. Since x is adjacent to a_1, b_1, b_3 and it is not adjacent to a_3, x_1, x_n, it violates Lemma 5.1 applied to Σ'. $\Box$

In the following results we assume that G contains a $\Sigma = 3PC(a_1a_2a_3, b_1b_2b_3) = \bar{C}_6$ with a Type t4d node. In fact by Theorem 8.1, we may assume that there are at least three nodes of Type t4d: node v_1 adjacent to a_1, a_2, b_1, b_3, node v_2 adjacent to a_2, a_3, b_1, b_2, and node v_3 adjacent to a_1, a_3, b_2, b_3. Furthermore, v_1v_2 and v_1v_3 are not edges. In fact, neither is v_2v_3, since otherwise $\{v_1, v_2, v_3, a_2, b_2, b_3\}$ induces and odd wheel with center b_2. By Corollary 10.4 and Theorem 10.1, nodes of $G \setminus \Sigma$ that have a neighbor in Σ are of Type p2, t4d or t6 w.r.t. Σ. We now show that all nodes of $G \setminus \Sigma$ are of Type p2, t4d or t6 w.r.t. Σ.

Lemma 10.5 *Let u be of Type t4d w.r.t. Σ and v of Type p2 w.r.t. Σ. Then uv is not an edge if and only if $N(v) \cap \Sigma \subseteq N(u) \cap \Sigma$.*

Proof: First we show that if u is adjacent to a_1, a_2, b_1, b_3 and v is adjacent to a_3, b_3, then uv is an edge. Suppose uv is not an edge. Let $S = (N(a_3) \cup N(b_3)) \setminus \{u, v\}$ and let $P = x_1, \ldots, x_n$ be a direct connection from u to v in $G \setminus S$. By definition of S, no node of P is of Type t4d or t6 w.r.t. Σ. If a_2 has no neighbor in P, then $P \cup \{a_2, a_3, b_3, u, v\}$ induces a $3PC(a_3b_3v, u)$. So a_2 has a neighbor in P, and similarly so does a_1. Let x_i be the node of P with highest index adjacent to a_1 or a_2. By Corollary 10.4, x_i must be of Type p2 w.r.t. Σ. If x_i is adjacent to a_2 and b_2, then $P_{x_ix_n} \cup P^1 \cup \{a_2, b_3, u, v\}$ induces a proper wheel with center u, contradicting Corollary 9.21. So x_i is adjacent to a_1 and b_1. Let $\Sigma' = 3PC(a_1b_1x_i, a_3b_3v)$ induced by $P_{x_ix_n} \cup P^1 \cup P^3 \cup v$. Then u is of Type t3p w.r.t. Σ', contradicting Lemma 9.4.

Next we show that if u is adjacent to a_1, a_2, b_1, b_3 and v is adjacent to a_1, b_1, then uv is not an edge. Assume uv is an edge. By Theorem 8.1, there exists a node w of Type t4d w.r.t. Σ adjacent to a_2, a_3, b_1, b_2 and not adjacent to u. By the above paragraph, vw is an edge. But then $\{u, v, w, b_1, b_2, b_3\}$ induces an odd wheel with center b_1. $\Box$

Lemma 10.6 *Nodes of $G \setminus \Sigma$ are of Type p2, t4d or t6 w.r.t. Σ.*

Proof: We show that if u is a node of $G \setminus \Sigma$ that has a neighbor in Σ, then there cannot exist a node x adjacent to u and not adjacent to Σ. Assume not.

First suppose that u is of Type t4d w.r.t. Σ, say adjacent to a_1, a_2, b_1, b_3. Let $S = (N(u) \cup N(a_1)) \setminus x$ and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. By Lemma 10.5 and definition of S, no node of P is of Type p2 or t6 w.r.t. Σ, or of Type t4d w.r.t. Σ adjacent to a_1. Let x_i be the node of P with lowest index that is of Type t4d. Then x_i is adjacent to a_2 and a_3, and hence $P_{x_1x_i} \cup \{u, x, a_1, a_2, a_3\}$ induces a proper wheel with center a_2, contradicting Corollary 9.21.

Next suppose that u is of Type p2 w.r.t. Σ, say adjacent to a_3, b_3. Let $S = (N(a_3) \cup N(u)) \setminus x$ and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. By Lemma 10.5 and definition of S, no node of P is of Type t4d or t6 w.r.t. Σ. So x_n is of Type p2 w.r.t. Σ, and no node of $P \setminus x_n$ has a neighbor in Σ. W.l.o.g. assume that x_n is adjacent to a_2, b_2. Let $\Sigma' = 3PC(a_2b_2x_n, a_3b_3u)$ induced by $P^2 \cup P^3 \cup P \cup \{u, x\}$. Note that $\Sigma' \neq \bar{C}_6$. By our assumption there is a node v_1 of Type t4d w.r.t. Σ adjacent to a_1, a_2, b_1, b_3. By Lemma 5.1, v_1 is of Type t2p or t4d w.r.t. Σ'. But this contradicts Theorem 9.1 or 8.1.

Finally suppose that u is of Type t6 w.r.t. Σ. Let $S = N(a_1) \cup a_1$ and let $P = x_1, \ldots, x_n$ be a direct connection from x to $\Sigma \setminus S$ in $G \setminus S$. By definition of S, no node of P is of Type t6 w.r.t. Σ. So x_n is of Type p2 or t4d w.r.t. Σ and no node of $P \setminus x_n$ has a neighbor in Σ. But then either x_n and x_{n-1} (if $n \neq 1$) or x_n and x (if $n = 1$) contradict the above paragraphs. □

Lemma 10.7 *Σ has exactly three Type p2 nodes, say u_1, u_2 and u_3, where u_i is adjacent to $a_i b_i$. Furthermore u_1, u_2, u_3 are pairwise adjacent.*

Proof: Let $S_1 = (N(a_1) \cup N(b_1)) \setminus \{a_2, b_3\}$. Since S_1 is not a double star cutset there exists a direct connection $P = x_1, \ldots, x_n$ from a_2 to b_3 in $G \setminus S_1$. By definition of S_1 and Lemma 10.6, every node of P is of Type p2. So $n = 2$, x_1 is adjacent to $a_2 b_2$ and x_2 is adjacent to $a_3 b_3$. Repeating the same argument with $S_2 = (N(a_2) \cup N(b_2)) \setminus \{a_1, b_3\}$, we get that Σ has three Type p2 nodes, say u_1, u_2 and u_3, where u_i is adjacent to $a_i b_i$.

Next we show that u_1, u_2 and u_3 are pairwise adjacent. W.l.o.g. assume that $u_2 u_3$ is not an edge. By our assumption there exist nonadjacent nodes v_1 and v_2, both of Type t4d w.r.t. Σ, such that v_1 is adjacent to a_1, a_2, b_1, b_3 and v_2 is adjacent to a_2, a_3, b_1, b_2. By Lemma 10.5, v_1 is adjacent to both u_2 and u_3, and v_2 is adjacent to u_3 but not to u_2. But then $\{v_1, v_2, u_2, u_3, b_2, a_2\}$ induces an odd wheel with center a_2.

Finally we show that there are exactly three Type p2 nodes. Assume w.l.o.g. that there exists a Type p2 node u_3' that is adjacent to $a_3 b_3$ and is distinct from u_3. By the above paragraph, $u_2 u_3$ and $u_2 u_3'$ are both edges. Let $\Sigma' = 3PC(a_2 b_2 u_2, a_3 b_3 u_3)$ induced by $\{u_2, u_3, a_2, a_3, b_2, b_3\}$. Then u_3' is of Type t2p or t3p w.r.t. Σ', contradicting Theorem 9.1 or Lemma 10.3. □

Lemma 10.8 *If u is of Type p2 w.r.t. Σ and v is of Type t6 w.r.t. Σ, then uv is an edge.*

Proof: Assume w.l.o.g. that u is adjacent to a_1, b_1 and that uv is not an edge. By Lemma 10.7 there is a node u_2 of Type p2 w.r.t. Σ adjacent to a_2, b_2 and uu_2 is an edge. Let $\Sigma' = 3PC(a_1 b_1 u, a_2 b_2 u_2)$ induced by $\{u, u_2, a_1, a_2, b_1, b_2\}$. By Lemma 5.1, v is of Type t5 w.r.t. Σ. But this contradicts Theorem 10.1. □

Lemma 10.9 *If u and u' are both of Type t4d w.r.t. Σ such that $N(u) \cap \Sigma = N(u') \cap \Sigma$, then uu' is not an edge.*

Proof: W.l.o.g. $N(u) \cap \Sigma = N(u') \cap \Sigma = \{a_1, a_2, b_1, b_3\}$. Suppose uu' is an edge. By Theorem 8.1 there exists a node v of Type t4d adjacent to a_2, a_3, b_1, b_2 and not adjacent to u. Let $\Sigma' = 3PC(va_2 a_3, b_1 u b_3)$ induced by $\{u, v, a_2, a_3, b_1, b_3\}$. Then u' is of Type t3p or t5 w.r.t. Σ', contradicting Theorem 10.1 or Lemma 10.3. □

Note that the six nodes of Σ together with the three nodes u_1, u_2, u_3 from Lemma 10.7 actually form six distinct $\bar{C}_6$ with their three Type p2 nodes. Each of these nine nodes is Type p2 in exactly two of the three $\bar{C}_6$. In addition, the Type t4d nodes w.r.t. Σ are Type t4d relative to all six of the $\bar{C}_6$. It follows from Lemma 10.5 that the adjacencies between the Type p2 nodes u_1, u_2, u_3 and the Type t4d nodes v_1, v_2, v_3 are totally determined. These six nodes together with the six nodes of Σ can be arranged on a 3×4 grid in such a way that the node in position (i, j) is adjacent to the node in position (p, q) if and only if $i \neq p$ and $j \neq q$. For example, we can set $a_1 = (3,3)$, $a_2 = (2,1)$, $a_3 = (1,2)$, $b_1 = (2,2)$, $b_2 = (1,3)$, $b_3 = (3,1)$ with the u_i's filling the remaining three positions (i, j) for $1 \leq i, j \leq 3$ and the v_i's in positions $(i, 4)$ for $1 \leq i \leq 3$. We call this 12-node graph $G_{3,4}$.

More generally, for $k \geq 3$ and $l \geq 4$, denote by G_{kl} the graph whose nodes are labeled (i, j) for $1 \leq i \leq k$ and $1 \leq j \leq l$, where an edge exists between (i, j) and (p, q) if and only if $i \neq p$ and $j \neq q$. The graph G_{kl} is the complement of the line graph of the complete bipartite graph K_{kl}. Any three rows and three columns of G_{kl} induce a graph on nine nodes which is a $\bar{C}_6$ plus its three Type p2 nodes. By symmetry every node of G_{kl} is of Type p2 w.r.t. at least one such $\bar{C}_6$.

Lemma 10.10 *Consider a maximal subgraph G_{kl} of G which is the complement of the line graph of a complete bipartite graph K_{kl}, with $k \geq 3$ and $l \geq 4$. Then every node $u \in V(G) \setminus V(G_{kl})$ is adjacent to all the nodes of G_{kl}.*

Proof: Consider any $\Sigma = \bar{C}_6$ in G_{kl} formed by three rows and three columns, say with nodes $(1,1), (1,2), (2,2), (2,3), (3,1), (3,3)$. By Lemma 10.7, u cannot be of Type p2 w.r.t. Σ since the three possible Type p2 nodes for Σ already exist in G_{kl}.

Suppose u is of Type t4d w.r.t. Σ. W.l.o.g. node u is adjacent to $(2,2), (2,3), (3,1), (3,3)$. Every node w of G_{kl} is of Type p2 w.r.t. some $\Sigma' = \bar{C}_6$ that contains node $(1,1)$. Since u is not adjacent to $(1,1)$, it follows from Lemma 10.6 that u is of Type t4d w.r.t. Σ'. By Lemma 10.5, the adjacency between u and w is determined. Specifically, node u is adjacent to the nodes of G_{kl} that are not in row 1 and is not adjacent to the nodes in row 1. Let us label node u by $(1, l+1)$. By Theorem 8.1 applied to Σ, G must contain nodes of Type t4d adjacent to $(1,1), (1,2), (3,1), (3,3)$ and to $(1,1), (1,2), (2,2), (2,3)$ respectively. Furthermore, these two nodes and u form a stable set. Therefore these two nodes are not in G_{kl}. By the same argument as above, their adjacencies with the nodes of G_{kl} are totally determined. Let us label them $(2, l+1)$ and $(3, l+1)$ respectively. Node $(i, l+1)$ is adjacent to all the nodes of G_{kl} except those in row i. By Theorem 8.1 applied to Σ' as defined above, there exist nodes $(i, l+1)$ in $V(G) \setminus V(G_{kl})$ that form a stable set for all $1 \leq i \leq k$ and that are adjacent with (p, q) in G_{kl} if and only if $i \neq p$. Therefore G contains a graph $G_{k,l+1}$, a contradiction to the maximality of G_{kl}. So node u is not of Type t4d w.r.t. Σ.

By Lemma 10.6, it follows that u is of Type t6 w.r.t. Σ. Since every node of G_{kl} belongs to a $\bar{C}_6$ of G_{kl}, it follows that u is adjacent to all the nodes of G_{kl}. $\square$

Theorem 10.2 follows from Lemma 10.10 since, if $G \neq G_{kl}$, then for any $u \notin V(G_{kl})$ the set $N((1,1)) \cup N(u) \setminus \{(1,2), (1,3)\}$ is a double star cutset separating $(1,2)$ from $(1,3)$.

Theorem 1.2 follows from Theorem 2.5, Theorem 6.1, Corollary 9.21, Theorem 10.1 and Theorem 10.2.

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